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On the domain decomposition method for parabolic problems

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The paper deals with studying the domain decomposition algorithm with overlapping subdomains. This algorithm is based on the splitting method and uses the additive presentation of some bilinear form. This method was described in [1] for two subdomains. In our consideration we formulate the decomposition algorithm for an arbitrary number of subdomains and give the error estimation in L_2-norm. The obtaning of this estimation is based on the analysis of the greed Green function of the implicit scheme operator, which is given in [2]. The method formulation is given in projection form with the finite element approximation. The approximate computation of the mass matrix is important in this paper. The lumping operators technique [3] is used for this purpose. The results of this investigation were published earlier in Russian in [4]. For briefness we give some statements without proofs, which may be found in [4].

1. Differential problem and discretization

Let Ω be a bounded polytop in R^m and B is some arbitrary subdomain of Ω. In the space $H^1(\Omega) \times H^1(\Omega)$ we consider one-parametric families of the bilinear forms

$$a_B(t; u, v) = \int_B \sum_{i,j=1}^{m} \lambda_{ij}(t, \bar{x}) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\bar{x}, \tag{1.1}$$

where $t \in [t_0, t_*]$ is a real parameter. For the functions $\lambda_{ij}(t, \bar{x})$ we assume that bilinear form (1.1) is symmetric, continuous and coercive, i.e., $\forall u, v \in H^1(\Omega)$ the inequalities

$$| a_B(t; u, v) | \leq a_0 \| u\|_{H^1(B)} \|v\|_{H^1(B)}, \tag{1.2}$$

$$a_B(t; u, u) \geq a_B |u|^2_{H^1(B)} \tag{1.3}$$

are valid, where a_0 and a_1 are positive numbers, which are independent of t. Here $|u|_{H^1(B)} = (\int_B |\nabla u|^2 d\bar{x})^{1/2}$ is a semi-norm in the space $H^1(B)$. Assumptions on smoothness of the functions $\lambda_{ij}(t, \bar{x})$ will be given in what follows. When $B = \Omega$ we will use notation $a(t; u, v) = a_\Omega(t; u, v)$. Then we introduce the family of the continues on t and continues in $L_2(\Omega)$ linear functionals $(f(t),u)$, where $(,)$ is the scalar product in $L_2(\Omega)$, $f:[t_0,t_*]\to L_2(\Omega)$. Henceforth, by $u(t)$ we denote the value of the function $u:[t_0,t_*]\to X$, which is the element of some Banach space X, and $\frac{du}{dt}(t)$ means a strong limit in X (if such limit exists) of the elements $[u(t)]_\tau\equiv(u(t+\tau)-u(t))/\tau$ for $\tau\to 0$.

Now we will formulate the parabolic Neumann problem, for which we will construct the domain decomposition algorithm [5]. Let $u_0\in L_2(\Omega)$ and $f\in L_2((t_0,t_*);H^{-1}(\Omega))$. It's necessary to find the function $u\in L_2((t_0,t_*);H^1(\Omega))$, such that $\frac{du}{dt}\in L_2((t_0,t_*);H^{-1}(\Omega))$ and $\forall v\in H^1(\Omega)$ the following equalities are valid:

$$\begin{aligned}(\frac{du}{dt}(t),v)+a(t;u(t),v) &= (f(t),v), \quad t\in(t_0,t_*], &(1.4)\\ (u(t_0),v) &= (u_0,v). &(1.5)\end{aligned}$$

Let $\mathcal{T}_h$ be a regular set of m-simplexes with "inverse assumption" [6]. For set $\mathcal{T}_h$ we will introduce the finite-dimensional space V_h with the piecewise linear basis $\{\varphi_i(\bar{x})\}_{i\in I}$, where $\varphi_i(\bar{x}_j)=\delta_{i,j}$, $i,j\in I$, $\bar{x}_j$ is a set of all different vertesies of the set $\mathcal{T}_h$. Then let $\Pi_h:C(\bar{\Omega})\to V_h$ be an interpolating operator defined by the formula $\Pi_h u(\bar{x})=\sum_{i\in I}u(\bar{x}_i)\varphi_i(\bar{x})$.

In accordance with paper [3] we introduce the lumping operator $P_{h,\mu}$ in the space V_h, for which the condition of the approximation relative to the scalar product in $L_2(\Omega)$ is valid. Let us consider the bilinear form

$$d_h(u,v)=(P_{h,\mu}u,P_{h,\mu}v), \quad u,v\in V_h. \tag{1.6}$$

As shown in [3], there exist numbers d_0 and d_1 independent of h, such that

$$\begin{aligned}|\,d_h(u,v)\,| &\le d_0\,\|\,u\,\|_{L_2(\Omega)}\|\,v\,\|_{L_2(\Omega)}, &(1.7)\\ d_h(u,u) &\ge d_1\,\|\,u\,\|^2_{L_2(\Omega)} &(1.8)\end{aligned}$$

for any $u,v\in V_h$.

Let us introduce some inequalities, which will be used futher. Let $u\in W_p^2(\Omega)$ where $p>m/2$. Then according to inclusion of $W_p^2(\Omega)$ into $C(\bar{\Omega})$ the function $\Pi_h u\in V_h$ is defined. In accordance with [3] for bilinear form (1.6) the following estimation of approximation is valid:

$$\begin{aligned}|\,(u,v)-d_h(\Pi_h u,v)\,| \le\ & ch^{k+l}\{|\,u\,|_{H^k(\Omega')}|\,v\,|_{H^l(\Omega')}\\ &+h^{2-k-l}(\text{meas}(\Omega'))^{\frac{p-2}{2p}}\,|\,u\,|_{W_p^2(\Omega')}\|\,v\,\|_{L_2(\Omega')}\},\end{aligned} \tag{1.9}$$

where $k, l = 0, 1$, $\Omega' = \text{supp}(v)$, number c does not depend on h, u, v and Ω'. Let us give two inequalities for functions from the space V_h. In the first one there is the so-called "inverse inequality" [6]:

$$| v |_{H^1(\Omega')} \leq ch^{-1} \| v \|_{L_2(\Omega')}, \tag{1.10}$$

where $\Omega' = \bigcup_{e \in \mathcal{T}'_h} e$, $\mathcal{T}'_h \subseteq \mathcal{T}_h$, number c does not depend on h, u and Ω'. And then there is the well-known type of the trace theorem inverse inequality:

$$| \gamma v |_{H^k(\Gamma')} \leq ch^{-1/2} | v |_{H^k(\Omega')}, \tag{1.11}$$

where Γ' is the boundary of the polytop Ω', γ is a trace operator.

Finally, let us introduce notations for two norms, which will be used further:

$$\|u\|_{(t',t'')} = [\|u\|^2_{L_2((t',t'');H^1(\Omega))} + h^2\|u\|^2_{L_2((t',t'');W^2_p(\Omega))}]^{1/2}, \tag{1.12}$$

$$\|u\|_{(*)} = [\|u\|^2_{C([t_0,t_*]);H^3(\Omega))} + h^2\|u\|^2_{C([t_0,t_*]);W^4_p(\Omega))}]^{1/2}. \tag{1.13}$$

2. Properties of the subdomains system

Let us introduce some notions, which will be used for formulation and analysis of the domain decomposition algorithm. Let $\{\Omega^{(k)}\}_{k=1}^s$ be the system of some opened sets in R^m. We will use the following notations

$$D^{(k,l)} = \bigcup_{n=k}^{l} \Omega^{(n)}, \quad G^{(k,l)} = \Omega \setminus D^{(k,l)}.$$

Then let us denote

$$\text{dist}(\Omega', \Omega'') = \inf_{\bar{x} \in \Omega', \bar{y} \in \Omega''} | \bar{x} - \bar{y} |_m,$$

where Ω' and Ω'' are arbitrary sets from R^m, $| \ |_m$ is the Euqlidian norm in R^m.

