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On energy estimates for preconditioning grid parabolic problems

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The paper deals with studying of some preconditioning operators providing unconditional convergence of difference schemes for solving parabolic problems. Three examples of such operators are considered. There are a preconditioner of the domain decomposition method of the Neumann–Dirichlet type and two preconditioners of the fictitious domain methods for the Neumann and the Dirichlet boundary value problems respectively.

1. Introduction

Regularization is the basis of a guarantee of unconditional stability of two-layer difference schemes [11, 12], i.e., there is considered the following family of schemes in the Euclidean space $\mathcal{E}_K$ of K dimension:

$$B_\tau \frac{\bar{u}^{n+1} - \bar{u}^n}{\tau} + A\bar{u}^n = \bar{f}^n, \quad n = 0, \ldots, N-1, \tag{1.1}$$

where A is a positive semi-definite matrix, $B_\tau = I + \sigma\tau B$ is a positive definite matrix, and I is an identity matrix, $\sigma > 0$. In this paper we consider only the symmetric matrices A and B. The matrix B plays the role of a regularizator. The standard *a priori* estimate, providing stability in the energy norm of any positive definite matrix D, contains $\|B\bar{v}\|_{D^{-1}}$ norm, where $D = B_\tau$ or $D = A$ [12, Theorems 6 or 7 on pp. 312, 313]. For splitting schemes the boundedness of this norm (for a small spatial step h) results in strong requirements on smoothness of the differential problem solution (the fourth spatial derivatives must exist). On the other hand, there are efficient procedures for inversion of the mesh operator of the Helmholtz equation in the rectangular region with a uniform grid such that their use does not require, generally, any additional smoothness in the generalized solution [3]. Many modern approaches for the design of iterative procedures with preconditioning operators of solution grid elliptic problems are based on the use of such a type of algorithms [13]. In this work, an attempt is undertaken for the transfer of a number of ideas, that are developed for the design of such preconditioning operator to construct schemes in the form of (1.1) with the efficient inversion of B_τ matrix.

As will be seen, the solution to this problem could be realized by the requirement of uniform in h energy equivalence of A and B operators:

$$c_1\langle A\bar{u},\bar{u}\rangle \le \langle B\bar{u},\bar{u}\rangle \le c_2\langle A\bar{u},\bar{u}\rangle \qquad \forall\,\bar{u}\in\mathcal{E}_K. \tag{1.2}$$

Here c_1 and c_2 are positive, independent of h numbers, with a possibility of effective solution to the system $B_\tau\bar{u}=\bar{g}$. However, we cannot succeed in designing a sapid example of B matrix, satisfying all these requirements simultaneously. On the other hand, it is sufficient to use essentially weaker than (1.2) conditions to provide *a priori* estimate, that will be presented in Section 3. Exactly, let us assume that there are the numbers $\alpha\ge 0$, $\beta>0$ and $\gamma>0$ independent of h, such that $\forall\,\bar{u}\in\mathcal{E}_K$

$$-\alpha\langle A\bar{u},\bar{u}\rangle \le \langle B\bar{u},\bar{u}\rangle \le \beta\langle A\bar{u},\bar{u}\rangle, \tag{1.3}$$

$$\gamma\langle A_\tau\bar{u},\bar{u}\rangle \le \langle B_\tau\bar{u},\bar{u}\rangle, \tag{1.4}$$

where $A_\tau=I+\sigma\tau A$. Let us note that we even do not assume nonnegativeness of B matrix. In Sections 4–6, some examples with such a matrix will be presented. The first example is the domain decomposition method with the adjoint conditions of the Neumann–Dirichlet type on the boundary between subdomains. Here we use the construction similar to [1]. The fictitious domain method [9] gives other examples. The Neumann problem, as a simple case, will be considered apart, and then we will study the Dirichlet problem with the symmetric extension of the matrix in "junior term" [7, 4]. The application of the fictitious domain method to nonstationary problems was considered in [5]. The error estimates in L_2-norm, corresponding to these examples, are presented in [6]. Here we give the error estimates in energy norm with the use of other *a priori* estimates. To prove all the statements we use the technique from [6] and do not present its here. For splitting schemes in general case such estimates are unknown.

Finally, a few words about the terminology. As the regularizator B replaces the condition number of the step operator of an explicit difference scheme, hence the analog of the elliptic case is called the preconditioning operator.

Before giving account of the main results, we will formulate the differential problem and present its approximation by the finite element method in the next section.

