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On one modification of Marchuk–Kuzin's scheme for parabolic equations with mixed derivatives

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In the article, we propose and study one modification of the finite element splitting algorithm for the solution to the Neumann boundary parabolic problem with mixed derivatives [1]. We consider the simplest equations with constant coefficients without advective terms. For the Neumann problem, the error estimate of the method from [1] contains the term $O(\tau/\sqrt{h})$, and the numerical experiments show that this estimate cannot be improved. We give the modification with the estimate $O(\tau+h)$. The approach is based on our results for the parabolic problems in nonrectangular domains [2, 3].

1. Description of the problem and discretization

Let $\Omega \subset R^2$ be a rectangle with the sides parallel to coordinate axis, Γ be the boundary of Ω. In the space $H^1(\Omega) \times H^1(\Omega)$, let us consider the bilinear form

$$a_0(u,v) = \int_\Omega \left(\Lambda \nabla u, \nabla v\right) d\bar{x}, \tag{1.1}$$

where Λ is the matrix of coefficients λ_{ij}, $i,j = 1,2$, such that the bilinear form $a_0(u,v)$ is continuos and $H^1(\Omega)$-elliptic. Bellow we consider the symmetric $a_0(u,v)$ with $\lambda_{12} = \lambda_{21}$. Moreover, let us define the bilinear form

$$m(\varphi,\psi) = \int_\Gamma \sigma(\bar{x})\varphi\psi\, ds,$$

where φ, $\psi \in L_2(\Gamma)$, $\sigma(\bar{x})$ is a piecewise continuos nonnegative function defined on Γ. Then in the space $H^1(\Omega) \times H^1(\Omega)$ the family of the continuos, $H^1(\Omega)$-elliptic, symmetric bilinear forms

$$a(u,v) = a_0(u,v) + m(\gamma u, \gamma v), \tag{1.2}$$

is defined, where γ is the trace operator onto Γ. And finally, we introduce the bilinear form $(f(t),v)$.

Let us formulate a parabolic problem to which we apply the splitting method. For $u_0 \in L_2(\Omega)$ and $f \in L_2(t_0,t_*;L_2(\Omega))$ it is necessary to find the function $u \in L_2(t_0, t_*; H^1(\Omega, \Gamma))$, such that $\frac{du}{dt} \in L_2(t_0, t_*; L_2(\Omega))$, and $\forall v \in H^1(\Omega, \Gamma_0)$, and $t \in (t_0, t_*)$ the following equalities are valid:

$$\begin{aligned} &\Big(\frac{du}{dt}(t), v\Big) + a(u(t), v) = (f(t), v), \\ &(u(t_0), v) = (u_0, v). \end{aligned} \tag{1.3}$$

Consider the finite element discretization. Let $\mathcal{T}_h$ be the regular triangulation that is based on the uniform rectangular mesh in Ω with the steps h_1 and h_2 corresponding to the variables x_1 and x_2. Every rectangular box is triangulated by a diagonal with the positive direction (from low-left to up-right) if $\lambda_{12} \geq 0$, and with the negative one (from up-left to low-right) if $\lambda_{12} < 0$. Let α be the angle between positive direction of the axis OX_1 and direction l of triangulating diagonals. Then

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x_1} \cos\alpha + \frac{\partial u}{\partial x_2} \sin\alpha, \tag{1.4}$$

where

$$\cos\alpha = \operatorname{sign}(\lambda_{12}) \frac{h_1}{\sqrt{h_1^2 + h_2^2}}, \quad \sin\alpha = \frac{h_2}{\sqrt{h_1^2 + h_2^2}}. \tag{1.5}$$

Let us denote $\eta = h_1/h_2$. In the space $H^1(\Omega) \times H^1(\Omega)$ define the bilinear forms

$$\begin{aligned} a_0^{(k)}(u, v) &= \int_\Omega \lambda_k \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k}\, d\bar{x}, \quad k = 1, 2, \\ a_0^{(3)}(u, v) &= \int_\Omega \lambda_3 \frac{\partial u}{\partial l} \frac{\partial v}{\partial l}\, d\bar{x}, \end{aligned} \tag{1.6}$$

where the functions λ_k $(k = 1, 2, 3)$ are given according to the equalities

$$\begin{aligned} \lambda_1 &= \lambda_{11} - \eta|\lambda_{12}|, \\ \lambda_2 &= \lambda_{22} - \eta^{-1}|\lambda_{12}|, \\ \lambda_3 &= \eta \cos^{-2}\alpha\, |\lambda_{12}|. \end{aligned} \tag{1.7}$$

The following additive presentation is valid:

$$a_0(u, v) = \sum_{k=1}^{3} a_0^{(k)}(u, v). \tag{1.8}$$

This equality follows from (1.6), (1.7). Let us reqire the nonnegativeness in $H^1(\Omega) \times H^1(\Omega)$ of the bilinear forms (1.6). For this aim it is sufficient that $\lambda_k \geq 0$, $k = 1, 2, 3$, or in accordance with (1.7)

$$\lambda_{11} \geq \eta|\lambda_{12}|, \quad \lambda_{22} \geq \eta^{-1}|\lambda_{12}|. \tag{1.9}$$