Definition 2.1. *The system of the subdomains $\{\Omega^{(k)}\}_{k=1}^s$ will be called ρ-regular system if the following conditions are valid:*

$$\bigcup_{k=1}^{s} \Omega^{(k)} = \Omega, \qquad \min_{k=1,\dots,s-1} \text{dist}(G^{(1,k)}, G^{(k+1,s)}) \geq \rho > 0.$$

Definition 2.2. *The system of the subdomains $\{\Omega^{(k)}\}_{k=1}^s$ will be called (ρ, h) – regular system if this system is ρ-regular and for any k there exists set $\mathcal{T}_{h,k} \subseteq \mathcal{T}_h$, such that $\bar{\Omega}^{(k)} = \bigcup_{e \in \mathcal{T}_{h,k}} e$.*

Remark 2.1. The distance to empty set will be assumed positive.

Now we will introduce some sets, which will be used further, and will formulate some properties of these sets without proofs. The appropriate proofs are given in paper [4]. Let

$$B^{(1)} = \emptyset, \quad B^{(k)} = \Omega^{(k)} \cup D^{(1,k-1)}, \ k = 2,\dots,s, \tag{2.1}$$

$$\Omega_0^{(k)} = \Omega^{(k)} \setminus B^{(k)}, \ k = 1,\dots,s-1. \tag{2.2}$$

Lemma 2.1. *The equalities*

$$\Omega_0^{(k)} \cap \Omega_0^{(l)} = \emptyset, \ k \neq l, \tag{2.3}$$

$$\bigcup\nolimits_{k=1}^{l} \Omega_0^{(k)} = D^{(1,l)} \tag{2.4}$$

are valid.

Now let us define the following sets:

$$\Omega^{(k,k)} = \Omega^{(k)}, \ k = 1,\dots,s, \tag{2.5}$$

$$B_0^{(l,k)} = \Omega^{(l-1,k)} \cap B^{(l)}, \tag{2.6}$$

$$\Omega^{(l,k)} = \Omega^{(l-1,k)} \setminus B_0^{(l,k)}, \ k = 1,\dots,s-1, \ l = k+1,\dots,s. \tag{2.7}$$

The properties of the sets $\Omega^{(l,k)}$ are given in the following statment.

Lemma 2.2. *The inclusions*

$$\Omega^{(l,k)} \subseteq \Omega^{(l-1,k)} \subseteq \dots \subseteq \Omega^{(k,k)}, \ k \leq l \tag{2.8}$$

and the equalities

$$\Omega^{(l,k)} \cap \Omega^{(l,n)} = \emptyset, \ l \geq 2, \ k,n \leq l, \ k \neq n, \tag{2.9}$$

$$\bigcup\nolimits_{k=1}^{l} \Omega^{(l,k)} = D^{(1,l)} \tag{2.10}$$

are valid.

Now we will consider the properties of the sets $B_0^{(l,k)}$.

Lemma 2.3. *The following equalities hold:*

$$B_0^{(l,k)} \cap B_0^{(l,n)} = \emptyset, \ l \geq 3, \ k,n < l, \ k \neq n, \tag{2.11}$$

$$B_0^{(l,k)} \cap B_0^{(n,k)} = \emptyset, \ l,n \geq 2, \ k < l,n, \ l \neq n, \tag{2.12}$$

$$B_0^{(l,k)} \cap \Omega^{(n,k)} = \emptyset, \ l,n \geq 2, \ k < l \leq n, \tag{2.13}$$

$$\bigcup_{k=1}^{l-1} B_0^{(l,k)} = B^{(l)}, \ l \geq 2. \tag{2.14}$$

Corollary 2.1. *The following inclusions are valid:*

$$\Omega^{(s,k)} \subseteq G^{(k+1,s)}, \quad k \leq s-1. \tag{2.15}$$

Let us introduce the sets

$$B^{(l,k)} = \bigcup_{n=l+1}^{s} B_0^{(n,k)}, \quad k \leq l \leq s-1. \tag{2.16}$$

The statement holds.

Lemma 2.4. *The following equalities are valid:*

$$B^{(l,k)} = \Omega^{(l,k)} \cap D^{(l+1,s)}, \quad k \leq l \leq s-1, \tag{2.17}$$

$$B^{(l,k)} \cap D^{(k+1,l)} = \emptyset, \quad k < l \leq s-1. \tag{2.18}$$

Corollary 2.2. *The equalities*

$$\Omega^{(s,k)} \cap B^{(k,k)} = \emptyset, \quad k \leq s-1, \tag{2.19}$$

$$\Omega^{(s,k)} \cup B^{(k,k)} = \Omega^{(k)}, \quad k \leq s-1 \tag{2.20}$$

hold.

From these statements the main result of this section follows.

Theorem 2.1. *Let $\{\Omega^{(k)}\}_{k=1}^{s}$ be (ρ, h)-regular system. Then the positive number ν exists and does not depend on ρ and h, and the sets of the m-simplexes $\mathcal{T}_h^{(k)}$ may be found, such that for $h \leq \nu\rho/2$ the following inequalities hold*

$$G^{(k+1,s)} \subseteq D^{(k)} \subseteq D^{(1,k)}, \quad k \leq s-1, \tag{2.21}$$

$$\mathrm{dist}(D^{(k)}, G^{(1,k)}) \geq \nu\rho/2, \quad k \leq s-1, \tag{2.22}$$

$$\mathrm{dist}(G^{(k)}, G^{(k+1,s)}) \geq \nu\rho, \quad k \leq s-1, \tag{2.23}$$

$$\mathrm{dist}(D^{(l)}, B^{(l,k)} \cap G^{(k)}) \geq \nu\rho/2, \quad k < l \leq s-1, \tag{2.24}$$

where $D^{(k)}$ is the set of the inner points of the set $\bigcup_{e \in \mathcal{T}_h^{(k)}} e$, $G^{(k)} = \Omega \setminus D^{(k)}$.

In conclusion of this section let us give one simple equality for characteristic functions of some sets. Let Ω' and Ω'' be arbitrary sets from R^m. Then the following equality is valid:

$$\chi_{\Omega' \cup \Omega''}(\bar{x}) = \chi_{\Omega'}(\bar{x}) + \chi_{\Omega''}(\bar{x}) - \chi_{\Omega' \cap \Omega''}(\bar{x}). \tag{2.25}$$

3. Additive scheme and error equations

First of all let us introduce some bilinear forms in the space $H^1(\Omega)\times H^1(\Omega)$ according to the sets from Section 2 . In accordance with notation (1.1) and formulas (2.1), (2.2), (2.5)-(2.7), (2.16) we will denote

$$\begin{aligned} a^{(k)}(t;u,v) &= a_{\Omega^{(k)}}(t;u,v), \\ a_0^{(k)}(t;u,v) &= a_{\Omega_0^{(k)}}(t;u,v), \\ a^{(l,k)}(t;u,v) &= a_{\Omega^{(l,k)}}(t;u,v), \quad k\le l, \\ b^{(k)}(t;u,v) &= a_{B^{(k)}}(t;u,v), \quad k\ge 2, \\ b_0^{(l,k)}(t;u,v) &= a_{B_0^{(l,k)}}(t;u,v), \quad k< l, \\ b^{(l,k)}(t;u,v) &= a_{B^{(l,k)}}(t;u,v), \quad k\le l, \end{aligned} \tag{3.1}$$

where $\{\Omega^{(k)}\}_{k=1}^{s}$ is (ρ,h)-regular system. In accordance with (1.3) these bilinear forms are non-negative.