2. Problem of formulation and discretization

Let Ω be a bounded open polygonal region in R^2 with the boundary $\Gamma=\Gamma_0\cup\Gamma_1$. We will denote

$$H^1(\Omega,\Gamma_0)=\{u\in H^1(\Omega)\mid u(\bar{x})=0,\ \ \bar{x}\in\Gamma_0\},$$

where $H^1(\Omega)$ is the Sobolev space. In the space $H^1(\Omega)\times H^1(\Omega)$, let us consider the bilinear form

$$a_\Omega(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{2} a_{ij}(\bar{x}) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0(\bar{x}) uv \Big) \, d\bar{x},$$

where $\bar{x} = (x_1, x_2)$ is a point in R^2. For the functions $a_{ij}(\bar{x})$ and $a_0(\bar{x}) \geq 0$ we assume that the bilinear form $a_\Omega(u,v)$ is continuous and $H^1(\Omega, \Gamma_0)$-elliptic. Let us introduce a family of the functionals $(f(t), v)_\Omega$, that are uniformly continuous in $t \in [t_0, t_*]$ in the space $H^{-1}(\Omega) \times H^1(\Omega)$. Here the inner product $(\cdot,\cdot)_\Omega$ in $L_2(\Omega)$ is expanded to the duality relation on $H^{-1}(\Omega) \times H^1(\Omega)$ [12]. Further the notation $u(t)$ means that $u : [t_0, t_*] \to X$, where X is a Banach space, and $\frac{du}{dt}(t)$ is a strong limit in X (if such a limit exists) of the elements $u_\tau(t) \equiv \frac{1}{\tau}(u(t+\tau) - u(t))$ for $\tau \to 0$.

Let us formulate a parabolic problem to which we will apply the schemes with preconditioning operators. For $u_0 \in L_2(\Omega)$ and $f \in L_2(t_0, t_*; H^{-1}(\Omega))$ it is necessary to find the function $u \in L_2(t_0, t_*; H^1(\Omega, \Gamma_0))$ such that $\frac{du}{dt} \in L_2(t_0, t_*; H^{-1}(\Omega))$ and $\forall\, v \in H^1(\Omega, \Gamma_0)$ the following equalities are valid:

$$\Big(\frac{du}{dt}(t), v\Big)_\Omega + a_\Omega(u(t), v) = (f(t), v)_\Omega, \tag{2.1}$$

$$(u(t_0), v)_\Omega = (u_0, v)_\Omega. \tag{2.2}$$

For simplicity, further we will study the examples with $a_{ii}(\bar{x}) = 1$, $a_{ij}(\bar{x}) = 0$ at $i \neq j$, and $a_0(\bar{x}) = 1$, i.e., $a_\Omega(u,v)$ is the inner product in $H^1(\Omega)$.

Consider the finite element spatial discretization. Let $\mathcal{T}_h$ be the regular triangulation of Ω [2], and $\{\varphi_i(\bar{x})\}_{i=1}^K$ be the piecewise-linear basis on $\mathcal{T}_h$. Then $H_h^1(\Omega) = \mathrm{span}\{\varphi_i(\bar{x})\}_{i=1}^K$ and $H_h^1(\Omega, \Gamma_0) = H_h^1(\Omega) \cap H^1(\Omega, \Gamma_0)$. Then, using lumping operators, let us introduce the mesh inner product $d_\Omega(u,v)$ in $L_2(\Omega)$, that is equivalent to the usual inner product uniformly in h, approximates it and leads to a diagonal mass matrix. Particularly the positive numbers ν and μ exist such that the following inequalities are valid:

$$\nu \|u\|^2_{L_2(\Omega)} \leq d_\Omega(u,u) \leq \mu \|u\|^2_{L_2(\Omega)}. \tag{2.3}$$

Moreover, we will use the bilinear form $a_{\tau,\Omega}(u,v) = d_\Omega(u,v) + \sigma\tau a_\Omega(u,v)$, that is defined in the space $H_h^1(\Omega) \times H_h^1(\Omega)$.

Now let us introduce the vector-matrix notations. Let $\mathcal{E}_K$ be the Euclidean space of dimension K with the inner product $\langle\cdot,\cdot\rangle_K$ and the norm $\|\cdot\|_K$, where K is the number of vertices of $\mathcal{T}_h$ in $\overline{\Omega} \setminus \Gamma_0$. The conformity between $\mathcal{E}_K$ and the space $H_h^1(\Omega, \Gamma_0)$ is determined by the following relation for the components of the vector $\bar{u} \in \mathcal{E}_K$: $(\bar{u})_i = \rho_i u(\bar{x}_i)$, $i = 1, \ldots, K$, where $\rho_i = \sqrt{d_\Omega(\varphi_i, \varphi_i)}$. For the right-hand side of the equations we will use vectors with the components $(\bar{f})_i = \frac{1}{\rho_i}(f, \varphi_i)_\Omega$. Then let A be a symmetric matrix of order K with the elements $(A)_{ij} = \frac{1}{\rho_i \rho_j} a_\Omega(\varphi_i, \varphi_j)$. It is not difficult to see that the following relations between mesh inner products in $L_2(\Omega)$, $H_h^1(\Omega)$ and $\mathcal{E}_K$ are valid $\forall\, u, v \in H_h^1(\Omega, \Gamma_0)$:

$$d_\Omega(u,v) = \langle \bar{u}, \bar{v}\rangle_K, \quad a_\Omega(u,v) = \langle A\bar{u}, \bar{v}\rangle_K. \tag{2.4}$$

In particular, from (2.4) it follows that the identity matrix I of order K corresponds to the bilinear form $d_\Omega(\cdot,\cdot)$ in the subspace $H_h^1(\Omega,\Gamma_0)\times H_h^1(\Omega,\Gamma_0)$. As the initial value for scheme (1.1) we will use the vector $\bar{u}^0$ with the components $\rho_i u_0(\bar{x}_i)$, where u_0 is the function from (2.2).

