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On the explicit-implicit domain decomposition method for parabolic problems

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The paper deals with studying the domain decomposition algorithm on two subdomains, where for one of them, which contains sufficiently small number of nodes, is used explicit scheme with small time step, and for another subdomain may be used effective direct algorithm (for example, subdomain is parallelepiped). This method is based on the splitting method, which is described in [1]. The algorithm formulation is given in projection form with the finite element approximation. The approximate computation of the mass matrix is important in this paper. The lumping operators technique [2] is used for this purpose.

1. Formulation of the problem and discretization

Let Ω be a bounded polytop in R^m, $\Omega^{(1)}$ and $\Omega^{(2)}$ be subdomains in Ω, such that the conditions

$$\Omega^{(1)} \cup \Omega^{(2)} = \Omega, \qquad \operatorname{dist}(\Omega_0^{(1)}, \Omega_0^{(2)}) \geq \rho > 0 \tag{1.1}$$

are valid, where

$$\Omega_0^{(k)} = \Omega^{(k)} \setminus B, \quad k = 1,2, \qquad B = \Omega^{(1)} \cap \Omega^{(2)},$$
$$\operatorname{dist}(\Omega', \Omega'') = \inf_{\bar{x}\in\Omega', \bar{y}\in\Omega''} |\bar{x} - \bar{y}|_m,$$

$|\cdot|_m$ is the Euclidian norm in R^m.

Let D be a subdomain in Ω. In the space $H^1(\Omega) \times H^1(\Omega)$ we consider one-parametric families of the bilinear forms

$$a_D(t; u, v) = \int_D \sum_{i,j=1}^m \lambda_{ij}(t, \bar{x}) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\bar{x}, \tag{1.2}$$

where $t \in [t_0, t_*]$ is a real parameter. For the functions $\lambda_{ij}(t, \bar{x})$ we assume that bilinear form (1.2) is symmetric, continuous and coercive, i.e., $\forall u, v \in H^1(\Omega)$ the inequalities

$$|a_D(t; u, v)| \leq a_0 \|u\|_{H^1(D)} \|v\|_{H^1(D)}, \tag{1.3}$$

$$a_D(t; u, u) \geq a_D |u|^2_{H^1(D)} \tag{1.4}$$

are valid, where a_0 and a_1 are positive numbers, which are independent of t. Here $|u|_{H^1(D)} = (\int_D |\nabla u|^2 d\bar{x})^{1/2}$ is a semi-norm in the space $H^1(D)$. Assumptions on smoothness of the functions $\lambda_{ij}(t, \bar{x})$ will be given in what follows. We will use the notations:

$$\begin{aligned} &a(t; u, v) = a_\Omega(t; u, v), && a^{(k)}(t; u, v) = a_{\Omega^{(k)}}(t; u, v), \\ &a_0^{(k)}(t; u, v) = a_{\Omega_0^{(k)}}(t; u, v), && b(t; u, v) = a_B(t; u, v). \end{aligned} \tag{1.5}$$

From conditions (1.1) and notations (1.5) we will obtain the following additive presentations:

$$a^{(k)}(t; u, v) = a_0^{(k)}(t; u, v) + b(t; u, v), \quad k = 1, 2, \tag{1.6}$$

$$a(t; u, v) = a^{(1)}(t; u, v) + a_0^{(2)}(t; u, v). \tag{1.7}$$

Then we introduce the family of the continues on t and continues in $L_2(\Omega)$ linear functionals $(f(t), u)$, where $(\cdot, \cdot)$ is the scalar product in $L_2(\Omega)$, $f : [t_0, t_*] \to L_2(\Omega)$. Henceforth, by $u(t)$ we denote the value of the function $u : [t_0, t_*] \to X$, which is the element of some Banach space X, and $\frac{du}{dt}(t)$ means a strong limit in X (if such limit exists) of the elements $[u(t)]_\tau \equiv (u(t + \tau) - u(t))/\tau$ for $\tau \to 0$.