Now let us formulate the difference Neumann problem. Let N be some integer number, $\tau=(t_*-t_0)/N$, $t_n=t_0+n\tau$, $n=1,\dots,N$. It is necessary to find two two-parametric sequences of the functions

$$\{u^{n+k/s},\ \hat{u}^{n+k/s},\ n=0,\dots,N-1,\ k=1,\dots,s\}$$

such that $u^{n+k/s}$, $\hat{u}^{n+k/s}\in V_h$ and $\forall v^{n+k/s}$, $\hat{v}^{n+k/s}\in V_h$ the following equalities are valid:

$$d_h(u^{n+1/s}-u^n,v^{n+1/s})+\tau a^{(1)}(t_{n+1};u^{n+1/s},v^{n+1/s})=0, \tag{3.2}$$

$$d_h(\hat{u}^{n+k/s}-u^{n+(k-1)/s},\hat{v}^{n+k/s})-\tau\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1};u^{n+l/s},\hat{v}^{n+k/s})=0, \tag{3.3}$$

$$\begin{aligned} d_h(u^{n+k/s}-\hat{u}^{n+k/s},v^{n+k/s})&+\tau a^{(k)}(t_{n+1};u^{n+k/s},v^{n+k/s}) \\ &=\tau(f^{n,k},v^{n+k/s}), \end{aligned} \tag{3.4}$$

where $k=2,\dots,s$, $f^{n,k}=0$, $k\le s-1$, $f^{n,s}=f(t_{n+1})$,

$$u^0=\Pi_h u_0. \tag{3.5}$$

For the correct definition of the right-hand side of equality (3.5) we will suppose that $u_0\in W_p^2(\Omega)$, $p>m/2$, which ensures the inclusion of $W_p^2(\Omega)$ into $C(\bar{\Omega})$. Scheme (3.2)-(3.4) requires to inverse matrices only in subdomains (steps (3.2),(3.4)) and step (3.3) has explicit realization. For further consideration we will give some other form of the scheme. Let us assume $\hat{v}^{n+k/s} = v^{n+k/s}$ and sum up equalities (3.3) and (3.4). As a result we will obtain for $k \geq 2$

$$\begin{aligned} &d_h(u^{n+k/s} - u^{n+(k-1)/s}, v^{n+k/s}) + \tau a^{(k)}(t_{n+1}; u^{n+k/s}, v^{n+k/s}) \\ &\quad - \tau \textstyle\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1}; u^{n+l/s}, v^{n+k/s}) = \tau(f^{n,k}, v^{n+k/s}). \end{aligned} \tag{3.6}$$

Let $u(t)$ be a solution to problem (1.4), (1.5). We assume that the following conditions hold:

$$\begin{aligned} &\lambda_{ij} \in C([t_0, t_*]; C^3(\bar{\Omega})), \quad u \in C([t_0, t_*]; W_p^4(\Omega)), \\ &du/dt \in L_2((t_0, t_*); W_p^2(\Omega)), \quad d^2u/dt^2 \in L_2(Q_t), \end{aligned} \tag{3.7}$$

where $Q_t = (t_0, t_*) \times \Omega$. Then let $u^{n+k/s}$ be a solution to (3.2), (3.6). According to (3.7), the function $\Pi_h u(t_{n+1})$ exists and, henceforth, we may introduce the sequence:

$$\xi^{n+k/s} = u^{n+k/s} - \Pi_h u(t_{n+1}) + \tau r^{n+k/s}, \quad k = 1, \ldots, s, \tag{3.8}$$

such that $\xi^{n+k/s}$, $r^{n+k/s} \in V_h$ and identities hold: $r^n(\bar{x}) \equiv r^{n+1}(\bar{x}) \equiv 0$. Then $\xi^n = u^n - \Pi_h u(t_n)$, and according to (3.5), $\xi^0 = 0$. From the differential equation (1.4) and difference equations (3.2), (3.6) we obtain the error equations in the following form:

$$\begin{aligned} &d_h(\xi^{n+1/s} - \xi^n, v^{n+1/s}) + \tau a^{(1)}(t_{n+1}; \xi^{n+1/s}, v^{n+1/s}) \\ &\quad = \tau g^{n,1}(v^{n+1/s}), \end{aligned} \tag{3.9}$$

$$\begin{aligned} &d_h(\xi^{n+k/s} - \xi^{n+(k-1)/s}, v^{n+k/s}) + \tau a^{(k)}(t_{n+1}; \xi^{n+k/s}, v^{n+k/s}) \\ &\quad - \tau \textstyle\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1}; \xi^{n+l/s}, v^{n+k/s}) = \tau g^{n,k}(v^{n+k/s}), \end{aligned} \tag{3.10}$$

where $k = 2, \ldots, s$. The functionals $g^{n,k}(v)$ are defined by the equalities:

$$g^{n,1}(v) = d_h(r^{n+1/s}, v) + \tau a^{(1)}(t_{n+1}; r^{n+1/s}, v) + \hat{g}^{n,1}(v), \tag{3.11}$$

$$\begin{aligned} g^{n,k}(v) &= d_h(r^{n+k/s} - r^{n+(k-1)/s}, v) + \tau a^{(k)}(t_{n+1}; r^{n+k/s}, v) \\ &\quad - \tau \textstyle\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1}; r^{n+l/s}, v) + \hat{g}^{n,k}(v), \quad k = 2, \ldots, s-1, \end{aligned} \tag{3.12}$$

$$g^{n,s}(v) = -d_h(r^{n+(s-1)/s}, v) - \tau \sum_{k=1}^{s-1} b_0^{(s,k)}(t_{n+1}; r^{n+k/s}, v) + \hat{g}^{n,s}(v), \tag{3.13}$$

where the functionals $\hat{g}^{n,k}(v)$ have the form:

$$\begin{aligned}
\hat{g}^{n,1}(v) &= -d_h(\Pi_h[u(t_n)]_\tau, v) - a_0^{(1)}(t_{n+1}; \Pi_h u(t_{n+1}), v), \\
\hat{g}^{n,k}(v) &= -a_0^{(k)}(t_{n+1}; \Pi_h u(t_{n+1}), v), \quad k = 2, \dots, s-1, \\
\hat{g}^{n,s}(v) &= (\tfrac{du}{dt}(t_{n+1}), v) + a(t_{n+1}; u(t_{n+1}), v) \\
&\quad - a_0^{(s)}(t_{n+1}; \Pi_h u(t_{n+1}), v).
\end{aligned}$$

For obtaining these formulas we used the following additive presentation:

$$\sum_{l=1}^{k-1} b_0^{(k,l)}(t; w, v) = b^{(k)}(t; w, v), \tag{3.14}$$

where w is an arbitrary function from the space $H^1(\Omega)$. Presentation (3.15) follows from equalities (2.11), (2.14) of Lemma 2.3, property of the characteristic functions (2.25) and expressions (3.1). From equalities (2.1), (2.2) and (3.15) we obtain

$$\begin{aligned}
a^{(1)}(t; w, v) &= a_0^{(1)}(t; w, v) \\
a^{(k)}(t; w, v) - \textstyle\sum_{l=1}^{k-1} b_0^{(k,l)}(t; w, v) &= a_0^{(k)}(t; w, v), \quad k \geq 2.
\end{aligned} \tag{3.15}$$

Then equalities (3.14) follow from (3.16). Let

$$z(t) = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}(\lambda_{ij} \frac{\partial u}{\partial x_j})(t).$$

Consider the sequence of the functions $w^n = [u(t_n)]_\tau + z(t_{n+1})$. According to smoothness conditions (3.7), $w^n \in W_p^2(\Omega)$. Then the function $\Pi_h w^n$ exists and, henceforth, the functions

$$\hat{r}^{n+k/s} = r^{n+k/s} - \Pi_h w^n \in V_h \tag{3.16}$$

may be defined. Now we will rewrite expressions (3.11)-(3.13), using functions (3.17) and the Green formula

$$a_0^{(k)}(t; u(t), v) = \sigma_0^{(k)}(t; v) + (\chi_{\Omega_0^{(k)}} z(t), v), \tag{3.17}$$

where

$$\sigma_0^{(k)}(t; v) = \int_{\partial\Omega_0^{(k)}} \frac{\partial u}{\partial n^{(k)}}(t) v d\sigma, \tag{3.18}$$

$$\frac{\partial u}{\partial n^{(k)}} = \sum_{i,j=1}^{m} \lambda_{ij} \frac{\partial u}{\partial x_i} \cos(\bar{n}^{(k)}, \bar{x}^j), \tag{3.19}$$

$\bar{n}^{(k)}$, $\bar{x}^j$ are the unit vectors of the external normal to $\Omega_0^{(k)}$ and of the coordinate j-axis. As a result we will obtain

$$g^{n,k}(v) = g_1^{n,k}(v) + g_2^{n,k}(v), \quad k = 1, \dots, s-1. \tag{3.20}$$