3. A priori estimates

The explicit presentation of the error structure is the basis for the *a priori* estimatebs given below. Let $\{\bar{w}^n\}_{n=0}^N$ be a sequence of the vectors from $\mathcal{E}_K$, and in accordance with (1.1) the vectors $\bar{\xi}^n = \bar{u}^n - \bar{w}^n$ satisfy the equations

$$B_\tau \frac{\bar{\xi}^{n+1} - \bar{\xi}^n}{\tau} + A\bar{\xi}^n = \bar{g}^n, \quad n = 0,\ldots,N-1,$$

where $\bar{g}^n = \bar{z}^n - \sigma\tau B\bar{w}_\tau^n$ and $\bar{z}^n = \bar{f}^n - \bar{w}_\tau^n - A\bar{w}^n$, $\bar{w}_\tau^n = \frac{1}{\tau}(\bar{w}^{n+1} - \bar{w}^n)$, $\bar{f}^n$ corresponds to the function $f(t_n)$. The following statement is valid:

Lemma 1. *From conditions* (1.3), (1.4) *at* $\sigma \geq \frac{1}{\gamma}$ *and* $\tau \leq 1$ *for arbitrary* $n = 1,\ldots N$, *it follows:*

$$\|\bar{\xi}^n\|_A \leq c_0\Big\{\|\bar{\xi}^0\|_A^2 + \sigma\tau\sum_{k=0}^{n-1}\Big(\|\bar{z}^k\|^2 + (\sigma\tau)^2(\alpha^2+\beta^2)\|\bar{w}_{\tau\tau}^k\|_A^2\Big)\Big\}^{1/2}, \tag{3.1}$$

where the positive number c_0 *does not depend on the parameters* α, β, γ, *and* σ. *If* B *is a nonnegative matrix* ($\alpha = 0$), *then the following estimate is also valid:*

$$\|\bar{\xi}^n\|_A \leq c_0\Big\{\|\bar{\xi}^0\|_A^2 + \sigma\tau\sum_{k=0}^{n-1}\Big(\|\bar{z}^k\|^2 + (\sigma\tau)^2\beta^2\|\bar{w}_{\tau\tau}^k\|_B^2\Big)\Big\}^{1/2}. \tag{3.2}$$

In the following sections Lemma 1 will be used in different situations, and therefore the errors will be separately estimated for each example. Let us note that to formulate the method in the form of (1.1) we are in need of the exact value of the parameter γ, because the choice of σ depends on it.

4. Domain decomposition method

For simplicity we will consider in this section the Neumann problem ($\Gamma_0 = \emptyset$). Let Ω be a union of non-overlapping subdomains Ω_1 and Ω_2 such that the conditions

$$\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}, \qquad \Omega_1 \cap \Omega_2 = \emptyset$$

are valid. Other restrictions on Ω will be indicated later. Further we will use the following notations: $S = \overline{\Omega}_1 \cap \overline{\Omega}_2$, $\Omega_{k0} = \overline{\Omega}_k \setminus S$, $k = 1, 2$. Let all the nodes of the grid be partitioned to three groups: the first group contains nodes lying in the set Ω_{10}, the second – nodes from S, and the third one contains nodes from Ω_{20}. In accordance with this order, the space $\mathcal{E} = \mathcal{E}_K$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$ may be presented as $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_0 \times \mathcal{E}_2$, where $\mathcal{E}_i$, $i = 0, 1, 2$, are the Euclidean spaces of orders K_i, $K_0 + K_1 + K_2 = K$ with the inner products $\langle \cdot, \cdot \rangle_i$ and the norms $\| \cdot \|_i$. Then A matrix has the block form

$$A = \begin{pmatrix} A_{11} & A_{10} & 0 \\ A_{10}^T & A_{00} & A_{02} \\ 0 & A_{02}^T & A_{22} \end{pmatrix}.$$

In the spaces $\mathcal{E}^{(1)} = \mathcal{E}_1 \times \mathcal{E}_0$ and $\mathcal{E}^{(2)} = \mathcal{E}_0 \times \mathcal{E}_2$ with the inner products $\langle \cdot, \cdot \rangle_{(k)}$ and the norms $\| \cdot \|_{(k)}$, $k = 1, 2$, respectively, the matrices

$$A^{(1)} = \begin{pmatrix} A_{11} & A_{10} \\ A_{10}^T & A_{00}^{(1)} \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} A_{00}^{(2)} & A_{02} \\ A_{02}^T & A_{22} \end{pmatrix},$$

where $A_{00}^{(1)} + A_{00}^{(2)} = A_{00}$, correspond to $a_k(u, v) = a_{\Omega_k}(u, v)$, $k = 1, 2$.