Now we will formulate the parabolic Neumann problem, for which we will construct the domain decomposition algorithm. Let $u_0 \in L_2(\Omega)$ and $f \in L_2((t_0, t_*); H^{-1}(\Omega))$. It's necessary to find the function $u \in L_2((t_0, t_*); H^1(\Omega))$, such that $\frac{du}{dt} \in L_2((t_0, t_*); H^{-1}(\Omega))$ and $\forall v \in H^1(\Omega)$ the following equalities are valid:

$$\Big(\frac{du}{dt}(t), v\Big) + a(t; u(t), v) = \big(f(t), v\big), \quad t \in (t_0, t_*], \tag{1.8}$$

$$\big(u(t_0), v\big) = \big(u_0, v\big). \tag{1.9}$$

Let us introduce some notations for discretization of problem (1.8), (1.9). Let $\mathcal{T}_h$ be a regular set of m-simplexes with "inverse assumption" [3]. For set $\mathcal{T}_h$ we will introduce the finite-dimensional space V_h with the piecewise linear basis $\{\varphi_i(\bar{x})\}_{i \in I}$, where $\varphi_i(\bar{x}_j) = \delta_{i,j}$, $i, j \in I$, $\bar{x}_j$ is a set of all different vertesies of the set $\mathcal{T}_h$. Then let $\Pi_h : C(\bar{\Omega}) \rightarrow V_h$ be an interpolating operator defined by the formula $\Pi_h u(\bar{x}) = \sum_{i \in I} u(\bar{x}_i)\varphi_i(\bar{x})$. In accordance with paper [2] we introduce the lumping operator $P_{h,\mu}$ in the space V_h, for which the condition of the approximation relative to the scalar product in $L_2(\Omega)$ is valid. Let us consider the bilinear form

$$d_h(u,v) = (P_{h,\mu}u, P_{h,\mu}v), \quad u, v \in V_h. \tag{1.10}$$

As shown in [2], there exist numbers d_0 and d_1 independent of h, such that

$$| \, d_h(u,v) \, | \leq d_0 \parallel u \parallel_{L_2(\Omega)} \parallel v \parallel_{L_2(\Omega)}, \tag{1.11}$$

$$d_h(u,u) \geq d_1 \parallel u \parallel^2_{L_2(\Omega)} \tag{1.12}$$

for any $u, v \in V_h$.

Let us give some inequalities, which will be used further. Let $u \in W^2_p(\Omega)$ where $p > m/2$. Then according to inclusion of $W^2_p(\Omega)$ into $C(\bar{\Omega})$ the function $\Pi_h u \in V_h$ is defined. In accordance with [2] for bilinear form (1.10) the following estimation of approximation is valid:

$$\begin{aligned} | \, (u,v) - d_h(\Pi_h u, v) \, | &\leq c h^{k+l} \{ \, | \, u \, |_{H^k(\Omega')} | \, v \, |_{H^l(\Omega')} \\ &+ h^{2-k-l} (\text{meas}(\Omega'))^{\frac{p-2}{2p}} \, | \, u \, |_{W^2_p(\Omega')} \| \, v \, \|_{L_2(\Omega')} \, \}, \end{aligned} \tag{1.13}$$

where $k, l = 0, 1$, $\Omega' = \text{supp}(v)$, number c does not depend on h, u, v and Ω'. Let us give the so-called "inverse inequality" for functions from the space V_h [3]:

$$| \, v \, |_{H^1(\Omega')} \leq c h^{-1} \parallel v \parallel_{L_2(\Omega')}, \tag{1.14}$$

where $\Omega' = \bigcup_{e \in \mathcal{T}'_h} e$, $\mathcal{T}'_h \subseteq \mathcal{T}_h$, number c does not depend on h, u and Ω'.

Finally, let us introduce notations for two norms, which will be used further:

$$\|u\|_{(t',t'')} = \left[\|u\|^2_{L_2((t',t'');H^1(\Omega))} + h^2 \|u\|^2_{L_2((t',t'');W^2_p(\Omega))} \right]^{1/2}, \tag{1.15}$$

$$\|u\|_{(*)} = \left[\|u\|^2_{C([t_0,t_*]);H^3(\Omega))} + h^2 \|u\|^2_{C([t_0,t_*]);W^4_p(\Omega))} \right]^{1/2}. \tag{1.16}$$

All these notations and inequalities were presented in [1].

2. Explicit-implicit scheme

Now let us formulate the difference Neumann problem. Let N and s be some integer numbers, $\tau = (t_* - t_0)/N$, $\tau_0 = \tau/s$ and $t_n = t_0 + n\tau$, $n = 1, \ldots, N$. It is necessary to find the sequence of the functions

$$\left\{ u^{n+k/2s}, \; k = 1, \ldots, s, \quad u^{n+1}, \; \hat{u}^{n+1}, \; n = 0, \ldots, N-1 \right\},$$

such that $u^{n+k/2s}, u^{n+1}, \hat{u}^{n+1} \in V_h$ and $\forall v^{n+k/2s}$, $k = 0,\ldots,s$, $v^{n+1} \in V_h$ the following equalities are valid:

$$\begin{aligned} &d_h(u^{n+(k+1)/2s} - u^{n+k/2s}, v^{n+k/2s}) \\ &\qquad + \tau_0 a^{(1)}(t_n; u^{n+k/2s}, v^{n+k/2s}) = 0, \quad k = 0,\ldots,s-1, \end{aligned} \tag{2.1}$$

$$d_h(\hat{u}^{n+1} - u^{n+1/2}, v^{n+1/2}) - \tau_0 \sum_{k=0}^{s-1} b(t_n; u^{n+k/2s}, v^{n+1/2}) = 0, \tag{2.2}$$

$$d_h(u^{n+1} - \hat{u}^{n+1}, v^{n+1}) + \tau a^{(2)}(t_n; u^{n+1}, v^{n+1}) = \tau(f(t_n), v^{n+1}), \tag{2.3}$$

$$u^0 = \Pi_h u_0. \tag{2.4}$$

Here we suppose that $u_0 \in W_p^2(\Omega)$, $p > m/2$, which ensures the inclusion of $W_p^2(\Omega)$ into $C(\bar{\Omega})$ and therefore the function $\Pi_h u_0$ exists. For further consideration let us present the system (2.1)–(2.3) in some other form. Let $v^{n+1} = v^{n+1/2}$ and sum up equations (2.2), (2.3). As a result we will obtain

$$\begin{aligned} &d_h(u^{n+1} - u^{n+1/2}, v^{n+1/2}) + \tau a^{(2)}(t_n; u^{n+1}, v^{n+1/2}) \\ &\qquad - \tau_0 \sum_{k=0}^{s-1} b(t_n; u^{n+k/2s}, v^{n+1/2}) = \tau(f(t_n), v^{n+1/2}). \end{aligned} \tag{2.5}$$

Let $u(t)$ be a solution to problem (1.8), (1.9). We assume that the following conditions hold:

$$\begin{aligned} &\lambda_{ij} \in C([t_0,t_*]; C^3(\bar{\Omega})), \qquad && u \in C([t_0,t_*]; W_p^4(\Omega)), \\ &du/dt \in L_2((t_0,t_*); W_p^2(\Omega)), && d^2u/dt^2 \in L_2(Q_t), \end{aligned} \tag{2.6}$$

where $Q_t = (t_0, t_*) \times \Omega$. According to (2.6), the function $\Pi_h u(t_{n+1})$ exists and, henceforth, we may introduce the sequence:

$$\begin{aligned} \xi^{n+k/2s} &= u^{n+k/2s} - \Pi_h u(t_n) + \tau_0 r^{n+k/2s}, \quad k = 0,\ldots,s, \\ \xi^{n+1} &= u^{n+1} - \Pi_h u(t_{n+1}), \end{aligned} \tag{2.7}$$

where $r^n(\bar{x}) \equiv 0$ and the functions $r^{n+k/2s} \in V_h$, $k = 1,\ldots,s$ will be defined later. Then according to (2.4) $\xi^0 = 0$. In accordance with equations (2.1), (2.5) let us write the scheme for the functions $\xi^{n+k/2s}$. We have

$$\begin{aligned} &d_h(\xi^{n+(k+1)/2s} - \xi^{n+k/2s}, v^{n+k/2s}) + \tau_0 a^{(1)}(t_n; \xi^{n+k/2s}, v^{n+k/2s}) \\ &\qquad = \tau_0 g^{n,k}(v^{n+k/2s}), \quad k = 0,\ldots,s-1, \end{aligned} \tag{2.8}$$

$$d_h(\xi^{n+1}-\xi^{n+1/2},v^{n+1/2})+\tau a^{(2)}(t_n;\xi^{n+1},v^{n+1/2}) \\ -\tau_0\sum_{k=0}^{s-1} b(t_n;\xi^{n+k/2s},v^{n+1/2})=\tau g^{n,s}(v^{n+1/2}). \tag{2.9}$$

Before writing the functionals $g^{n,k}(v)$ let us introduce some notations. Let

$$\hat{r}^{n+k/2s}=r^{n+k/2s}-k\Pi_h z(t_n),\quad k=0,\dots,s,$$

where

$$z(t)=-\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\Big(\lambda_{ij}\frac{\partial u}{\partial x_i}\Big)(t).$$