The functionals $g_1^{n,k}(v)$ are given by the equalities

$$\begin{aligned} g_1^{n,1}(v) &= d_h(\hat{r}^{n+1/s}, v) + \tau a^{(1)}(t_{n+1}; \hat{r}^{n+1/s}, v) - \sigma^{n,1}(v), \\ g_1^{n,k}(v) &= d_h(\hat{r}^{n+k/s} - \hat{r}^{n+(k-1)/s}, v) + +\tau a^{(k)}(t_{n+1}; \hat{r}^{n+k/s}, v) \\ &\quad - \textstyle\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1}; \hat{r}^{n+l/s}, v) - \sigma^{n,k}(v), \\ &\quad k = 2, \dots, s-1, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \sigma^{n,1}(v) &= \sigma_0^{(1)}(t_{n+1}; v) - (\chi_{G^{(1,1)}} P_{h,\mu} \Pi_h z(t_{n+1}), P_{h,\mu} v), \\ \sigma^{n,k}(v) &= \sigma_0^{(k)}(t_{n+1}; v) + (\chi_{\Omega_0^{(k)}} P_{h,\mu} \Pi_h z(t_{n+1}), P_{h,\mu} v), \\ &\quad k = 2, \dots, s-1, \end{aligned} \tag{3.22}$$

$P_{h,\mu}$ is a lamping operator. Here we used the equality $G^{(1,1)} = \Omega \setminus \Omega_0^{(1)}$, which follows from (2.1), (2.2) and the evident equality $D^{(1,1)} = \Omega^{(1)}$. The functionals $g_2^{n,k}(v)$ have form

$$g_2^{n,k}(v) = \alpha^{n,k}(v) + \beta^{n,k}(v) + \tau\gamma^{n,k}(v), \;\; k = 1, \dots, s-1, \tag{3.23}$$

where, according to (3.14), the functionals $\alpha^{n,k}(v)$, $\beta^{n,k}(v)$ and $\gamma^{n,k}(v)$ are given by formulas

$$\begin{aligned} \alpha^{n,k}(v) &= (\chi_{\Omega_0^{(k)}} P_{h,\mu} \Pi_h z(t_{n+1}), P_{h,\mu} v) - (\chi_{\Omega_0^{(k)}} z(t_{n+1}), v), \\ \beta^{n,k}(v) &= a_0^{(k)}(t_{n+1}; u(t_{n+1}) - \Pi_h u(t_{n+1}), v), \\ \gamma^{n,k}(v) &= a_0^{(k)}(t_{n+1}; \Pi_h w^n, v). \end{aligned} \tag{3.24}$$

Let us consider the functional $g^{n,s}(v)$. From Lemma 2.1 and property (2.25) the additive presentation

$$a(t_{n+1}; u(t_{n+1}), v) = \sum_{k=1}^{s} a_0^{(k)}(t_{n+1}; u(t_{n+1}), v) \tag{3.25}$$

follows. Using equalities (3.18), (3.22), (3.24) and (3.25) the third formula from equality (3.14) may be written in the form

$$\begin{aligned}\hat{g}^{n,s}(v) &= (\tfrac{du}{dt}(t_{n+1}), v) + d_h(\Pi_h z(t_{n+1}), v) + \beta^{n,s}(v) \\ &+ \textstyle\sum_{k=1}^{s-1}(\sigma^{n,k}(v) - \alpha^{n,k}(v)).\end{aligned} \tag{3.26}$$

Then let us substitute equalities (3.17) and (3.26) into the last formula from equality (3.14) with using formula (3.15) for $k = s$. As a result we have

$$g^{n,s}(v) = g_1^{n,s}(v) + g_2^{n,s}(v), \tag{3.27}$$

where

$$\begin{aligned}g_1^{n,s}(v) &= -d_h(\hat{r}^{n+(s-1)/s}, v) - \tau \textstyle\sum_{k=1}^{s-1} b_0^{(s,k)}(t_{n+1}; \hat{r}^{n+k/s}, v) \\ &- \textstyle\sum_{k=1}^{s-1} \sigma^{n,k}(v),\end{aligned} \tag{3.28}$$

$$g_2^{n,s}(v) = \alpha^{n,s}(v) - \textstyle\sum_{k=1}^{s-1} \alpha^{n,k}(v) + \beta^{n,s}(v) + \tau\gamma^{n,s}(v),$$

and the functionals $\alpha^{n,s}(v)$, $\beta^{n,s}(v)$ and $\gamma^{n,s}(v)$ are given by the formulas

$$\begin{aligned}\alpha^{n,s}(v) &= (\tfrac{du}{dt}(t_{n+1}), v) - d_h(\Pi_h[u(t_{n+1})]_\tau, v), \\ \beta^{n,s}(v) &= a_0^{(s)}(t_{n+1}; u(t_{n+1}) - \Pi_h u(t_{n+1}), v), \\ \gamma^{n,s}(v) &= -b^{(s)}(t_{n+1}; \Pi_h w^n, v).\end{aligned} \tag{3.29}$$

In further consideration we will use the following equality:

$$\sum_{k=2}^{r}\sum_{l=1}^{k-1} b_0^{(k,l)}(t; u_l, v_l) = \sum_{k=1}^{r-1}\sum_{l=k+1}^{r} b_0^{(l,k)}(t; u_k, v_k), \quad r \geq 2, \tag{3.30}$$

which holds for arbitrary sequences $\{u_k\}_{k=1}^{s-1}$ and $\{v_k\}_{k=1}^{s-1}$. Then in accordance with formula (2.16), equality (2.12) of Lemma 2.3 and property (2.25) the following equality holds:

$$\sum_{l=k+1}^{s} b_0^{(l,k)}(t; u_k, v_k) = b^{(k,k)}(t; u_k, v_k).$$

From this formula and from (3.31) for $r = s$, $u_k = \hat{r}^{n+k/s}$ and $v_k \equiv v$ we have

$$\begin{aligned}&\sum_{k=2}^{s-1}\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1}; \hat{r}^{n+l/s}, v) \\ &= \sum_{k=1}^{s-1}[b^{(k,k)}(t_{n+1}; \hat{r}^{n+k/s}, v) - b_0^{(s,k)}(t_{n+1}; \hat{r}^{n+k/s}, v)].\end{aligned} \tag{3.31}$$

Let us sum up equalities (3.21) over k from 1 to $s-1$ and for obtaining result add (3.28). Using (3.32) and Corollary 2.2 for $l=k$ we will obtain the following presentation of the functional $g_1^{n,s}(v)$:

$$g_1^{n,s}(v)=\sum_{k=1}^{s-1}[\tau a^{(s,k)}(t_{n+1};\hat{r}^{n+k/s},v)-g_1^{n,k}(v)]. \tag{3.32}$$

Let us define the functions $\hat{r}^{n+k/s}$ from the following conditions: $\forall v\in V_h$

$$g_1^{n,k}(v)=\tau a^{(s,k)}(t_{n+1};\hat{r}^{n+k/s},v),\quad k=1,\dots,s-1, \tag{3.33}$$

and according to (3.33), $g_1^{n,s}(v)=0$. It means that from formulas (3.20), (3.27), (3.33) and (3.34) we obtain

$$\begin{aligned} g^{n,k}(v) &= g_2^{n,k}(v)+\tau a^{(s,k)}(t_{n+1};\hat{r}^{n+k/s},v),\quad k=1,\dots,s-1,\\ g^{n,s}(v) &= g_2^{n,s}(v), \end{aligned} \tag{3.34}$$

where the functionals $g_2^{n,k}(v)$ are given by equalities (3.23) and (3.29). Therefore, we have comletely defined the right-hand side of error equations (3.9), (3.10). At first we will estimate $H^1(\Omega^{(s,k)})$-semi-norms of the functions $\hat{r}^{n+k/s}$ to continue the account of error estimation analysis To this question we will devote the separate section.