Let $\omega_2 = \Omega_2 \cap (\cup_{\bar{x}_i \in S} \operatorname{supp} \mu_i)$ (see Section 2) and $d_1(u, v) = d_{\Omega_1}(u, v) + d_{\omega_2}(u, v)$, $d_2(u, v) = d_{\Omega_2}(u, v) - d_{\omega_2}(u, v)$. From (2.3) the inequalities

$$d_1(u, u) \geq \nu \|u\|^2_{L_2(\Omega_1)}, \quad d_2(u, u) \leq \mu \|u\|^2_{L_2(\Omega_2)} \tag{4.1}$$

follow. Then in the spaces $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ the matrices

$$D^{(1)} = \begin{pmatrix} I_1 & 0 \\ 0 & I_0 \end{pmatrix}, \quad D^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}$$

correspond to the bilinear forms $d_k(u, v)$, $k = 1, 2$, where I_i, $i = 0, 1, 2$, are identity matrices of orders K_i. And finally, the matrices $A_\tau^{(k)} = D^{(k)} + \sigma\tau A^{(k)}$ correspond to the bilinear forms $a_{\tau,k}(u, v) = a_{\tau,\Omega_k}(u, v)$.

Now we consider the approach from [1], but using the bilinear form $a_{\tau,2}(u, v)$ instead of $a_2(u, v)$. Let us find for all $u \in H_h^1(\Omega_2)$ the expansion $u = u_H + u_P$ such that $u_P \in H_h^1(\Omega_2, S)$ and the equality

$$a_{\tau,2}(u_H, v) = 0 \quad \forall\, v \in H_h^1(\Omega_2, S) \tag{4.2}$$

holds. It is not difficult to show that for a function $u \in H_h^1(\Omega_2)$ and corresponding vector $\bar{u}^{(2)} = (\bar{u}_0^T, \bar{u}_2^T)^T \in \mathcal{E}^{(2)}$, the vector presentation of this expansion is defined by the equalities

$$\bar{u}_{P,0} = 0, \quad \bar{u}_{P,2} = \left(\frac{1}{\sigma\tau} I_2 + A_{22}\right)^{-1} A_{02}^T \bar{u}_0 + \bar{u}_2.$$

Then $\bar{u}_H^{(2)} = \bar{u}^{(2)} - \bar{u}_P^{(2)}$ and (4.2) may be rewritten in the form

$$\langle A_\tau^{(2)} \bar{u}_H^{(2)}, \bar{v}^{(2)} \rangle_{(2)} = 0, \quad \forall\, \bar{v}^{(2)} \in \mathcal{E}^{(2)}, \ \bar{v}_0^{(2)} = 0.$$

Let us consider in $H_h^1(\Omega) \times H_h^1(\Omega)$ space the bilinear form

$$b_\tau(u, v) = a_{\tau,1}(u, v) + a_{\tau,2}(u_P, v_P).$$

To this form the matrix $B_\tau = I + \sigma\tau B$ of order K, where

$$B = \begin{pmatrix} A_{11} & A_{10} & 0 \\ A_{10}^T & A_{00}^{(1)} + A_{02}(\frac{1}{\sigma\tau} I_2 + A_{22})^{-1} A_{02}^T & A_{02} \\ 0 & A_{02}^T & A_{22} \end{pmatrix}, \tag{4.3}$$

corresponds. Let

$$Q_\tau = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_0 & 0 \\ 0 & (\frac{1}{\sigma\tau} I_2 + A_{22})^{-1} A_{02}^T & I_2 \end{pmatrix}.$$

Then $B_\tau = Q_\tau^T D_\tau Q_\tau$, where $D_\tau = I + \sigma\tau D$,

$$D = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A_{22} \end{pmatrix}.$$

The indicated presentation means that an inversion of B_τ is reduced to one inversion of $A_\tau^{(1)}$ (solution to the Neumann problem in Ω_1) and two inversions of $I_2 + \sigma\tau A_{22}$ (solutions to the Dirichlet problem in Ω_2). These inversions may be realized efficiently [3] for the following structure of Ω, for example. Let

$$\Omega^{(i)} = \{(x_1, x_2) \mid x^{(i-1)} < x_1 < x^{(i)}, \ y^{(i)} < x_2 < z^{(i)}\}, \ i = 1, \dots, 2m+1,$$

and

$$\Omega_1 = \bigcup_{j=0}^{m} \Omega^{(2j+1)}, \quad \Omega_2 = \bigcup_{j=1}^{m} \Omega^{(2j)}.$$

Moreover, let $z^{(1)} - y^{(1)} \geq z^{(2)} - y^{(2)}$, $z^{(2j)} - y^{(2j)} \leq z^{(2j+1)} - y^{(2j+1)}$, $j = 1, \dots m$. Then we have the Neumann problems for odd rectangles and the mixed problems with the Neumann conditions on the horizontal sides, and the Dirichlet conditions on the vertical sides for even rectangles. It is assumed that the step size in variable x_2 is h_2 in the whole domain Ω (a restriction for the numbers $z^{(i)}$ and $y^{(i)}$). Then let the step sizes in variable x_1 in the subdomains $\Omega^{(i)}$ be $h_1^{(i)} = (x^{(i)} - x^{(i-1)})/p_i$, where p_i are integer numbers.