Let $\theta(\bar{x})$ be the angle between the vector $\bar{x}_1$ and the external normal to Γ. Let us define the bilinear forms

$$m_k(\varphi, \psi) = \int_\Gamma \sigma_k(\bar{x})\varphi\psi\, ds, \quad k = 1,2,3, \tag{1.10}$$

where $\varphi,\ \psi \in L_2(\Gamma)$,

$$\begin{aligned} \sigma_1 &= \frac{\lambda_1 \cos^2\theta}{\omega}\sigma, \qquad \sigma_2 = \frac{\lambda_2 \sin^2\theta}{\omega}\sigma, \\ \sigma_3 &= \frac{\lambda_3(\cos^2\alpha\cos^2\theta + \sin^2\alpha\sin^2\theta)}{\omega}\sigma, \end{aligned} \tag{1.11}$$

$$\omega = \cos^2\theta(\lambda_1 + \lambda_3\cos^2\alpha) + \sin^2\theta(\lambda_2 + \lambda_3\sin^2\alpha). \tag{1.12}$$

Then in the space $H_1(\Omega) \times H_1(\Omega)$ the bilinear forms

$$a_k(u,v) = a_0^{(k)}(u,v) + m_k(\gamma u, \gamma v), \quad k = 1,2,3, \tag{1.13}$$

are defined.

And finally, in the space $H_1(\Gamma) \times L_2(\Gamma)$ let us introduce the bilinear forms

$$\begin{aligned} b_k(\varphi,\psi) &= \int_\Gamma \nu_k \frac{\partial u}{\partial s} v\, ds, \quad k = 1,2, \\ b(\varphi,\psi) &= b_1(\varphi,\psi) + b_2(\varphi,\psi), \end{aligned} \tag{1.14}$$

where

$$\nu_1 = \frac{\lambda_{12}\lambda_1\cos^2\theta}{\omega}, \quad \nu_2 = \frac{\lambda_{12}\lambda_2\sin^2\theta}{\omega}. \tag{1.15}$$

As it is easy to see the following additive presentation is valid:

$$a(u,v) = \sum_{k=1}^{3} a_k(u,v),$$

and this one is the basis for splitting method formulation. Note that the set of numbers I of all verticies of the triangulation $\mathcal{T}_h$ may be ordered such that $\forall i \in I$, $s \in Z$, and $i + s \in I$ at $|s| > 1$ the equality

$$\frac{\partial\varphi_i}{\partial l}\frac{\partial\varphi_{i+s}}{\partial l} = 0, \quad \bar{x} \in \Omega, \tag{1.16}$$

holds. Here $\{\varphi_i(\bar{x})\}_{i\in I}$ is the system of piecewise linear functions corresponding to the set $\mathcal{T}_h$, $\varphi_i(\bar{x}_j) = \delta_{ij}$, $i,j \in I$.

Let us discuss the third boundary condition on Γ. Under sufficient smoothness of the solution to the problem (1.3) the following equality is valid:

$$\frac{\partial u}{\partial N} + \sigma(\bar{x})u = 0, \quad \bar{x} \in \Gamma, \tag{1.17}$$

where $\frac{\partial u}{\partial N} = \sum_{i,j=1}^{2} \lambda_{ij} \frac{\partial u}{\partial x_i} \cos(\bar{x}_j, \bar{n})$, $\bar{x}_j$ is the unit vector of j-th coordinate axis, and $\bar{n}(\bar{x})$ is the unit vector of the external normal to Γ. The expression for the co-normal derivative may be rewritten with respect to $\theta(\bar{x})$ in the following form:

$$\frac{\partial u}{\partial N} = \sum_{i=1}^{2} \lambda_{i1} \frac{\partial u}{\partial x_i} \cos\theta + \sum_{i=1}^{2} \lambda_{i2} \frac{\partial u}{\partial x_i} \sin\theta. \tag{1.18}$$

Let Γ_p, $p = \overline{1,4}$, be the sides of $\overline{\Omega}$, and $\Gamma_1, \Gamma_3 \parallel OX_2$. Let us suppose that a direction of the sides Γ_1, Γ_4 coincide with the directions of corresponding axis (Γ_2, Γ_3 have opposite directions to Γ_4, Γ_1). Let $\bar{x}'_p = (x'_{1p}, x'_{2p})$, $\bar{x}''_p = (x''_{1p}, x''_{2p})$ be the begining and the end of the side Γ_p. Then the domain Ω may be presented in the following form:

$$\Omega = \{\bar{x} \in R^2 \mid x_1 \in (x'_{14}, x''_{14}), \; x_2 \in (x'_{21}, x''_{21})\}. \tag{1.19}$$