4. Estimation of auxiliary greed functions

The functions $\hat{r}^{n+k/s}$ are the solutions of equations (3.34). Let us modify this system using the functionals

$$\tilde{g}^{n,l}(v)=\sum_{k=1}^{l}[g_1^{n,k}(v)-\tau a^{(s,k)}(t_{n+1};\hat{r}^{n+k/s},v)],\ \ l\le s-1.$$

Then system (3.34) is equivalent to the problem

$$\tilde{g}^{n,l}(v)=0,\quad v\in V_h,\quad l\le s-1. \tag{4.1}$$

From equality (3.21), presentation (2.16), Corollary 2.2, equality (3.31) for $r=l$ and property (2.25) it follows

$$\tilde{g}^{n,l}(v)=d_h(\hat{r}^{n+l/s},v)+\tau\sum_{k=1}^{l}b^{(l,k)}(t_{n+1};\hat{r}^{n+k/s},v)-\tilde{\sigma}^{n,l}(v), \tag{4.2}$$

where $\tilde{\sigma}^{n,l}(v) = \sum_{k=1}^{l} \sigma^{n,k}(v)$. Lemma 1.1 and the Neumann condition give the equalities

$$\sum_{k=1}^{l} \sigma_0^{(k)}(t;v) = \int_{\partial D^{(1,l)}} \frac{\partial u}{\partial n_l}(t) v d\sigma = \int_{\partial G^{(1,l)}} \frac{\partial u}{\partial n_l}(t) v d\sigma.$$

Here $\frac{\partial u}{\partial n_l}$ is conormal derivative over external normal to $D^{(1,l)}$. From these equalities and formulas (3.19), (3.22) the presentation of the functional $\tilde{g}^{n,l}(v)$, which we will use further, follows. We have

$$\tilde{\sigma}^{n,l}(v) = \tilde{\sigma}_1^{n,l}(v) + \tilde{\sigma}_2^{n,l}(v), \tag{4.3}$$

where

$$\tilde{\sigma}_1^{n,l}(v) = \int_{\partial G^{(1,l)}} \frac{\partial u}{\partial n_l}(t_{n+1}) v d\sigma, \tag{4.4}$$

$$\tilde{\sigma}_2^{n,l}(v) = -(\chi_{G^{(1,l)}} P_{h,\mu} \Pi_h z(t_{n+1}), P_{h,\mu} v). \tag{4.5}$$

Now we will introduce vector-matrix notations, which are more convinient for further consideration. Let $\mathcal{E}$ be the Euclidian vector space corresponding to some order of the set I and $\langle\ \rangle$, $\|\ \|$ are scalar product and norm in the space $\mathcal{E}$. Then let $\rho_i^2 = d_h(\varphi_i, \varphi_i)$, $i \in I$, $\bar{r}^k$ and $\bar{\sigma}^k$ are vectors with components $\rho_i \hat{r}^{n+k/s}(\bar{x}_i)$ and $\tilde{\sigma}^{n,k}(\varphi_i)/\rho_i$. Let $A^{(l,k)}$ be a square matrix with elements $b^{(l,k)}(t_{n+1}; \varphi_i, \varphi_j)/\rho_i \rho_j$, $k \le l \le s-1$. (Since all consideration will be conduct for only one time step we will not use index n). From symmetry and non-negativity of bilinear forms (3.1) the similar properties for matrix $A^{(l,k)}$ follow. Then for eigenvalues we will use the following well-known condition: $\lambda(A^{(l,k)}) \in [0, \lambda_0/h^2]$, where number λ_0 does not depend on h and τ. Moreover, we will use matrix $A_\tau^{(k)} = E + \tau A^{(k,k)}$, $k \le s-1$, where E is the identity operator in $\mathcal{E}$. In vector-matrix notations problem (4.1), (4.2) has the form:

$$A_\tau^{(1)} \bar{r}^1 = \bar{\sigma}^1, \tag{4.6}$$

$$A_\tau^{(l)} \bar{r}^l = \bar{\sigma}^l - \tau \sum_{k=1}^{l-1} A^{(l,k)} \bar{r}^k, \quad l = 2, \ldots, s-1. \tag{4.7}$$

Let us introduce some additional notations. Let $B \subseteq \bar{\Omega}$. Then $I_B = \{i \in I | \bar{x}_i \in B\}$. If $\bar{u}$ is some vector from $\mathcal{E}$ with the components u_i, then $\bar{u}_B$ is the vector with the components $u_{B,i} = u_i$ if $i \in I_B$ and $u_{B,i} = 0$ if $i \notin I_B$. Now we will formulate the statement, which is a foundation of all following considerations. In some other form this result was obtained by Kuznetsov in [2].

Lemma 4.1. *Let B, B' B'' be arbitrary subdomains from Ω and the following condition holds:*

$$\mathrm{dist}(B', B'') \geq \rho_0 > 0. \tag{4.8}$$

Let A be a non-negative matrix with the elements $a_B(t; \varphi_i, \varphi_j)/\rho_i\rho_j$, $A_\tau = E + \tau A$ and $\bar{g}$ is an arbitrary vector from $\mathcal{E}$. Besides, the following inequalities are valid:

$$2h^2 \leq \tau\lambda_0 \leq \left(\frac{\rho_0}{\ln(2/\varepsilon)}\right)^2, \tag{4.9}$$

where ε is an arbitrary positive number. Then for solution of the problem $A_\tau \bar{u} = \bar{g}_{B''}$ the estimation

$$\|\bar{u}_{B'}\| \leq \varepsilon \|\bar{g}_{B''}\| \tag{4.10}$$

is valid.

Now for solution of system (4.6), (4.7) the following result holds.

Lemma 4.2. *If the condition $2h^2 \leq \tau\lambda_0$ holds, then for solution of problem (4.6), (4.7) the estimation*

$$\|\bar{r}^l\| \leq \sum_{k=1}^{l} \kappa^{l-k} \bar{\sigma}^k, \quad l \leq s-1 \tag{4.11}$$

is valid, where $\kappa = 2h^{-1}\sqrt{\tau\lambda_0}$.

The proofs of Lemmas 4.1, 4.2 are presented in [4].

Now we will estimate the norms of the vectors $\bar{r}^l$. In accordance with presentation (4.3)-(4.5) the equality

$$\bar{\sigma}^l = \bar{\sigma}^l_{\bar{G}^{(1,l)}} \tag{4.12}$$

is valid. Let us consider the sequence of the sets $\{D^{(k)}\}_{k=1}^{s-1}$ and the parameter ν, which are defined by Theorem 2.1. We will estimate the norms of the vectors $\bar{r}^l_{\bar{D}^{(l)}}$ by induction over l from 1 to $s-1$. For $l = 1$ we will apply Lemma 4.1 to equation (4.6), which according to (4.12) may be rewritten in the form

$$A^{(1)}_\tau \bar{r}^1 = \bar{\sigma}^1_{\bar{G}^{(1,1)}}, \tag{4.13}$$

with the following notations: $B = \bar{B}^{(1,1)}$, $B' = D^{(1)}$, $B'' = \bar{G}^{(1,1)}$, $\rho_0 = \nu\rho/2$. Then in accordance with inequality (2.22) from Theorem 2.1 condition (4.8) of Lemma 4.1 holds. Henceforth, for any $\varepsilon > 0$ from inequalities (4.9) estimation (4.10) follows. In our notations this estimation has form

$$\|\bar{r}^1_{\bar{D}^{(1)}}\| \leq \varepsilon \|\bar{\sigma}^1\|. \tag{4.14}$$

Let us assume that for some sequence of numbers $\varepsilon_k > 0,\ k \le l-1$ the inequalities

$$\|\bar{r}^k_{\bar{D}(k)}\| \le \varepsilon_k \sum_{r=1}^{k} \|\bar{\sigma}^r\| \tag{4.15}$$

are valid. The numbers ε_k will be defined later. Let us introduce the vectors $\bar{r}^{l,k}$, $k \le l$, which are solutions of the following greed problems:

$$\begin{aligned} A^{(l)}_\tau \bar{r}^{l,k} &= -\tau A^{(l,k)} \bar{r}^k,\ \ k \le l-1, && (4.16)\\ A^{(l)}_\tau \bar{r}^{l,l} &= \bar{\sigma}^l. && (4.17)\end{aligned}$$