Let us find the parameters α, β, and γ from conditions (1.3), (1.4). We use the following notation:

$$\Lambda_{00}^{(2)} = A_{00}^{(2)} - A_{02}\left(\frac{1}{\sigma\tau}I_2 + A_{22}\right)^{-1} A_{02}^T,$$

where $\Lambda_{00}^{(2)}$ is a positive semi-definite matrix of order K_0. Then $B = A - C$, where

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_{00}^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a positive semi-definite matrix of order K. From nonnegativeness of C matrix the right condition of (1.3) follows when $\beta = 1$. Then let us find α such that the left condition of (1.3) holds. It means that $\langle C\bar{u}, \bar{u}\rangle \le (1+\alpha)\langle A\bar{u}, \bar{u}\rangle$. For the Rayleigh relation we have the inequality

$$R(\bar{u}) \equiv \frac{\langle C\bar{u}, \bar{u}\rangle}{\langle A\bar{u}, \bar{u}\rangle} \le \frac{\langle \Lambda_{00}^{(2)}\bar{u}_0, \bar{u}_0\rangle_0}{\langle A^{(1)}\bar{u}^{(1)}, \bar{u}^{(1)}\rangle_{(1)}}, \tag{4.4}$$

where

$$\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_0 \\ \bar{u}_2 \end{bmatrix} \in \mathcal{E}, \quad \bar{u}_0 \in \mathcal{E}_0, \quad \bar{u}^{(1)} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_0 \end{bmatrix} \in \mathcal{E}^{(1)}.$$

As it is known, the inequality $\langle \Lambda_{00}^{(2)}\bar{u}_0, \bar{u}_0\rangle_0 \le \frac{1}{\sigma\tau}\langle A_\tau^{(2)}\bar{v}^{(2)}, \bar{v}^{(2)}\rangle_{(2)}$ is valid $\forall\, \bar{v}^{(2)} \in \mathcal{E}^{(2)}$ such that $\bar{v}_0 = \bar{u}_0$ (see, for example, Lemma 1 from [10]). In accordance with (4.1)

$$\langle A_\tau^{(2)}\bar{v}^{(2)}, \bar{v}^{(2)}\rangle_{(2)} \le \mu\|v\|^2_{L_2(\Omega_2)} + \sigma\tau\|v\|^2_{H^1(\Omega_2)}$$

(μ is a number from (4.1)). Then the estimate for the Rayleigh relation

$$R(\bar{u}) \le \frac{\mu\|v\|^2_{L_2(\Omega_2)} + \sigma\tau\|v\|^2_{H^1(\Omega_2)}}{\sigma\tau\|u\|^2_{H^1(\Omega_1)}}, \quad v(\bar{x}) = u(\bar{x}), \;\; \bar{x} \in S, \tag{4.5}$$

holds. Thus, there arises a standard problem of extension of piecewise-linear functions from Ω_1 onto Ω_2 with the minimal norm. The following statement is valid:

Lemma 2. *There exists a positive number c independent of h and τ such that $\forall\, \bar{u} \in \mathcal{E}$ at $\sigma\tau \le \delta_0^2$, where $\delta_0 = \frac{1}{2}\min_{i=1,\dots,2m+1}(x^{(i)} - x^{(i-1)})$, the estimate $R(\bar{u}) \le c/\sqrt{\sigma\tau}$ holds.*

Corollary 1. *From Lemma 2 and presentation (4.4) it follows that $\alpha = \alpha_0/\sqrt{\sigma\tau}$, where the number α_0 does not depend on h, τ, and σ.*

Finally, let us find γ by estimating the Rayleigh relation

$$R_\tau(\bar{u}) \equiv \frac{\langle A_\tau \bar{u}, \bar{u}\rangle}{\langle B_\tau \bar{u}, \bar{u}\rangle}.$$

Let λ_0 be the maximal eigenvalue, and $\bar{w}$ be the corresponding eigenvector for the problem

$$A_\tau \bar{v} = \lambda B_\tau \bar{v}. \tag{4.6}$$

Then $R_\tau(\bar{u}) \le \lambda_0$, and the trivial case $\lambda_0 \le 1$ immediately leads to the estimate $\gamma \ge 1$. Let $\lambda_0 > 1$. Since $B = A - C$, then $B_\tau = A_\tau - \sigma\tau C$, and problem (4.6) may be rewritten in the form

$$(\lambda - 1)A_\tau \bar{v} = \lambda\sigma\tau C\bar{v}.$$

It is not difficult to show that from the condition $\lambda_0 > 1$ the equality $\bar{w} = S_\tau \bar{w}^{(1)}$, where

$$S_\tau = \begin{pmatrix} I_1 & 0 \\ 0 & I_0 \\ 0 & -(\frac{1}{\sigma\tau}I_2 + A_{22})^{-1}A_{02}^T \end{pmatrix},$$

follows. Taking into account $S_\tau^T A_\tau S_\tau = A_\tau^{(1)} + \sigma\tau S_\tau^T C S_\tau$, we will obtain $(\lambda_0 - 1)A_\tau^{(1)}\bar{w}^{(1)} = \sigma\tau S_\tau^T C S_\tau \bar{w}^{(1)}$. As $\langle S_\tau^T C S_\tau \bar{w}^{(1)}, \bar{w}^{(1)}\rangle_{(1)} = \langle \Lambda_{00}^{(2)}\bar{w}_0, \bar{w}_0\rangle_0$, the inequality