The existence of the functions $\Pi_h z(t_n)$ follows from conditions (2.6). Additive presentation (1.7) and Green's formula gives an equality

$$a^{(1)}(t_n;u(t_n),v)=\big(z(t_n),v\big)-a_0^{(2)}(t_n;u(t_n),v). \tag{2.10}$$

Then the functionals $g^{n,k}(v)$ are defined in the following form:

$$g^{n,k}(v)=g_1^{n,k}(v)+g_2^{n,k}(v),\quad k=0,\dots,s, \tag{2.11}$$

where

$$g_1^{n,k}(v)=d_h(\hat{r}^{n+(k+1)/2s}-\hat{r}^{n+k/2s},v)+\tau_0 a^{(1)}(t_n;\hat{r}^{n+k/2s},v) \\ +a_0^{(2)}(t_n;u(t_n),v),\quad k=0,\dots,s-1, \tag{2.12}$$

$$g_1^{n,s}(v)=-\frac{1}{s}\Big(d_h(\hat{r}^{n+1/2},v)+\tau_0\sum_{k=0}^{s-1} b(t_n;\hat{r}^{n+k/2s},v)\Big) \\ -a_0^{(2)}(t_n;u(t_n),v), \tag{2.13}$$

$$g_2^{n,k}(v)=-\alpha_1^n(v)+\beta_1^n(v)+k\tau_0\gamma_1^n(v),\quad k=0,\dots,s-1, \tag{2.14}$$

$$g_2^{n,s}(v)=\alpha_1^n(v)+\alpha_2^n(v)+\beta_2^n(v)+\tau\gamma_2^n(v). \tag{2.15}$$

The functionals $\alpha_l^n(v)$, $\beta_l^n(v)$, $\gamma_l^n(v)$, $l=1,2$ are given by equalities

$$\begin{aligned}
\alpha_1^n(v) &= (z(t_n),v)-d_h(\Pi_h z(t_n),v),\\
\alpha_2^n(v) &= \Big(\frac{du}{dt}(t_n),v\Big)-d_h(\Pi_h[u(t_n)]_\tau,v),\\
\beta_1^n(v) &= a^{(1)}(t_n;u(t_n)-\Pi_h u(t_n),v),\\
\beta_2^n(v) &= a_0^{(2)}(t_n;u(t_n)-\Pi_h u(t_n),v),\\
\gamma_1^n(v) &= a^{(1)}(t_n;\Pi_h z(t_n),v),\\
\gamma_2^n(v) &= \frac{1-s}{2s}b(t_n;\Pi_h z(t_n),v)-a^{(2)}(t_n;\Pi_h[u(t_n)]_\tau,v).
\end{aligned} \tag{2.16}$$

Here we used integral identity (1.8), additive presentations (1.6), (1.7) and equality (2.10).

Let us consider the question on a choice of the functions $\hat{r}^{n+k/2s}$. Dividing equalities (2.12) on s sum up them over k from 0 to $s-1$ and the result sum with equality (2.13). Using equality (1.6) for $k=1$ we will obtain the following presentation of the functional $g_1^{n,s}(v)$:

$$g_1^{n,s}(v) = \frac{1}{s}\sum_{k=0}^{s-1}\Big(\tau_0 a_0^{(1)}(t_n;\hat{r}^{n+k/2s},v) - g_1^{n,k}(v)\Big). \tag{2.17}$$

Let us define the functions $\hat{r}^{n+k/2s}$ from the following conditions: $\forall v \in V_h$

$$g_1^{n,k}(v) = \tau_0 a_0^{(1)}(t_n;\hat{r}^{n+k/2s},v), \quad k=0,\dots,s-1. \tag{2.18}$$

From equalities (2.11), (2.17) and (2.18) it follows

$$\begin{aligned} g^{n,k}(v) &= \tau_0 a_0^{(1)}(t_n;\hat{r}^{n+k/2s},v) + g_2^{n,k}(v), \quad k=0,\dots,s-1 \\ g^{n,s}(v) &= g_2^{n,s}(v), \end{aligned} \tag{2.19}$$

where the functionals $g_2^{n,k}(v)$, $k=0,\dots,s$ are given by equalities (2.14)–(2.16). In accordance with equalities (1.6), (2.12) system (2.18) may be written in the following reccurent form:

$$\begin{aligned} d_h(\hat{r}^{n+(k+1)/2s},v) &= d_h(\hat{r}^{n+k/2s},v) - \tau_0 b(t_n;\hat{r}^{n+k/2s},v) \\ &\quad - a_0^{(2)}(t_n;u(t_n),v), \quad k=0,\dots,s-1. \end{aligned} \tag{2.20}$$

Therefore, we have comletely defined the right-hand side of error equations (2.8), (2.9).