Our analysis will be based on the following

Lemma. *Let $u(t) \in H^2(\Omega)$ be the solution to problem* (1.3) *and $\lambda_k \in C^1(\overline{\Omega})$, $k = 1, 2, 3$. Let*

$$z_k(t) = -\frac{\partial}{\partial x_k} \lambda_k \frac{\partial u}{\partial x_k}(t), \quad k = 1, 2, \qquad z_3(t) = -\frac{\partial}{\partial l} \lambda_3 \frac{\partial u}{\partial l}(t).$$

Then for bilinear forms (1.13), (1.14) *the equalities*

$$a_k(u(t), v) + b_k(u(t), v) = (z_k(t), v), \quad k = 1, 2, \tag{1.20}$$

$$a_3(u(t), v) - b(u(t), v) = (z_3(t), v) \tag{1.21}$$

hold.

Proof. Firstly, let us obtain equality (1.20) for $k = 1$. In accordance with presentation (1.19)

$$(z_1(t), v) = \int_{x'_{21}}^{x''_{21}} \Big(\int_{x'_{14}}^{x''_{14}} \frac{\partial}{\partial x_1} \lambda_1 \frac{\partial u(t)}{\partial x_1} v \, dx_1 \Big) dx_2.$$

After integration by parts we have

$$(z_1(t), v) = a_0^{(1)}(u(t), v) - \int_{\Gamma_1 \cup \Gamma_3} \lambda_1 \frac{\partial u(t)}{\partial x_1} v \, dx_2. \tag{1.22}$$

Let us consider the second term on the right-hand side of (1.22) more detailly. Introduce the local orthogonal coordinate system (n, s) with the origin at the point $\bar{x}'_p$ $(p = 1, 3)$. Then the equalities

$$n = (x_1 - x'_{1p})\cos\theta + (x_2 - x'_{2p})\sin\theta,$$
$$s = -(x_1 - x'_{1p})\sin\theta + (x_2 - x'_{2p})\cos\theta$$

hold. It is easy to see that

$$\begin{aligned}\frac{\partial u}{\partial x_1} &= \frac{\partial u}{\partial n}\cos\theta - \frac{\partial u}{\partial s}\sin\theta,\\ \frac{\partial u}{\partial x_2} &= \frac{\partial u}{\partial n}\sin\theta + \frac{\partial u}{\partial s}\cos\theta.\end{aligned} \tag{1.23}$$

According to (1.18) from the latter equalities it follows

$$\frac{\partial u}{\partial N} = (\lambda_{11}\cos^2\theta + \lambda_{22}\sin^2\theta)\frac{\partial u}{\partial n} + \lambda_{12}(\cos^2\theta - \sin^2\theta)\frac{\partial u}{\partial s}.$$

Substitute this equality into boundary condition (1.17) and use notations (1.7) and (1.12). Then we obtain the following formula for normal derivative:

$$\frac{\partial u}{\partial n} = -\frac{\lambda_{12}(\cos^2\theta - \sin^2\theta)}{\omega}\frac{\partial u}{\partial s} + \frac{\sigma u}{\omega}. \tag{1.24}$$

And finally, the use of equality (1.24) and the fact that $\cos^2\theta(\bar{x}) = 1$, $\sin^2\theta(\bar{x}) = 0$ for $\bar{x} \in \Gamma_1 \cup \Gamma_3$ in accordance with the first formula from (1.23) leads to the following expression on $\Gamma_1 \cup \Gamma_3$:

$$\frac{\partial u}{\partial x_1} = -\frac{\lambda_{12}\cos\theta}{\omega}\frac{\partial u}{\partial s} - \frac{\sigma\cos\theta}{\omega}u. \tag{1.25}$$

Those, we obtain $dx_2 = \cos\theta\, ds$,

$$\int_{\Gamma_1\cup\Gamma_3}\lambda_1\frac{\partial u(t)}{\partial x_1}v\,dx_2 = \int_{\Gamma_1\cup\Gamma_3}\Big(\nu_1\frac{\partial u}{\partial s} + \sigma_1 u\Big)v\,ds.$$

Let us note that $\sigma_1 = \nu_1 = 0$ on Γ_2 and Γ_4. Then according to (1.10), (1.14) we have

$$\int_{\Gamma_1\cup\Gamma_3}\lambda_1\frac{\partial u(t)}{\partial x_1}v\,dx_2 = -m_1(u,v) - b_1(u,v).$$

Substituting this equality into (1.22), we obtain (1.20) for $k = 1$. The case $k = 2$ may be considered the same way. Then integration by parts gives

$$(z_3(t), v) = a_0^{(3)}(u,v) - \int\limits_{\Gamma_1\cup\Gamma_3}\cos\alpha\lambda_3\frac{\partial u}{\partial l}v\,dx_2 - \int\limits_{\Gamma_2\cup\Gamma_4}\sin\alpha\lambda_3\frac{\partial u}{\partial l}v\,dx_1. \tag{1.26}$$