$$\langle S_\tau^T C S_\tau \bar{w}^{(1)}, \bar{w}^{(1)}\rangle_{(1)} \le \frac{1}{\sigma\tau}\langle A_\tau^{(2)}\bar{v}^{(2)}, \bar{v}^{(2)}\rangle_{(2)} \quad \forall\, \bar{v}^{(2)} \in \mathcal{E}^{(2)},\ \bar{v}_0 = \bar{w}_0$$

holds and the following estimate of the Rayleigh relation is valid:

$$R_\tau(\bar{u}) \le \lambda_0 \le 1 + \frac{\langle A_\tau^{(2)}\bar{v}^{(2)}, \bar{v}^{(2)}\rangle}{\langle A_\tau^{(1)}\bar{w}^{(1)}, \bar{w}^{(1)}\rangle} \qquad \forall\, \bar{u} \in \mathcal{E},\ \forall\, \bar{v}^{(2)} \in \mathcal{E}^{(2)},\ \bar{v}_0 = \bar{w}_0.$$

Analogously to (4.5) and in accordance with (4.1) from this inequality follows the estimate

$$R_\tau(\bar{u}) \le 1 + \frac{\mu\|v\|^2_{L_2(\Omega_2)} + \sigma\tau\|v\|^2_{H^1(\Omega_2)}}{\nu\|w\|^2_{L_2(\Omega_1)} + \sigma\tau\|w\|^2_{H^1(\Omega_1)}}, \quad v(\bar{x}) = w(\bar{x}),\ \bar{x} \in S, \tag{4.7}$$

where w is the function in $H_h^1(\Omega)$ corresponding to the vector $\bar{w}$. A solution to the extension problem in this case gives

Lemma 3. *Let $\sigma\tau \leq \frac{1}{2}\mu\delta_0^2$, where δ_0 is the number from Lemma 2. Then the following estimate is valid $\forall\, \bar{u} \in \mathcal{E}$:*

$$R_\tau(\bar{u}) \leq 1 + 2\frac{\mu}{\nu},$$

where μ and ν are the numbers from inequalities (4.1).

Corollary 2. *From Lemma 3 and* (4.7) *it follows that $\gamma = (1 + 2\mu/\nu)^{-1}$.*

Remark 1. For the mesh described above it is not difficult to find the numbers μ and ν.

Thus, in accordance with Corollaries 1 and 2 the parameters from conditions (1.3), (1.4) are as follows:

$$\alpha = \frac{\alpha_0}{\sqrt{\sigma\tau}}, \quad \beta = 1, \quad \gamma = \left(1 + 2\frac{\mu}{\nu}\right)^{-1}.$$

Now we consider the convergence of scheme (1.1) with B matrix from (4.3). Before we formulate the theorem let us indicate the smoothness conditions for problem (2.1), (2.2):

$$u \in H^1(t_0, t_*; H^2(\Omega)), \quad \frac{d^2u}{dt^2} \in L_2(t_0, t_*; H^1(\Omega)). \tag{4.8}$$

Now with the use of (3.1) from Lemma 1 may be proved the following

Theorem 1. *Let conditions* (4.8) *hold for problem* (2.1), (2.2). *Then the numbers c and τ_0 independent of h, τ and $u(t)$ exist such that for the solution to problem* (1.1), (4.3) *at $\sigma \geq 1 + 2\frac{\mu}{\nu}$ and $\tau \leq \tau_0$ the following error estimate is valid:*

$$\max_{n=1,\dots,N} \|u^n - u(t_n)\|_{H^1(\Omega)} \leq c(M_h h + M_\tau \sqrt{\tau}), \tag{4.9}$$

where

$$M_h = \|u\|_{H^1(t_0,t_*;H^2(\Omega))}, \quad M_\tau = M_h + \left\|\frac{d^2u}{dt^2}\right\|_{L_2(t_0,t_*;H^1(\Omega))}.$$

Remark 2. Let us note that the structure of Ω does not allow one to use the Nitsche method [2] for obtaining estimates in L_2-norm, and we applied the finite element estimates only in energy norms.

5. Fictitious domain method (Neumann problem)

Let G be a rectangle and $\Omega \subset G$. Introduce the uniform rectangular mesh $\mathcal{T}_{h,G}$ in G such that $\mathcal{T}_h \subset \mathcal{T}_{h,G}$. Let us consider the scheme

$$B_{G,\tau}\frac{\bar{U}^{n+1}-\bar{U}^n}{\tau}+A_G\bar{U}^n=\bar{F}^n, \quad n=0,\dots,N-1, \tag{5.1}$$