3. Error analysis

We begin the error analysis from the obtaining of the integral identity, which is a foundation of the stability analysis. Let us $v^{n+k/2s} = 2\tau_0\xi^{n+(k+1)/2s}$, $k=0,\dots,s-1$ and $v^{n+1/2} = 2\tau\xi^{n+1}$, and then we will sum equations (2.8) over k from 0 to $s-1$ and the result sum with (2.9). Here we use additive presentation (1.6) and evident equality

$$2a_D(t;u,v) = a_D(t;u,u) + a_D(t;v,v) - a_D(t;u-v,u-v).$$

As a result we obtain

$$\begin{aligned} & d_h(\xi^{n+1},\xi^{n+1}) - d_h(\xi^n,\xi^n) + d_h(\eta^s,\eta^s) + \sum_{k=0}^{s-1} J_h(t_n;\eta^k) \\ & + \tau_0 \sum_{k=0}^{s-1} (a^{(1)}(t_n;\xi^{n+(k+1)/2s},\xi^{n+(k+1)/2s}) \\ & + a_0^{(1)}(t_n;\xi^{n+k/2s},\xi^{n+k/2s})) + \tau(a^{(2)}(t_n;\xi^{n+1},\xi^{n+1}) \\ & + a_0^{(2)}(t_n;\xi^{n+1},\xi^{n+1})) + \tau_0 \sum_{k=0}^{s-1} b(t_n;\xi^{n+1}-\xi^{n+k/2s},\xi^{n+1}-\xi^{n+k/2s}) \\ & = 2\tau_0 \sum_{k=0}^{s-1} g^{n,k}(\xi^{n+(k+1)/2s}) + 2\tau g^{n,s}(\xi^{n+1}), \end{aligned} \tag{3.1}$$

where $\eta^k = \xi^{n+(k+1)/2s} - \xi^{n+k/2s}$, $k = 0,\dots,s-1$, $\eta^s = \xi^{n+1} - \xi^{n+1/2}$, and the functional $J_h(t;v)$ is presented in the form

$$J_h(t;v) = d_h(v,v) - \tau_0 a^{(1)}(t;v,v). \tag{3.2}$$

Our further actions are connected with estimation of the functionals $g^{n,k}(v)$. For this aim we use standard technique based on using of the Cauchy-Buniakovsky and the ε-inequalities, estimation (1.13) and the well-known results about interpolation in Sobolev's spaces [3]. As a result we obtain the following estimations:

$$|\alpha_1^n(v)| \le \frac{\varepsilon_1}{2d_1} d_h(v,v) + \frac{c_1}{\varepsilon_1} h^2 \|u\|^2_{(*)}, \tag{3.3}$$

$$|\alpha_2^n(v)| \le \frac{\varepsilon_2}{d_1} d_h(v,v) + \frac{\tau}{4\varepsilon_2} \left\| \frac{d^2u}{dt^2} \right\|^2_{L_2((t_n,t_{n+1})\times\Omega)} + \frac{c_2}{\varepsilon_2} \tau^{-1} h^2 \left\| \frac{du}{dt} \right\|^2_{(t_n,t_{n+1})}, \tag{3.4}$$

$$|\beta_1^n(v)| \le \frac{\varepsilon_3}{2} a^{(1)}(t_n;v,v) + \frac{c_3}{\varepsilon_3} h^2 \|u\|^2_{C([t_0,t_*];W_p^2(\Omega^{(1)}))}, \tag{3.5}$$

$$|\beta_2^n(v)| \le \frac{\varepsilon_4}{2} a_0^{(2)}(t_n;v,v) + \frac{c_4}{\varepsilon_4} h^2 \|u\|^2_{C([t_0,t_*];W_p^2(\Omega_0^{(2)}))}, \tag{3.6}$$

$$|\gamma_1^n(v)| \le \frac{\varepsilon_5}{2} a^{(1)}(t_n;v,v) + \frac{c_5}{\varepsilon_5} \|u\|^2_{(*)}, \tag{3.7}$$

$$|\gamma_2^n(v)| \le \frac{\varepsilon_6}{2} a^{(2)}(t_n;v,v) + \frac{c_6}{\varepsilon_6} \left(\|u\|^2_{(*)} + \tau^{-1} \left\| \frac{du}{dt} \right\|_{(t_n,t_{n+1})} \right), \tag{3.8}$$

where the norms $\|\cdot\|_{(t_n,t_{n+1})}$ and $\|\cdot\|_{(*)}$ are defined by expressions (1.15), (1.16). Moreover for estimation of the functional $\gamma_2^n(v)$ we used the fact that $B \subset \Omega^{(2)}$ and therefore $b(t;v,v) \le a^{(2)}(t;v,v)$. In accordance with presentations (2.14), (2.15) inequalities (3.3)-(3.8) allow to estimate the functionals $g_2^{n,k}(v)$.