Transform expressions (1.23) in accordance with (1.24). As a result we have

$$\frac{\partial u}{\partial x_1} = -\frac{\partial u}{\partial s}\sin\theta, \quad \frac{\partial u}{\partial x_2} = -\frac{\lambda_{12}\sin\theta}{\omega}\frac{\partial u}{\partial s} - \frac{\sigma\sin\theta}{\omega}u$$

on Γ_2, Γ_4, as $\cos\theta(\bar{x}) = 0$, $\sin^2\theta(\bar{x}) = 1$ for $\bar{x} \in \Gamma_2 \cup \Gamma_4$. Then

$$\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial s}\cos\theta$$

on Γ_1, Γ_3, as $\cos^2\theta(\bar{x}) = 1$, $\sin\theta(\bar{x}) = 0$ for $\bar{x} \in \Gamma_1 \cup \Gamma_3$. Recalling that $dx_2 = \cos\theta\, ds$ on Γ_1, Γ_3 and $dx_1 = -\sin\theta\, ds$ on Γ_2, Γ_4, substitute these equalities into boundary integrals from (1.26). After simple transformations with the use of notations (1.7), (1.5), (1.11), and (1.15) equality (1.26) gives

$$(z_3(t), v) = a_0^{(3)}(u,v) - \int_{\Gamma_1\cup\Gamma_3}\Big(-(\nu_1+\nu_2)\frac{\partial u}{\partial s}\sigma_3 u\Big)v\, ds + \int_{\Gamma_2\cup\Gamma_4}\Big(-(\nu_1+\nu_2)\frac{\partial u}{\partial s} + \sigma_3 u\Big)v\, ds.$$

Then according to (1.14), (1.10), and (1.13), we obtain (1.21). □

At the end of this section introduce some finite element notations. Let $V_h = \mathrm{span}\{\varphi_i(\bar{x})\}_{i\in I}$ and $\Pi_h : C(\overline{\Omega}) \to V_h$ be interpolating operator: $\Pi_h u(\bar{x}) = \sum_{i\in I} u(\bar{x}_i)\varphi_i(\bar{x})$. With the use of lumping operator [4] let us introduce mesh scalar product $d_h(u,v)$ in $L_2(\Omega)$. In the future we will use such propeties of the bilinear form $d_h(u,v)$ as continuity in $L_2(\Omega)\times L_2(\Omega)$ and $L_2(\Omega)$-ellipticity. It means that

$$|d_h(u,v)| \le d_0\|u\|_{L_2(\Omega)}\|v\|_{L_2(\Omega)}, \tag{1.27}$$

$$d_h(u,v) \ge d_1\|u\|_{L_2(\Omega)}, \tag{1.28}$$

where d_0, d_1 do not depend on h and functions u, v.

2. The method of formulation and convergence

Here we propose the scheme for the solution to the problem from the previous section. We suppose that $\sigma(\bar{x}) \ge \sigma_0 > 0\ \forall\, \bar{x} \in \Gamma$ and $\lambda_{12} > 0$. Then $\nu_k > 0$, $k = 1, 2$. Let us denote $F_p^k(f_1, f_2) = (\sqrt{\nu_k}f_2 - \sqrt{\nu_1}f_1)(\bar{x}'_p)$, $k = 1, 2$, $p = \overline{1,4}$, $f_1, f_2 \in V_h$. Let $\tau = (t_* - t_0)/N$ and $t_n = t_0 + n\tau$, where N is some interger number, $n = \overline{1,N}$. We give the method of formulation as the following difference problem. It is necessary to find the family of functions

$$\{u^{n+k/3},\ k = 1,2,3,\ n = \overline{0, N-1}\},$$

such that $u^{n+k/3} \in V_h$ and $\forall\, v^{n+k/3} \in V_h$ the equalities

$$d_h\Big(\frac{u^{n+k/3}-u^{n+(k-1)/3}}{\tau}, v^{n+k/3}\Big) + a_k(u^{n+k/3}, v^{n+k/3}) + b_k(u^{n+k/3}, v^{n+k/3}) + \sum_{p=k,k+2} F_p^k(u^{n+(k-1)/3}, u^{n+k/3})\sqrt{\nu_k}v^{n+k/3}(\bar{x}'_p) = 0, \quad k = 1,2,$$

$$d_h\Big(\frac{u^{n+1} - u^{n+2/3}}{\tau}, v^{n+1}\Big) + a_3(u^{n+1}, v^{n+1}) - b(u^{n+1}, v^{n+1}) - \sum_{p=1,3} F_p^2(u^{n+2/3}, u^{n+1})\sqrt{\nu_1}v^{n+1}(\bar{x}'_p) = (f(t_{n+1}), v^{n+1}) \qquad (2.1)$$

hold. Here $n = \overline{0, N-1}$,

$$u^0 = \Pi_h u_0, \qquad (2.2)$$

and $u_0 \in H^2(\Omega)$. The conditions of a smoothness of the solution $u(t)$ will be given bellow. The role of the terms with the values of mesh functions in the corners of Ω is to provide a stability. It has different form at $\lambda_{12} < 0$, but with the similar analysis.