where $\bar{U}^n, \bar{F}^n \in \mathcal{E}_G$. The Euclidean space $\mathcal{E}_G$ of K_G dimension corresponds to the space $H_h^1(G)$ and may be presented as $\mathcal{E}_G = \mathcal{E} \times \mathcal{E}_2$, where $\mathcal{E}$ and $\mathcal{E}_2$ have orders K and K_2 respectively. The presentation $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_0$ corresponds to K_1 vertices from Ω and K_0 vertices from Γ. Then $\bar{U} \in \mathcal{E}_G$ vectors have the form $\bar{U} = (\bar{u}^T, \bar{u}_2^T)^T$, where $\bar{u} = (\bar{u}_1^T, \bar{u}_0^T)^T$. In (5.1) we assume that $\bar{f}_2^n = 0$ and the matrix A_G have only one non-zero block A of order K, corresponding to the bilinear form $a_\Omega(u,v)$. And finally, $B_{G,\tau} = I_G + \sigma\tau B_G$, where B_G matrix corresponds to the inner product in $H^1(G)$, and I_G is the identity matrix of order K_G. Let us remind that we consider the bilinear form $a_\Omega(u,v)$ coinciding with the inner product in $H^1(\Omega)$. It means that

$$A_G = \begin{pmatrix} A_{11} & A_{10} & 0 \\ A_{10}^T & A_{00} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_G = \begin{pmatrix} A_{11} & A_{10} & 0 \\ A_{10}^T & A_{00} + A_{00}^{(2)} & A_{02} \\ 0 & A_{02}^T & A_{22} \end{pmatrix}.$$

Remark 3. The conformity between $\mathcal{E}_G$ and $H_h^1(G)$ is established with the use of the values $\rho_i = \sqrt{d_G(\varphi_i, \varphi_i)}$ (see Section 2).

It is not difficult to note that scheme (5.1) is equivalent to the equation in $\mathcal{E}$:

$$B_\tau \frac{\bar{u}^{n+1}-\bar{u}^n}{\tau} + A\bar{u}^n = \bar{f}^n, \quad n = 0,\dots,N-1,$$

and

$$\bar{u}_2^{n+1} = \bar{u}_2^n - \left(\frac{1}{\sigma\tau}I_2 + A_{22}\right)^{-1} A_{02}^T(\bar{u}^{n+1} - \bar{u}^n),$$

where $B_\tau = I + \sigma\tau B$ is the matrix of order K,

$$B = A + \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_{00}^{(2)} \end{pmatrix}. \tag{5.2}$$

The positive semi-definite matrix $\Lambda_{00}^{(2)}$ is defined in the previous section. In accordance with (5.2) the left inequality from (1.3) and (1.4) is immediate at $\alpha = 0$ and $\gamma = 1$. We find the value of β from the estimate of the Rayleigh relation

$$R(\bar{u}) \equiv \frac{\langle B\bar{u}, \bar{u}\rangle}{\langle A\bar{u}, \bar{u}\rangle} = 1 + \frac{\langle \Lambda_{00}^{(2)}\bar{u}_0, \bar{u}_0\rangle_0}{\langle A\bar{u}, \bar{u}\rangle}. \tag{5.3}$$

Comparing (5.3) with (4.4), we see that the estimate of $R(\bar{u})$ follows from the proof of Lemma 2 with obvious modifications. Thus, we have

$$\alpha = 0, \quad \beta = \frac{\beta_0}{\sqrt{\sigma\tau}}, \quad \gamma = 1,$$

where β_0 does not depend on h and τ. Now we may formulate the convergence theorem.

Theorem 2. *Let the conditions of Theorem 1 hold. Then estimate* (4.9) *is valid at* $\sigma \geq 1$.

Remark 4. If $\rho = \mathrm{dist}(\Gamma, \partial G) > 0$ and ρ does not depend on h and τ, then the extension onto $H_h^1(G, \partial G)$ (the Dirichlet problem in G) may be considered.

6. Fictitious domain method (Dirichlet problem)

In this section we will use the extension according to [7]. Let us consider scheme (5.1), assuming $B = B_G$ and $A = B_G + \frac{1}{\varepsilon}D_G$, $\varepsilon > 0$, where the matrix D_G is defined by the equality

$$\langle D_G\bar{u}, \bar{v}\rangle_G = (u, v)_{G\setminus\Omega} \quad \forall\, \bar{u}, \bar{v} \in \mathcal{E}_G.$$

Here $\langle\cdot,\cdot\rangle_G$ is the inner product in $\mathcal{E}_G$. Then we denote $A_\tau = I_G + \sigma\tau A$, $B_\tau = I_G + \sigma\tau B$. In this section we assume that $\mathcal{E}_G$ corresponds to $H_h^1(G, \partial G)$, and B matrix corresponds to the Dirichlet problem. From positive definiteness of B and positive semi-definiteness of D_G inequalities (1.3) immediately follow at $\alpha = 0$ and $\beta = 1$. Then from the inequality $d_G(u, u) \geq \nu\|u\|^2_{L_2(G\setminus\Omega)}$ (see (2.3)) it follows

$$\langle A_\tau\bar{u}, \bar{u}\rangle_G \leq (1 + \delta)\langle B_\tau\bar{u}, \bar{u}\rangle_G,$$

where $\delta = \frac{\sigma\tau}{\nu\varepsilon}$ is any given number. The latter inequality means that $\gamma = (1 + \delta)^{-1}$ and $\varepsilon = \frac{\sigma}{\nu\delta}\tau$. Thus, inequalities (1.3), (1.4) are valid at

$$\alpha = 0, \quad \beta = 1, \quad \gamma = (1 + \delta)^{-1}.$$

Let us note that, as distinguished from previous examples in the considered case, all parameters α, β, and γ do not depend on τ. But, as will be seen, the error estimate is the same as in Theorems 1 and 2.