Let us consider the functional $a_0^{(1)}(t_n; \hat{r}^{n+k/2s}, v)$, $k = 1, \ldots, s-1$, including to the right-hand side of the first equality from (2.19). The following result is valid:

Lemma 3.1. *If* $s \leq \rho/h + 1$, *then*

$$a_0^{(1)}(t_n; \hat{r}^{n+k/2s}, v) = 0, \quad v \in V_h, \quad k = 1, \ldots, s-1.$$

PROOF. Let $\Omega_1^{(2)} = \bar{\Omega}_0^{(2)}$ and $I_1^{(2)}$ be the numbers of the points $\bar{x}_i \in \Omega_1^{(2)}$. Then let us construct the sequences of the sets $I_l^{(2)}$, $\Omega_l^{(2)}$, $l = 1, \ldots, k-1$. Let us suppose

$$I_k^{(2)} = \{i \in I \mid \forall j \in I_{k-1}^{(2)}, \quad \mathrm{supp}(\varphi_i) \cap \mathrm{supp}(\varphi_j) \neq \emptyset\}, \qquad (3.9)$$

$$\mathcal{T}_{h,k} = \{e \in \mathcal{T}_h \mid \exists i \in I_k^{(2)}, \quad \bar{x}_i \in e\}, \qquad \Omega_k^{(2)} = \bigcup_{e \in \mathcal{T}_{h,k}} e. \qquad (3.10)$$

Let us note that $I_k^{(2)} \supset I_{k-1}^{(2)}$, $\Omega_k^{(2)} \supset \Omega_{k-1}^{(2)}$ and the sets $\Omega_k^{(2)}$ are closed. Continuing this process, we will construct the sequences of the extending sets $I_1^{(2)} \subset \ldots \subset I_{s-1}^{(2)}$ and $\Omega_1^{(2)} \subset \ldots \subset \Omega_{s-1}^{(2)}$. Then for an arbitrary $\bar{x} \in \Omega_k^{(2)}$ the point $\bar{y} \in \Omega_{k-1}^{(2)}$ exists, such that the inequality $|\bar{x} - \bar{y}|_m \leq h$ is valid. Let $\bar{x} \in \Omega_k^{(2)} \setminus \Omega_{k-1}^{(2)}$. (If $\bar{x} \in \Omega_{k-1}^{(2)}$, we may put $\bar{x} = \bar{y}$.) According to (3.10) there exists $j \in I_{k-1}^{(2)}$ – number of point from e, which contains the point $\bar{x}$. Then $\bar{x}_j \in \Omega_{k-1}^{(2)}$ and the inequality $|\bar{x} - \bar{x}_j|_m \leq h$ is valid. Let $\bar{y} = \bar{x}_j$. From the obtaining fact and the condition of lemma it follows that for an arbitrary $\bar{x} \in \Omega_{s-1}^{(2)}$ the point $\bar{y} \in \Omega_1^{(2)}$ exists and

$$|\bar{x} - \bar{y}|_m \leq (s-2)h < \rho. \qquad (3.11)$$

Let us show correctness of the inequality

$$\mathrm{dist}(\Omega_0^{(1)}, \Omega_k^{(2)}) > 0, \quad k = 1, \ldots, s-1. \qquad (3.12)$$

It is sufficiently to prove this inequality only for the set $\Omega_{s-1}^{(2)}$, since this set includes all sets $\Omega_k^{(2)}$. Let us assume that inequality (3.12) is not correct. It means that $\bar{\Omega}_0^{(1)} \cap \Omega_{s-1}^{(2)} \neq \emptyset$ and let $\bar{x} \in \bar{\Omega}_0^{(1)} \cap \Omega_{s-1}^{(2)}$. Then the point $\bar{y} \in \Omega_1^{(2)}$ exists so that inequality (3.11) is valid. This fact contradicts to condition (1.1). Obtaining contradiction proves inequality (3.12). Let us sum up equality (2.20) over k from 0 to $l-1$, assuming $v = \varphi_i$. Then we will obtain

$$\begin{aligned}\hat{r}^{n+l/2s}(\bar{x}_i) &= -\tau_0 \sum_{k=0}^{l-1} \sum_{j\in I} \rho_i^{-2} \hat{r}^{n+k/2s}(\bar{x}_j) b(t_n; \varphi_j, \varphi_i) \\ &\quad - l \sum_{j\in I} \rho_i^{-2} a_0^{(2)}(t_n; u(t_n), \varphi_i), \quad l = 1, \ldots, s-1,\end{aligned} \tag{3.13}$$