In accordance with equation (4.7) the following presentation holds:

$$\bar{r}^l = \sum_{k=1}^{l} \bar{r}^{l,k}. \tag{4.18}$$

Then the evident equality $\bar{r}^k = \bar{r}^k_{\bar{D}(k)} + \bar{r}^k_{\hat{G}(k)}$ is valid, where $\hat{G}^{(k)} = \bar{G}^{(k)} \setminus (\bar{G}^{(k)} \cap \bar{D}^{(k)})$. From this equality it follows that solution of system (4.16) may be written in the form

$$\bar{r}^{l,k} = \bar{r}^{l,k}_1 + \bar{r}^{l,k}_2, \tag{4.19}$$

where the vectors $\bar{r}^{l,k}_1$ and $\bar{r}^{l,k}_2$ are solutions of the following problems:

$$A^{(l)}_\tau \bar{r}^{l,k}_1 = -\tau A^{(l,k)} \bar{r}^k_{\bar{D}(k)},\ \ k \le l-1, \tag{4.20}$$

$$A^{(l)}_\tau \bar{r}^{l,k}_2 = -\tau A^{(l,k)} \bar{r}^k_{\hat{G}(k)},\ \ k \le l-1. \tag{4.21}$$

According to assumption (4.15) the estimation

$$\|\bar{r}^{l,k}_1\| \le \frac{1}{4}\kappa^2 \varepsilon_k \sum_{r=1}^{k} \|\bar{\sigma}^r\|,\quad k \le l-1 \tag{4.22}$$

is valid. Let us consider equation (4.21). Let $\bar{g}^{l,k} = \tau A^{(l,k)} \bar{r}^k_{\hat{G}(k)}$. Since the inequality $A^{(l,k)}_{i,j} \ne 0$ (elements of the matrix $A^{(l,k)}$) is equivalent to the condition $i, j \in I_{B(l,k)}$, then the following equality holds:

$$\bar{g}^{l,k} = \bar{g}^{l,k}_{\bar{B}(l,k)}. \tag{4.23}$$

As it is easy to see, $\hat{G}^{(k)} \cap \Omega$ is an opened set and, using the structure of the matrix $A^{(l,k)}$, we obtain the equality

$$\bar{g}^{l,k} = \bar{g}^{l,k}_{\hat{G}(k)} \tag{4.24}$$

Let us introduce the set $\hat{B}^{(l,k)} = \bar{B}^{(l,k)} \cap \bar{G}^{(k)}$. Since $\bar{G}^{(k)} = \hat{G}^{(k)}$, then according to (4.23), (4.24) system (4.21) may be rewritten in the form

$$A_\tau^{(l)} \bar{r}_2^{l,k} = \bar{g}_{\hat{B}^{(l,k)}}, \quad k \le l-1. \tag{4.25}$$

Let $B = \bar{B}^{(l,l)}$, $B' = D^{(l)}$, $B'' = \hat{B}^{(l,k)}$ and $\rho_0 = \nu\rho/2$. Then from inequality (2.24) of Theorem 2.1 the inequality (4.8) follows. From Lemma 4.1 and according to estimation of eigevalues of the matrix $A^{(l,k)}$, we will obtain the following estimation:

$$\|\bar{r}_{2,D^{(l)}}^{l,k}\| \le \frac{1}{4}\kappa^2 \varepsilon \|\bar{r}^k\|, \quad k \le l-1.$$

The right-hand side of this inequality may be estimated by Lemma 4.2. We have

$$\|\bar{r}_{2,D^{(l)}}^{l,k}\| \le \frac{1}{4}\kappa^2 \varepsilon \sum_{r=1}^{k} \kappa^{k-r} \|\bar{\sigma}^r\|, \quad k \le l-1.$$

From this estimation, presentation (4.19) and inequality (4.22) the estimation

$$\|\bar{r}_{\bar{D}^{(l)}}^{l,k}\| \le \frac{1}{4}\kappa^2 \sum_{r=1}^{k} (\varepsilon_k + \kappa^{k-r}\varepsilon) \|\bar{\sigma}^r\|, \quad k \le l-1 \tag{4.26}$$

follows. The estimation of the vector $\bar{r}_{\bar{D}^{(l)}}^{l,l}$ may be obtained from Lemma 4.1 similarly inequality (4.14). We have

$$\|\bar{r}_{\bar{D}^{(l)}}^{l,l}\| \le \varepsilon \|\bar{\sigma}^l\|. \tag{4.27}$$

In accordance with presentation (4.18) we will sum up estimations (4.26) over k from 1 to $l-1$ and the result sum with estimation (4.27). After simple transformations we will obtain

$$\|\bar{r}_{D^l}^{(l)}\| \le \varepsilon \|\bar{\sigma}^l\| + \frac{1}{4}\kappa^2 \sum_{k=1}^{l-1} \left(\sum_{r=k}^{l-1} \varepsilon_r + \frac{\kappa^{l-k}-1}{\kappa-1}\varepsilon \right) \|\bar{\sigma}^k\|. \tag{4.28}$$

Let

$$\varepsilon_k = \kappa^{2(k-1)}\varepsilon, \quad k \le s-1. \tag{4.29}$$

Then it is easy to obtain the estimation

$$\kappa^2 \left(\sum_{r=k}^{l-1} \varepsilon_r + \frac{\kappa^{l-k}-1}{\kappa-1}\varepsilon \right) \le \varepsilon_l \left(\frac{\kappa^2}{\kappa^2-1} + \frac{\kappa}{\kappa-1}\kappa^{2-l} \right).$$

According to (4.9), $\kappa \geq \sqrt{8}$, $\frac{\kappa}{\kappa-1} \leq (8+\sqrt{8})/7$, $\frac{\kappa^2}{\kappa^2-1} \leq 8/7$ and the right-hand side of the latter inequality may be estimated by $4\varepsilon_l$. Therefore, from inequality (4.28) the estimation

$$\|\bar{r}^{(l)}_{D^l}\| \leq \varepsilon_l \sum_{k=1}^{l} \|\bar{\sigma}^k\|. \tag{4.30}$$

follows. In accordance with (4.14), (4.15) and (4.30) the correctness of this estimation is valid for an arbitrary $l = 1, \ldots, s-1$.

Now we will estimate the norms $\|\bar{\sigma}^k\|$. Denote $I_{k,1} = I_{\partial G^{(1,k)}}$, $I_{k,2} = I_{\bar{G}^{(1,k)}}$ and

$$J_{k,1} = \sum_{i \in I_{k,1}} \left(\rho_i^{-1} \int_{\partial G^{(1,k)}} \frac{\partial u}{\partial n}(t_{n+1}) \varphi_i d\sigma \right)^2, \tag{4.31}$$

$$J_{k,2} = \sum_{i \in I_{k,2}} [\rho_i^{-1}(\chi_{G^{(1,k)}} P_{h,\mu} \Pi_h z(t_{n+1}), P_{h,\mu}\varphi_i)]^2. \tag{4.32}$$

According to (4.3)-(4.5), the inequality

$$\|\bar{\sigma}^k\|^2 \leq 2(J_{k,1} + J_{k,2}) \tag{4.33}$$

is valid. The first term in the right-hand side of this inequality we will estimate using the Cauchy-Buniakovsky inequality, inequality (1.11), condition (1.8) for the function φ_i and the trace theorem [5]. As a result we will obtain

$$J_{k,1} \leq c_1 h^{-1} \|u(t_{n+1})\|^2_{H^2(G^{(1,k)})}, \tag{4.34}$$

where c_1 does not depend on h, τ, ρ, u and that is very important for the consideration of "small subdomains" (Remark 5.2) on diameter of the subdomain $G^{(1,k)}$. The estimation of $J_{k,2}$ is a result of the using inequality (1.9) and condition (1.8). We have

$$\begin{aligned} J_{k,2} \leq\ & c_2 \Big(\|u(t_{n+1})\|^2_{H^2(G^{(1,k)})} + h^2 \|u(t_{n+1})\|^2_{H^3(G^{(1,k)})} \\ & + h^4 (\mathrm{meas}(G^{(1,k)}))^{\frac{p-2}{p}} \|u(t_{n+1})\|^2_{W^4_p(G^{(1,k)})} \Big). \end{aligned} \tag{4.35}$$

From inequalities (4.33)-(4.35) the estimation of $\|\bar{\sigma}^k\|$ evidently follows. Let us substitute this estimation into (4.30). Then according to formula (4.29), we will obtain the following inequality:

$$\|\bar{r}^l_{\bar{D}^{(l)}}\|^2 \leq c' \varepsilon^2 \kappa^{4(l-1)} h^{-1} \left(\|u\|^2_{C([t_0,t_*];H^2(\Omega))} + h^2 \|u\|^2_{(*)} \right).$$

Here we use notation (1.13). Using the left inclusions (2.21) from Theorem 2.1, Corollary 2.1, condition (1.8), inequality (1.10) and the latter estimation we will obtain the total result of this section, which we will formulate in the following form:

Lemma 4.3. *There exists the positive number ν, such that the conditions of Lemma 4.1 with $\rho_0 = \nu\rho/2$ and any positive number ε hold. Then the following inequality is valid:*

$$|\hat{r}^{n+l/s}|^2_{H^1(\Omega^{(s,l)})} \leq c\varepsilon^2\kappa^{4(l-1)}h^{-3}\left(\|u\|^2_{C([t_0,t_*];H^2(\Omega))} + h^2\|u\|^2_{(*)}\right),$$

where number c does not depend on h, τ, ρ, diameter of the subdomain $\Omega^{(s,l)}$ and the function u. Let us remind that $\kappa = 2h^{-1}\sqrt{\tau\lambda_0}$.