For obtaining the error estimate we need some auxiliary constructions. In $H^1(G) \times H^1(G)$ space, let us introduce a one-parameter family of the bilinear forms

$$a_G^\varepsilon(u,v) = a_G(u,v) + \frac{1}{\varepsilon}(u,v)_{G\setminus\Omega}. \tag{6.1}$$

We need the auxiliary generalized elliptic problem to find $u_\varepsilon \in H^1(G,\partial G)$ such that

$$a_G^\varepsilon(u_\varepsilon, v) = (g,v)_G \quad \forall\, v \in H^1(G,\partial G),$$

where $g \equiv 0$ in $G \setminus \Omega$. Then let $u_{\varepsilon,h}$ be the Ritz projection of the function u_ε on the subspace $H_h^1(G,\partial G)$ corresponding to the bilinear form (6.1). Moreover, let $\tilde{u}$ be the solution to the problem

$$a_\Omega(\tilde{u}, v) = (g,v)_\Omega \quad \forall\, v \in H^1(\Omega,\Gamma),$$

extended onto the subdomain $G\setminus\Omega$ by zero. Then $\tilde{u}_h$ is the Ritz projection of the function $\tilde{u}$ on the subspace $H_h^1(\Omega,\Gamma)$ respectively to the inner product in $H^1(\Omega)$, and $\tilde{u}_h \equiv 0$ in $G \setminus \Omega$. We use the notations $\varphi_\varepsilon = u_\varepsilon - \tilde{u}$ and $\varphi_{\varepsilon,h} = u_{\varepsilon,h} - \tilde{u}_h$. Let us formulate the following statement.

Lemma 4. *Let $\tilde{u} \in H^2(\Omega)$. Then the following estimate is valid:*

$$\|\varphi_{\varepsilon,h}\|_{L_2(G)}^2 \le c\sqrt{\varepsilon}(\sqrt{\varepsilon} + h^2)\|\tilde{u}\|_{H^2(\Omega)}^2,$$

where the positive number c does not depend on ε and h.

With the use of this lemma and inequality (3.2) of Lemma 1 we may prove the convergence theorem.

Theorem 3. *Let the conditions of Theorem 1 hold. Then the following estimate is valid at $\sigma \ge 1 + \delta,\ \delta > 0$:*

$$\max_{n=1,\dots,N} \|u^n - u(t_n)\|_{H^1(\Omega)} \le c(M_h h + M_\tau\sqrt{\tau}),$$

where the numbers M_h and M_τ are defined in Theorem 1.

References

[1] J.H. Bramble, J.E. Pasciak, A.H. Schatz, *An iterative method for elliptic problems on regions partitioned into substructures*, Math. of Comp., **46**, 1986, 361–369.

[2] Ph. Ciarlet, *The Finite Element Method for Elliptic Problems*, North–Holland, 1978.

[3] I.E. Kaporin, *A modified marching algorithm for solution to the Dirichlet difference problem for Poisson's equation in the rectangle*, Raznostnye Metody Mat. Fiz., Eds. Yu.P. Popov and E.S. Nikolaev, Moscow, 1980, 11–21 (in Russian).

[4] A.N. Konovalov, *On one variant of the fictitious domain method*, Some Problems of Numerical and Applied Mathematics, Nauka, Novosibirsk, 1975, 191–199 (in Russian).

[5] A.N. Konovalov, *Fictitious domain method in two-phase incompressible fluid filtration problems with taking into account capillary forces*, Numerical Methods in Continuum Mechanics, **3**, No. 5, 1972, 52–67 (in Russian).

[6] Yu.M. Laevsky, *Preconditioning operators for grid parabolic problems*, Rus. J. of Numer. Anal. and Math. Modell., **11**, 1996, 497–515.

[7] V.I. Lebedev, *Difference analogies of orthogonal expansions of the principal differential operators and of some boundary value problems of mathematical physics*, Zhurn. Vychisl. Matem. and Matem. Fiz., **4**, 1964, 449–465 (in Russian).

[8] J.-L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Application*, Dunod, Paris, 1968.

[9] G.I. Marchuk, *Methods of Numerical Mathematics*, Springer, New York, 1982.

[10] A.M. Matsokin and S.V. Nepomnyaschikh, *On using the bordering method for solving systems of mesh equations*, Sov. J. of Numer. Anal. and Math. Modell., **4**, 1989, 488–492.

[11] A.A. Samarsky, *On a regularization of difference schemes*, Zhurn. Vychisl. Matem. and Matem. Fiz., **7**, 1967, 62–93 (in Russian)

[12] A.A. Samarsky, *Introduction into the Theory of Difference Schemes*, Nauka, Moscow, 1971 (in Russian).

[13] A.A. Samarsky and E.S. Nikolaev, *Methods of Solution of Grid Equations*, Nauka, Moscow, 1978 (in Russian).