where $\rho_i = (d_h(\varphi_i, \varphi_i))^{1/2}$. Let $I_l^{(1)} = I \setminus I_l^{(2)}$. By induction we will show that $\hat{r}^{n+l/2s}(\bar{x}_i)$ for $i \in I_l^{(1)}$, $l = 1, \ldots, s-1$. In the first place the evident equalities are valid:

$$a_0^{(2)}(t_n; u(t_n), \varphi_i) = 0, \quad i \in I_l^{(1)}, \quad l = 1, \ldots, s-1. \tag{3.14}$$

Since $\hat{r}^n(\bar{x}) \equiv 0$, from (3.13) and (3.14) it follows that $\hat{r}^{n+1/2s}(\bar{x}_i) = 0$ for $i \in I_1^{(1)}$. Let $\hat{r}^{n+k/2s}(\bar{x}_i) = 0$ for $i \in I_k^{(1)}$, $k \le l-1$. Then for $i \in I_l^{(1)}$ according to (3.14) equalities (3.13) assume the following form:

$$\hat{r}^{n+l/2s}(\bar{x}_i) = -\tau_0 \sum_{k=0}^{l-1} \sum_{j\in I_k^{(2)}} \rho_i^{-2} \hat{r}^{n+k/2s}(\bar{x}_j) b(t_n; \varphi_j, \varphi_i).$$

According to (3.9) $b(t_n; \varphi_j, \varphi_i) \neq 0$ for $j \in I_k^{()2}$ only for $i \in I_{k+1}^{(2)}$. It means that $\hat{r}^{n+l/2s}(\bar{x}_i) \neq 0$ only for $i \in I_l^{(2)}$ and $\hat{r}^{n+l/2s}(\bar{x}_i) = 0$ for $i \in I_l^{(1)}$. Therefore $\hat{r}^{n+l/2s}(\bar{x}) \not\equiv 0$ only in the domain $\Omega_l^{(2)}$ and in accordance with (3.12) $\hat{r}^{n+l/2s}(\bar{x}) = 0$, when $\bar{x} \in \bar{\Omega}_0^{(1)}$. By this lemma is proved. □

From Lemma 3.1 and equalities (2.19) it follows that $g^{n,k}(v) = g_2^{n,k}(v)$, $k = 0, \ldots, s$ for $s \le \rho/h + 1$. Therefore inequalities (3.3)–(3.8) allow to estimate the functionals $g^{n,k}(v)$, which are the right-hand side of integral identity (3.1). Let us use these estimations, supposing that

$$\varepsilon_1 = d_1/12, \ \varepsilon_2 = d_1/24, \ \varepsilon_3 = 1/2, \ \varepsilon_4 = 1, \ \varepsilon_5 = 1/2\tau, \ \varepsilon_6 = 1/\tau,$$

and inequality

$$d_h(\xi^{n+(k+1)/2s}, \xi^{n+(k+1)/2s}) \le 2d_h(\xi^n, \xi^n) + s(s+1) \sum_{l=0}^{k} d_h(\eta^l, \eta^l).$$

The latter inequality follows from the evident presentation $\xi^{n+(k+1)/2s} = \xi^n + \sum_{l=0}^k \eta^l$ and from the Cauchy-Buniakovsky inequality. As a result of non-difficult transformations we obtain

$$(1-\tau/6)d_h(\xi^{n+1},\xi^{n+1})+\sum_{k=0}^{s-1}\Big\{J_h(t_n;\eta^k)-\frac{\tau(s+1)}{12}d_h(\eta^k,\eta^k)\Big\} \\ \leq (1+\tau/6)d_h(\xi^n,\xi^n)+\tau\psi^n, \tag{3.15}$$

where by evident way ψ^n is defined from the right-hand sides of inequalities (3.3)–(3.8). From (3.15) it follows that inequality

$$J_h(t;v)\geq \frac{\tau(s+1)}{12}d_h(v,v), \quad t\in[t_0,t_*], \quad v\in V_h \tag{3.16}$$

is a sufficient condition for stability of method (2.1)–(2.4). Let us use inverse inequality (1.14), which according to conditions (1.3), (1.12) may be written in the form:

$$a(t;v,v)\leq \lambda h^{-2}d_h(v,v), \quad t\in[t_0,t_*], \quad v\in V_h, \tag{3.17}$$