5. Error analysis

We will begin the error analysis of the domain decomposition method from the obtaining of the integral identity, which is a foundation of the stability analysis. Let us $v^{n+k/s} = 2\tau\xi^{n+k/s}$ and then we will sum equations (3.10) over k from 2 to s and the result sum with (3.9). Then we will obtain

$$\begin{aligned} d_h(\xi^{n+1},\xi^{n+1}) &+ \textstyle\sum_{k=1}^s d_h(\xi^{n+k/s} - \xi^{n+(k-1)/s}, \xi^{n+k/s} - \xi^{n+(k-1)/s}) \\ &+ 2\tau\textstyle\sum_{k=1}^s a^{(k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}) \\ &- 2\tau\textstyle\sum_{k=2}^s\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1};\xi^{n+l/s},\xi^{n+k/s}) \\ &= d_h(\xi^n,\xi^n) + 2\tau\textstyle\sum_{k=1}^s g^{n,k}(\xi^{n+k/s}). \end{aligned} \tag{5.1}$$

Then from equality

$$\begin{aligned} 2b_0^{(k,l)}(t_{n+1};\xi^{n+l/s},\xi^{n+k/s}) &= b_0^{(k,l)}(t_{n+1};\xi^{n+l/s},\xi^{n+l/s}) \\ &+ b_0^{(k,l)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}) \\ &- b_0^{(k,l)}(t_{n+1};\xi^{n+k/s}-\xi^{n+l/s},\xi^{n+k/s}-\xi^{n+l/s}), \end{aligned}$$

equality (3.31), formula (2.16), Lemma 2.3 and condition (2.25) the equality

$$\begin{aligned} &2\sum_{k=2}^{s}\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1};\xi^{n+l/s},\xi^{n+k/s}) \\ &= \sum_{k=1}^{s-1} b^{(k,k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}) + \sum_{k=2}^{s} b^{(k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}) \\ &- \sum_{k=2}^{s}\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1};\xi^{n+k/s}-\xi^{n+l/s},\xi^{n+k/s}-\xi^{n+l/s}) \end{aligned}$$

follows. According to Corollary 2.2 and equality (2.5) for $k = s$, the following equality holds:

$$\begin{aligned}\sum_{k=1}^{s} a^{(k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}) - \sum_{k=1}^{s-1} b^{(k,k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s})\\ = \sum_{k=1}^{s} a^{(s,k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}).\end{aligned}$$

Finally, from (2.2) the equality

$$\begin{aligned}\sum_{k=1}^{s} a^{(k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}) - \sum_{k=2}^{s} b^{(k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s})\\ = \sum_{k=1}^{s} a_0^{(k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s})\end{aligned}$$

follows. Then using three last equalities the identity (5.1) may be rewritten in the form

$$\begin{aligned}& d_h(\xi^{n+1},\xi^{n+1}) + \textstyle\sum_{k=1}^{s} d_h(\xi^{n+k/s} - \xi^{n+(k-1)/s},\xi^{n+k/s} - \xi^{n+(k-1)/s})\\ &+ \tau \textstyle\sum_{k=1}^{s}[a^{(s,k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s}) + a_0^{(k)}(t_{n+1};\xi^{n+k/s},\xi^{n+k/s})]\\ &+ \tau \textstyle\sum_{k=2}^{s}\sum_{l=1}^{k-1} b_0^{(k,l)}(t_{n+1};\xi^{n+k/s} - \xi^{n+l/s},\xi^{n+k/s} - \xi^{n+l/s})\\ &= d_h(\xi^{n},\xi^{n}) + 2\tau \textstyle\sum_{k=1}^{s} g^{n,k}(\xi^{n+k/s}).\end{aligned} \tag{5.2}$$

In accordance with presentation of the functionals $g^{n,k}(v)$ (3.23), (3.29), (3.35) an estimation of the right-hand side of identity (5.2) is a successive estimation of the functionals $a^{(s,k)}(t_{n+1};\hat{r}^{n+k/s},v)$, $\alpha^{n,k}(v)$, $\beta^{n,k}(v)$, $\gamma^{n,k}(v)$. The functionals $\alpha^{n,k}(v)$, $k \le s-1$ may be estimated with the use of estimation for the lumping operators (1.9), ε-inequality and condition (1.8). As a result we will obtain

$$|\alpha^{n,k}(v)| \le \frac{\varepsilon_1}{2d_1} d_h(v,v) + \frac{c_1}{\varepsilon_1} h^2 \|u\|_{(*)}^2, \quad k = 1,\ldots,s-1, \tag{5.3}$$

where the norm $\| \ \|_{(*)}$ is given by formula (1.13). Then for an estimation of the functional $\alpha^{n,s}(v)$ we will use the presentation

$$\alpha^{n,s}(v) = \alpha_1^{n,s}(v) + \alpha_2^{n,s}(v), \tag{5.4}$$

where

$$\begin{aligned}\alpha_1^{n,s}(v) &= (\frac{du}{dt}(t_{n+1}) - [u(t_{n+1})]_\tau, v),\\ \alpha_2^{n,s}(v) &= ([u(t_{n+1})]_\tau, v) - d_h(\Pi_h[u(t_{n+1})]_\tau, v).\end{aligned}$$

From the Cauchy-Buniakowvsky inequality, the ε-inequality and condition (1.8) the inequality

$$|\alpha_1^{n,s}(v)| \leq \frac{\varepsilon_2}{2d_1} d_h(v,v) + \frac{\tau}{4\varepsilon_2} \left\| \frac{d^2u}{dt^2} \right\|^2_{L_2(Q_t)} \tag{5.5}$$

follows. For an estimation of the functional $\alpha_2^{n,s}(v)$ we will use the inequality (1.9), the ε-inequality and condition (1.8). Then

$$|\alpha_2^{n,s}(v)| \leq \frac{\varepsilon_2}{2d_1} d_h(v,v) + \frac{c_2}{\varepsilon_2} \tau^{-1} h^2 \left\| \frac{du}{dt} \right\|^2_{(t_n,t_{n+1})}, \tag{5.6}$$

where the norm $\| \ \|_{(t_n,t_{n+1})}$ is given by expression (1.12). In accordance with (5.4), inequalities (5.5), (5.6) give the estimation for the functional $\alpha^{n,s}(v)$. Then the estimation of the functionals $\beta^{n,k}(v)$, $k = 1,\ldots,s$ follows from condition (1.2) and the traditional error estimation of the finite element method [6]. Using the ε-inequality we obtain the estimation

$$|\beta^{n,k}(v)| \leq \frac{\varepsilon_3}{2} a_0^{(k)}(t_{n+1}; v,v) + \frac{c_3}{\varepsilon_3} h^2 \|u\|^2_{C(t_0,t_*);W_p^2(\Omega))}. \tag{5.7}$$

Let us estimate the functional $\gamma^{n,k}(v)$. For $k \leq s-1$ we have the following presentation:

$$\gamma^{n,k}(v) = \gamma_1^{n,k}(v) + \gamma_2^{n,k}(v), \tag{5.8}$$

where

$$\gamma_1^{n,k}(v) = a_0^{(k)}(t_{n+1}; \Pi_h[u(t_{n+1})]_\tau, v), \quad \gamma_2^{n,k}(v) = a_0^{(k)}(t_{n+1}; \Pi_h z(t_{n+1}), v).$$