Let us introduce the sequence of the functions

$$\{\xi^n = u^n - \Pi_h u(t_n),\ \xi^{n+k/3} = u^{n+k/3} - \Pi_h u(t_{n+1}) + \tau r^{n+k/3},\ k = 1,2\}_{n=0}^{N-1},$$

with elements from V_h. The functions $r^{n+k/3} \in V_h$ will be introduced bellow. Here according to (2.2), $\xi^0 = 0$. Let us write the equations for the functions $\xi^{n+k/3}$, $k = 1,2,3$. In accordance with (2.1) we have

$$d_h\Big(\frac{\xi^{n+k/3}-\xi^{n+(k-1)/3}}{\tau}, v^{n+k/3}\Big) + a_k(\xi^{n+k/3}, v^{n+k/3}) + b_k(\xi^{n+k/3}, v^{n+k/3}) + \sum_{p=k,k+2} F_p^k(\xi^{n+(k-1)/3}, \xi^{n+k/3})\sqrt{\nu_k}v^{n+k/3}(\bar{x}'_p) = g_k^n(v^{n+k/3}),\ k = 1,2,$$

$$d_h\Big(\frac{\xi^{n+1} - \xi^{n+2/3}}{\tau}, v^{n+1}\Big) + a_3(\xi^{n+1}, v^{n+1}) - b(\xi^{n+1}, v^{n+1}) - \sum_{p=1,3} F_p^2(\xi^{n+2/3}, \xi^{n+1})\sqrt{\nu_1}v^{n+1}(\bar{x}'_p) = g_3^n(v^{n+1}). \qquad (2.3)$$

Here

$$g_k^n(v) = \alpha_k^n(v) + \beta_k^n(v) + \tau\gamma_k^n(v) - \delta_k^n(v) + \tau g_k^{n*}(v), \quad k = 1,2,$$
$$g_3^n(v) = \alpha_3^n(v) + \beta_3^n(v) + \sum_{k=1,2}(\delta_k^n(v) - \alpha_k^n(v)) - \tau\sum_{p=1,3}\sqrt{\nu_2}r^{n+2/3}(\bar{x}'_p), \quad (2.4)$$

where

$$g_1^{n*} = \nu_1 \sum_{p=1,3}\Big(r^{n+1/3} - \Pi_h[u(t_n)]_\tau\Big)(\bar{x}'_p)v(\bar{x}'_p), \quad [u(t_n)]_\tau \equiv \frac{u(t_{n+1}) - u(t_n)}{\tau},$$

$$g_2^{n*} = \sum_{p=2,4}\Big(\sqrt{\nu_2}r^{n+2/3} - \sqrt{\nu_1}r^{n+1/3}\Big)(\bar{x}'_p)v(\bar{x}'_p),$$

$$\begin{aligned}
\alpha_k^n(v) &= d_h\Big(\Pi_h z_k(t_{n+1}), v\Big) - \Big(z_k(t_{n+1}), v\Big),\\
\beta_k^n(v) &= (a_k + b_k)\Big(u(t_{n+1}) - \Pi_h u(t_{n+1}), v\Big),\\
\gamma_k^n(v) &= (a_k + b_k)(r^{n+k/3}, v),\\
\delta_1^n(v) &= d_h\Big(\Pi_h[u(t_n)]_\tau + \Pi_h z_1(t_{n+1}) - r^{n+1/3}, v\Big),\\
\delta_2^n(v) &= d_h\Big(\Pi_h z_2(t_{n+1}) + r^{n+1/3} - r^{n+2/3}, v\Big),\\
\alpha_3^n(v) &= \Big(\frac{du}{dt}(t^{n+1}), v\Big) - d_h\Big(\Pi_h[u(t_n)]_\tau, v\Big),\\
\beta_3^n(v) &= (a_3 - b)\Big(u(t_{n+1}) - \Pi_h u(t_{n+1}), v\Big).
\end{aligned} \tag{2.5}$$

To obtain (2.5) we use the lemma. The functions $r^{n+k/3} \in V_h$ are defined from the conditions

$$\begin{aligned}
&r^{n+1/3}(\bar{x}'_p) = \Pi_h[u(t_n)]_\tau(\bar{x}'_p), && p = 1, 3,\\
&\sqrt{\nu_1} r^{n+1/3}(\bar{x}'_p) = \sqrt{\nu_2} r^{n+2/3}(\bar{x}'_p), && p = 2, 4,\\
&r^{n+2/3}(\bar{x}'_p) = 0, && p = 1, 3.
\end{aligned}$$