where the number λ does not depend on h, t and the function v ($\lambda=ca_0/d_1$, c, a_0, d_1 are the numbers from inequalities (1.14), (1.3), (1.12)). Let the following conditions hold:

$$h\leq\rho, \quad \tau\leq\rho_0 h, \quad s=[\lambda_0\tau/h^2]+1, \tag{3.18}$$

where the parameters λ_0 and ρ_0 are given by equalities

$$\lambda_0=\lambda+\rho^2/4, \quad \rho_0=\rho/\lambda_0. \tag{3.19}$$

Let us show that from (3.18) it follows the condition of Lemma 3.1 and inequality (3.16). At first the evident inequality $s\leq\lambda_0\tau/h^2+1$ holds. From this inequality in accordance with the second condition from (3.18) it follows the condition of Lemma 3.1 $s\leq\rho/h+1$. Then, since $s\geq\lambda_0\tau/h^2$, two first conditions from (3.18) give the inequality

$$\tau\Big(\frac{\lambda}{sh^2}+\frac{s+1}{12}\Big)\leq\lambda/\lambda_0+\rho\rho_0/4=1. \tag{3.20}$$

In particular, from this inequality it follows that $\frac{\tau(s+1)}{12}<1$. Then multiplying inequality (3.20) by the functional $d_h(v,v)$, using inverse inequality in the form (3.17) and taking into account that $\tau=s\tau_0$, we obtain the estimation

$$\tau_0 a(t;v,v)\leq\Big(1-\frac{\tau(s+1)}{12}\Big)d_h(v,v).$$

Evidently, from this estimation we achieve the inequality (3.16). Moreover, two first conditions from (3.18) give the estimation

$$\tau \leq \rho_0 \rho \leq 4\rho^2/(4\lambda + \rho^2) \leq 4,$$

from which it follows that $(1 + \tau/6)/(1 - \tau/6) \leq 1 + \tau$. Using the latter inequality and inequality (3.16) let us intensify estimation (3.15). As a result we obtain

$$d_h(\xi^{n+1}, \xi^{n+1}) \leq (1 + \tau) d_h(\xi^n, \xi^n) + 3\tau\psi^n.$$

By standard scheme this inequality leads to the resulting error estimation of the explicit-implicit domain decomposition method. So, the following result is valid.

Theorem 3.1. *Let for problem* (1.8), (1.9), *conditions of smoothness* (2.6) *hold and let subdomains* $\Omega^{(k)}$, $k = 1, 2$ *satisfy conditions* (1.1). *Then from conditions* (3.18) *it follows the error estimation*

$$\max_{1 \leq n \leq N} \|u^n - u(t_n)\|_{L_2(\Omega)} \leq c(M_h h + M_\tau \tau),$$

where the number c does not depend on h, τ, ρ and the function u,

$$M_h = \|u\|_{C([t_0,t_*];W_p^2(\Omega))} + \|u\|_{(*)} + \left\|\frac{du}{dt}\right\|_{(t_0,t_*)},$$

$$M_\tau = \|u\|_{(*)} + \left\|\frac{du}{dt}\right\|_{(t_0,t_*)} + \left\|\frac{d^2u}{dt^2}\right\|_{L_2(Q_t)},$$

the norms $\|\cdot\|_{(t_0,t_*)}$ *and* $\|\cdot\|_{(*)}$ *are given by formulas* (1.15), (1.16).

Finally, let us count a number of arithmetical operations, necessary for realization of method (2.1)–(2.4). Let $K^{(l)}$, $l = 1, 2$ be a number of greed-points from subdomains $\bar{\Omega}^{(l)}$, and $K^{(1)} \ll K^{(2)}$. Let us suppose that for realization of implicit step (2.3) it is possible to use some economical algorithm, for which the number of arithmetical operations in one time-step is proportional to $K^{(2)}$. Then the total number of arithmetical operations is $K = O(N(sK^{(1)} + K^{(2)}))$. We have $K^{(2)} = c^{(2)}h^{-m}$ and let $K^{(1)} = c^{(1)}h^{-m+\alpha}$, $\alpha \in [0, 1]$. Then $\rho = h^\alpha$ and according to condition of Lemma 3.1 $s = O(h^{\alpha-1})$, and from the second condition (3.18) $N = O(h^{-1-\alpha})$. As a result we obtain

$$K = O\big(c^{(1)}h^{-m-2+\alpha} + c^{(2)}h^{-m-1-\alpha}\big).$$

This formula shows that computing costs for realization explicit and implicit steps have the same order when $\alpha = 1/2$.

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