Similarly to estimation (5.7), it's not difficult to obtain

$$|\gamma_1^{n,k}(v)| \leq \frac{\varepsilon_4}{2} a_0^{(k)}(t_{n+1}; v,v) + \frac{c_4}{\varepsilon_4} \tau^{-1} \left\| \frac{du}{dt} \right\|^2_{(t_n,t_{n+1})}, \quad k \leq s-1, \tag{5.9}$$

$$|\gamma_2^{n,k}(v)| \leq \frac{\varepsilon_4}{2} a_0^{(k)}(t_{n+1}; v,v) + \frac{c_5}{\varepsilon_4} \|u\|^2_{(*)}, \quad k \leq s-1. \tag{5.10}$$

In accordance with presentation (5.8) from inequalities (5.9), (5.10) the estimations of the functionals $\gamma^{n,k}(v)$, $k = 1,\ldots,s-1$ follow. The estimation for the functional $\gamma^{n,s}(v)$ may be found in similar manner, but with using the presentation

$$\gamma^{n,s}(v) = a_0^{(s)}(t_{n+1}; \Pi_h w^n, v) - a^{(s,s)}(t_{n+1}; \Pi_h w^n, v),$$

which follows from equalities (2.2), (2.5) and property (2.25). As a result we will obtain

$$\begin{aligned} |\gamma^{n,s}(v)| \;\le\; & \tfrac{\varepsilon_4}{2}(a_0^{(s)}(t_{n+1};v,v)+a^{(s,s)}(t_{n+1};v,v)) \\ & +\tfrac{c_6}{\varepsilon_4}\left(\tau^{-1}\left\|\tfrac{du}{dt}\right\|^2_{(t_n,t_{n+1})}+\|u\|^2_{(*)}\right). \end{aligned} \tag{5.11}$$

Finally, we should estimate the second term in the right-hand side of presentation of the functional $g^{n,k}(v)$ (3.35). From the Cauchy-Buniakovsky inequality, the ε-inequality, condition (1.2) and Lemma 4.3 the estimation

$$\begin{aligned} |a^{(s,k)}(t_{n+1};\hat{r}^{n+k/s},v)| \le & \frac{\varepsilon_5}{2}a^{(s,k)}(t_{n+1};v,v) \\ & +\tfrac{c_7}{\varepsilon_5}\sigma_k(h,\tau,\varepsilon)\left(\|u\|^2_{C([t_0,t_*];H^2(\Omega))}+h^3\|u\|^2_{(*)}\right), \end{aligned} \tag{5.12}$$

follows. Here $\sigma_k(h,\tau,\varepsilon)=\varepsilon^2h^{-3}\kappa^{4(k-1)}$, $\kappa=2h^{-1}\sqrt{\tau\lambda_0}$. The left inequality of conditions (4.9) gives the estimation $\sigma_k(h,\tau,\varepsilon)\le(4\lambda_0)^{2(s-1)}\varepsilon^2h^{1-4s}$, $k\le s-1$. Since ε is an arbitrary positive number let us

$$\varepsilon = 2h^{2s-1/2}. \tag{5.13}$$

Then the inequality

$$|\sigma_k(h,\tau,\varepsilon)|\le\sigma_0, \tag{5.14}$$

is valid, where σ_0 does not depend on h, τ and ρ. From inequalities (5.3), (5.5)-(5.7), (5.9)-(5.12) the estimation of the functionals $g^{n,k}(v)$ follows. Let us substitute this estimation into identity (5.2). Using the equality

$$\begin{aligned} &\sum_{k=1}^{s-1}\sum_{l=k+1}^{s} d_h(\xi^{n+l/s}-\xi^{n+(l-1)/s},\xi^{n+l/s}-\xi^{n+(l-1)/s}) \\ &\quad= \sum_{k=1}^{s}(s-1)d_h(\xi^{n+k/s}-\xi^{n+(k-1)/s},\xi^{n+k/s}-\xi^{n+(k-1)/s}), \end{aligned}$$

assuming

$$\varepsilon_1=d_1/(s-1)\tau,\;\; \varepsilon_2=d_1/2,\;\; \varepsilon_3=1/2,\;\; \varepsilon_4=1/4\tau,\;\; \varepsilon_5=1/\tau$$

and according to non-negativity of bilinear forms (3.1), we will obtain the inequality

$$d_h(\xi^{n+1},\xi^{n+1})\le(1+\tau)d_h(\xi^n,\xi^n)+\tau\psi^n, \tag{5.15}$$

where $\tau\le1$ and according to (5.14) ψ^n is evidently presented from right-hand sides of the estimations (5.3), (5.5)-(5.7), (5.9)-(5.12). The use of the greed Gronwall Lemma [7] and condition (1.8) to inequality (5.15) gives the following estimation:

$$\|\xi^n\|_{L_2(\Omega)} \leq c'(M_h h + M_\tau \tau), \tag{5.16}$$

where

$$\begin{aligned} M_h &= \|u\|_{\dot{C}([t_0,t_*];W_p^2(\Omega))} + \|\tfrac{du}{dt}\|_{(t_0,t_*)}, \\ M_\tau &= \|u\|_{(*)} + \|\tfrac{du}{dt}\|_{(t_0,t_*)} + \|\tfrac{d^2u}{dt^2}\|_{L_2(Q_t)}. \end{aligned} \tag{5.17}$$

Finally, from (5.16) the triangle inequality and the error estimation of the finite element method the resulting estimation follows. Before we formulate corresponding theorem we will recall all conditions for parameters h and τ. There are the inequality $\tau \leq 1$, the condition of the Theorem 2.1 $h \leq \nu\rho/2$ and conditions (4.9) of Lemma 4.1, which according to formula (5.13) may be rewritten in the form:

$$2h^2 \leq \lambda_0 \tau \leq \left(\frac{\nu\rho}{(4s-1)\ln h}\right)^2. \tag{5.18}$$

The left inequality from (5.18) is not too restricted, since the inverse inequality means that explicit scheme may be used. The right inequality is a condition of stability, which is essentially weaker than a condition of stability of explicit scheme. Here the role of the quantity h^2 (in a condition of stability of explicit scheme) plays the quantity $(\rho \ln h)^2$. Therefore, we have proved the following

Theorem 5.1. *Let for problem* (1.4), (1.5) *conditions* (3.7) *hold and let* $\{\Omega^{(k)}\}_{k=1}^s$ *be* (ρ, h)*-regular system. Then the positive number* ν*, which does not depend on parameters* h*,* τ *and* ρ*, exists such that if the condition* $\tau \leq 1$*,* $h \leq \nu\rho/2$ *and* (5.18) *hold, then the following estimation is valid:*

$$\max_{1\leq n\leq N} \|u^n - u(t_n)\|_{L_2(\Omega)} \leq c(M_h h + M_\tau \tau),$$

where the number c *does not depend on* h*,* τ ρ *and the function* $u(t)$*, the numbers* M_h *and* M_τ *are defined by formulas* (5.17).

Remark 5.1. This result may be easy expanded for the third boundary value problem. But the consideration of the Dirichlet problem requires additional constructions connecting with the necessary condition $r^{n+k/s} = 0,\ \bar{x} \in \partial\Omega$.

Remark 5.2. Let us consider a question about the domain decomposition to "small subdomains". A possibility of such decomposition with algorithm (3.2)-(3.5) is motived by the independece of the subdomains size the number c in the formulation of Theorem 5.1. Let us $\rho = h^\alpha$, $\alpha \in (0,1)$. This assumption means a consideration of the "small subdomains", including multi-connecting subdomains, for which the diameter of every component of connection is the quantity $O(h^\alpha)$. The conditions of Theorem 5.1 $h \leq \nu\rho/2$ and the right one from (5.18) may be rewritten in the form

$$h \leq (\nu/2)^{1/(1-\alpha)}, \quad \tau \leq c_0(h^\alpha \ln h)^2,$$

where the number $c_0 = \nu^2/(4s-1)^2\lambda_0$ does not depend on h, τ and ρ. So in order to receive the condition $\tau = O(h)$ for sufficiently small h it must assume $\alpha < 1/2$.

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