Let

$$\begin{aligned}
r^{n+1/3} &= \Pi_h\Big(\mu_1[u(t_n)]_\tau + \mu_2 z_1(t_{n+1})\Big),\\
r^{n+2/3} &= \Pi_h \mu_2\Big([u(t_n)]_\tau + z_1(t_{n+1}) + z_2(t_{n+1})\Big),
\end{aligned} \tag{2.6}$$

where $\mu_1(\bar{x})$, $\mu_2(\bar{x})$ are the cutting functions such that $\mu_1(\bar{x}'_p) = 0$ at $p = 1, 3$, $\mu_2(\bar{x}'_p) = 0$ at $p = \overline{1,4}$. As it is easy to note for such definition of $r^{n+k/3}$ the conditions mentioned above are valid.

Now in the domain Ω let us introduce new triangulation $\mathcal{T}_\rho$, where $\rho = \rho(h, \tau)$, and $ch \le \rho$, where $c > 1$. For the set $\mathcal{T}_\rho$ define the space V_ρ with the piecewise linear basis $\{\varphi_i^\rho(\bar{x})\}_{i \in I_\rho}$. Let i_p be the indexes of points $\bar{x}'_p$ in triangulation $\mathcal{T}_\rho$. Let

$$\mu_1(\bar{x}) = 1 - \sum_{p=1,3} \varphi_{i_p}^\rho(\bar{x}), \quad \mu_2(\bar{x}) = 1 - \sum_{p=1}^{4} \varphi_{i_p}^\rho(\bar{x}). \tag{2.7}$$

As a result formulae (2.4) may be rewritten in the form

$$\begin{aligned}
g_k^n(v) &= \alpha_k^n(v) + \beta_k^n(v) + \tau\gamma_k^n(v) - \delta_k^n(v), \quad k = 1, 2,\\
g_3^n(v) &= \alpha_3^n(v) + \beta_3^n(v) + \sum_{k=1}^{2}(\delta_k^n(v) - \alpha_k^n(v)),
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}\delta_1^n(v) &= d_h\Big(\Pi_h(1-\mu_1)[u(t_n)]_\tau + \Pi_h(1-\mu_2)z_1(t_{n+1}), v\Big),\\ \delta_2^n(v) &= d_h\Big(\Pi_h(\mu_1-\mu_2)[u(t_n)]_\tau + \Pi_h(1-\mu_2)z_2(t_{n+1}), v\Big).\end{aligned} \tag{2.9}$$

Equalities (2.5)–(2.9) completely give us the rihgt-hand side of the error equations (2.3). In equations (2.3) suppose $v^{n+k/3} = 2\tau\xi^{n+k/3}$, $k = 1, 2, 3$. According to $\nu_k = \text{const}$ the following equalities are valid:

$$b_k(v,v) = \int_\Gamma \nu_k \frac{\partial u}{\partial s} v\, ds = \frac{\nu_k}{2} \sum_{p=k,k+2} \Big(v^2(\bar{x}_p'') - v^2(\bar{x}_p')\Big), \quad k = 1, 2. \tag{2.10}$$

Let us introduce nonnegative functionals

$$\Xi_1(v_1) = d_h(v_1, v_1) + \tau\nu_1 \sum_{p=1,3} v_1^2(\bar{x}_p'),$$

$$\Xi_2(v_1, v_2) = d_h(v_2 - v_1, v_2 - v_1) + \tau \sum_{p=2,4} (\sqrt{\nu_2}v_2 - \sqrt{\nu_1}v_1)^2(\bar{x}_p'),$$

$$\Xi_3(v_1, v_2) = d_h(v_2 - v_1, v_2 - v_1) + \tau \sum_{p=1,3} (\sqrt{\nu_1}v_2 - \sqrt{\nu_2}v_1)^2(\bar{x}_p'),$$

where $v_1, v_2 \in V_h$. From equations (2.3) and equality (2.10) we obtain

$$\begin{aligned}&\Xi_1(\xi^{n+1}) + \Xi_1(\xi^{n+1/3} - \xi^n) + \Xi_2(\xi^{n+1/3}, \xi^{n+2/3}) + \Xi_3(\xi^{n+2/3}, \xi^{n+1}) +\\ &2\tau \sum_{k=1}^3 a_k(\xi^{n+k/3}, \xi^{n+k/3}) = \Xi_1(\xi^n) + 2\tau \sum_{k=1}^3 g_k^n(\xi^{n+k/3}).\end{aligned} \tag{2.11}$$

Further d_1 is the constant from inequality (1.28). Give the estimates of functionals from (2.5):

$$\begin{aligned}|\alpha_k^n(v)| \le\ & \frac{\varepsilon_k^\alpha}{d_1} d_h(v,v) + \frac{c}{\varepsilon_k^\alpha} h^2 \|u\|^2_{C(t_0,t_*;H^3(\Omega))} +\\ & \frac{c}{\varepsilon_k^\alpha} h^4 \|u\|^2_{C(t_0,t_*;H^4(\Omega))}, \quad k = 1, 2;\end{aligned}$$

$$\begin{aligned}|\alpha_3^n(v)| \le\ & \frac{\varepsilon_3^\alpha}{d_1} d_h(v,v) + \frac{c}{\varepsilon_3^\alpha}\left[\tau \left\|\frac{d^2u}{dt^2}\right\|^2_{L_2(t_n,t_{n+1}\times\Omega)} +\right.\\ & \left.\tau^{-1}h^2\left(\left\|\frac{du}{dt}\right\|^2_{L_2(t_n,t_{n+1};H^1(\Omega))} + h^2\left\|\frac{du}{dt}\right\|^2_{L_2(t_n,t_{n+1};H^2(\Omega))}\right)\right];\end{aligned}$$

$$\begin{aligned}|\beta_k^n(v)| \le\ & \frac{\varepsilon_k^\beta}{2}\Big(a_0^k(v,v) + m_k(v,v) + \int_\Gamma v^2\, ds\Big) +\\ & \frac{c}{\varepsilon_k^\beta} h^2 \Big(\|u\|^2_{C(t_0,t_*;H^2(\Omega))} + \|u\|^2_{C(t_0,t_*;H^2(\Gamma))}\Big), \quad k = 1, 2, 3;\end{aligned}$$

$$|\gamma_k^n(v)| \le \frac{\varepsilon_k^\gamma}{2}\Big(a^k(v,v) + \int_{\Gamma_k \cup \Gamma_{k+2}} v^2 ds\Big) + \frac{c}{\varepsilon_k^\gamma}\frac{1}{\rho}\Big(\frac{1}{\tau}\Big\|\frac{du}{dt}\Big\|^2_{L_2(t_n,t_{n+1};H^2(\Omega))} + \|u\|^2_{C(t_0,t_*;H^4(\Omega))}\Big), \quad k = 1,2;$$

$$|\delta_k^n(v)| \le \frac{\varepsilon_k^\delta}{2} d_h(v,v) + \frac{c}{\varepsilon_k^\delta}\rho^2\Big(\frac{1}{\tau}\frac{du}{dt}\Big\|^2_{L_2(t_n,t_{n+1};H^2(\Omega))} + \|u\|^2_{C(t_0,t_*;H^4(\Omega))}\Big), \quad k = 1,2.$$

Then from (2.8) we have the estimates of the functionals g_k^n. Substitute these estimates in (2.11). Let us note that the parameters ε (from ε-inequalities) may be chosen such that after its substitution into (2.11) we will obtain the inequality

$$\Xi_1(\xi^{n+1}) + \Xi_1(\xi^{n+1/3} - \xi^n) \le \Xi_1(\xi^n) + 2\tau\frac{\varepsilon_3^\alpha}{d_1} d_h(\xi^{n+1},\xi^{n+1}) + 2\tau\Psi^n, \quad (2.12)$$

where the term Ψ^n depends on the norms of the solution. Moreover, it is important that ε_k^α, ε_k^γ, ε_k^δ, $k = 1,2$, are the values of the order $1/\tau$, and $\varepsilon_k^\beta \sim O(1)$ $(k = 1,2,3)$. Here we essentially make use of the fact that the number c_0 exists, such that $0 < c_0\sigma_0 \le \sigma_k$, $k = 1,2,3$. It means that

$$c_0\sigma_0 \int_{\Gamma_k \cup \Gamma_{k+2}} v^2 ds \le m_k(v,v), \quad k = 1,2,$$
$$c_0\sigma_0 \int_\Gamma v^2 ds \le m_3(v,v).$$

Chosing $\varepsilon_3^\alpha = d_1/4$ from (2.12) we obtain

$$\Xi_1(\xi^{n+1}) \le \Xi_1(\xi^n) + 4\tau\Psi^n.$$

The use of the Gronwall grid lemma and condition (1.28) gives the estimate

$$\|\xi^n\|_{L_2(\Omega)} \le c(M_h h + M_\tau \tau + M_\Phi \Phi(\rho,\tau)), \quad (2.13)$$

where

$$M_h = \Big\|\frac{du}{dt}\Big\|_{L_2(t_0,t_*;H^2(\Omega))} + \|u\|_{C(t_0,t_*;H^4(\Omega))},$$
$$M_\tau = \Big\|\frac{d^2u}{dt^2}\Big\|_{L_2((t_0,t_*)\times\Omega)}, \qquad M_\Phi = M_\tau. \quad (2.14)$$

The function $\Phi(\rho,\tau)$ has a form $\Phi(\rho,\tau) = (\tau\rho^2 + \tau^2/\rho)^{1/2}$ and, as a function of the argument ρ, it has a minimal value at $\rho = (\tau/2)^{1/3}$, or $\rho \sim \tau^{1/3}$. Then instead of (2.13) the following estimate is valid:

$$\|\xi^n\|_{L_2(\Omega)} \le c(M_h h + M'_\tau \tau^{5/6}),$$

where

$$M'_\tau = M_\tau + (2^{-2/3} + 2^{1/3})^{1/2} M_\Phi. \tag{2.15}$$

Using the tringle inequality and the interpolation theorem we obtain the final error estimate of the splitting scheme including a mesh tangent derivative:

Theorem. *Let for problem* (1.3) *the conditions*

$$u \in C(t_0, t_*; H^4(\Omega)), \quad \frac{du}{dt} \in L_2(t_0, t_*; H^2(\Omega)), \quad \frac{d^2u}{dt^2} \in L_2((t_0, t_*) \times \Omega)$$

be valid. Then for the solution to problem (2.1), (2.2) *at* $\tau \le 1$ *we have the estimate*

$$\max_{1 \le n \le N} \|u^n - u(t_n)\|_{L_2(\Omega)} \le c(M_h h + M'_\tau \tau^{5/6}),$$

where c *is the number independent of* τ, h, *and the function* $u(t)$ *and the quantaties* M_h, M'_τ *are given by equalities* (2.14), (2.15).

3. Numerical example

The scheme described above includes additional three-points relations on the boundary, and it's special realization is similar to the realization of the method from [3]. Now we illustrate proposed technique by the following mixed boundary value problem with the known solution:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \lambda_0\Big(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\Big), && (t, x_1, x_2) \in (0,1) \times \Omega, \\
&u(t, x_1, x_2) = g(\bar{x}), && (t, x_1, x_2) \in (0,1) \times \Gamma_0, \\
&\frac{\partial u}{\partial n} = 0, && (t, x_1, x_2) \in (0,1) \times \Gamma_1, \\
&u(0, x_1, x_2) = \sin \pi(x_1 - 2x_2), && (x_1, x_2) \in \Omega,
\end{aligned}$$

Here $\dfrac{\partial u}{\partial n} = \dfrac{\partial u}{\partial x_1} + \dfrac{1}{2}\dfrac{\partial u}{\partial x_2}$ is the co-normal derivative,

$$\begin{aligned}
\Omega &= \{(x_1, x_2) \mid 0 \le x_1 \le 1,\ -1/2 \le x_2 \le 1/2\}, \\
\Gamma_0 &= \{(x_1, x_2) \mid 0 \le x_1 \le 1,\ x_2 = -1/2\} \cup \\
&\quad\ \{(x_1, x_2) \mid 0 \le x_1 \le 1,\ x_2 = 1/2\} \cup \\
&\quad\ \{(x_1, x_2) \mid x_1 = 0,\ -1/2 \le x_2 \le 1/2\}, \\
\Gamma_1 &= \{(x_1, x_2) \mid x_1 = 1,\ -1/2 \le x_2 \le 1/2\}.
\end{aligned}$$

The solution to this problem is the function

$$u(t, x_1, x_2) = e^{-3\pi^2\lambda_0 t} \sin \pi(x_1 - 2x_2).$$

For calculations we suppose $\lambda_0 = 0.03$. For this value L_2-norm of the solution decreases approximately in e times in the time $t = 1$. In the domain Ω a uniform mesh with the step h is introduced. For the L_2-norm error the following notation is used:

$$\varepsilon = h\Big(\sum_{i\in I_0}(u^N(\bar{x}_i) - u(t,\bar{x}_i))^2 + \frac{1}{2}\sum_{i\in I_1}(u^N(\bar{x}_i) - u(t,\bar{x}_i))^2\Big)^{1/2},$$

$N\tau = 1$, I_0 is the set of the numbers of inner verticies, I_1 is the set of the numbers of the verticies from Γ_1. Let ε_1 be the error of the Marchuk–Kuzin scheme in one-cyclic reduction, and ε_2 be the error of the scheme described in Section 2. In the table bellow the results are given at $\tau = h$.

$\log_2 h$	-3	-4	-5	-6	-7	-8
ε_1	.013780	.006116	.003371	.002128	.001424	.000972
ε_2	.012411	.004489	.001807	.000795	.000371	.000179

Those, on the simplest example we can see the role of $\sqrt{h}$ in the estimate $\|\varepsilon_1\|_{L_2(\Omega)}$. Given results allow us to see that with decreasing h in 2 times for ε_1, especially on the detail meshes with the greate accuracy, $\sqrt{2}$ is realized. For the scheme with the derivative along the boundary Γ_1 we have a convergence $O(\tau)$. Let us note that in the considered example the assumption from previous section on the third boundary condition is not valid, but we see from the table that our modification gives good results. Whereas in numerical expirements it is difficult to see the difference between quantaties of orders τ and $\tau^{5/6}$, we do no definite conclusion on an optimality of the estimate $O(\tau^{5/6})$, based on these results.

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