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On the explicit-implicit domain decomposition method without overlapping for parabolic problems*

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The paper deals with studying the domain decomposition algorithm on two subdomains, where for one of them, which contains sufficiently small number of nodes, is used explicit scheme with small time step, and for another subdomain may be used effective direct algorithm (for example, subdomain is parallelepiped). This method is based on the splitting method. The algorithm formulation is given in projection form with the finite element approximation. The lumping operators technique is used for this purpose.

1. Introduction

In the paper [1] the domain decomposition method without iterations, which use decomposition for problems with nonideal contact with application of penalty method, was cosidered. The third boundary value problem in separate subdomain is solved by splitting scheme. It is supposed that subdomain is m-parallelepiped. However, if the domain is a union of some numbers of m-parallelepipeds and of arbitrary polytops in R^m with sufficiently small quantity of verticies, then the explicit-implicit scheme with small step on time may be used in the same subdomain. For domain decomposition method with overlapping such approach was realised in the paper [2]. The realization of this idea for the method without overlapping constitutes the subject matter of this paper.

In this paper we will consider the case of only two subdomains. On the one hand it is made for simplification of the formulas, and on the other hand it justified the parallel algorithm of domain decomposition. The question about parallel algorithm was discussed in the paper [1]. Note that, as in [1], one may use a compound grid in proposed algorithm, i.e., the grids in separate subdomain do not connect with each other. Numeral experements, which are adduced at the end of the paper, illustrate the

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asimptotical behaviour of parameter s, which characterizes the small step on time $\tau_l = \tau/s$.

2. Formulation of the problem

Let Ω be a bounded domain in R^m $(m = 2,3)$, Ω_1 and Ω_2 be subdomain in Ω, such that the conditions

$$\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}, \quad \Omega_1 \cap \Omega_2 = 0, \tag{2.1}$$

are valid and $\Gamma_{1,2} = \bar{\Omega}_1 \cap \bar{\Omega}_2$. Let us introduce some notions, wich will be used further. Let $\Gamma_p = \partial\Omega_p$ be a boundary of subdomain Ω_p, $p = 1,2$, $Q_t = (t_0, t_*) \times \Omega$, $Q_{t,p} = (t_0, t_*) \times \Omega_p$ be subdomain in Q_t. In the space $H^1(\Omega_p) \times H^1(\Omega_p)$ we cosider m one-parametric families of the biliniar forms

$$a_p^{(k)}(t; u, v) = \int_{\Omega_p} \lambda_k(t, \bar{x}) \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} d\bar{x}, \quad k = 1, \ldots, m, \tag{2.2}$$

where $t \in [t_0, t_*]$ is a real parameter. We assume that the functions $\lambda_k(t, \bar{x})$ are continuous in $Q_{t,p}$ (but not in Q_t) and are limited from below positive number λ_0. Assumptions on smoothness of these functions will be given in what follows. Let us introduce in $H^1(\Omega_p)$ a family of linear functionals as a duality relation in $H^{-1}(\Omega_p) \times H^1(\Omega_p)$, i.e., $l_p(t; v) = (f_p(t), v)_p$, where $(,)$ is the scalar product in $L_2(\Omega_p)$, $f_p : [t_0, t_*] \to H^{-1}(\Omega_p)$. Henceforth, by $u(t)$ we denote the value of the function $u : [t_0, t_*] \to X$, which is the element of some Banach space X, and $\frac{du}{dt}(t)$ means a strong limit in X (if such limit exists) of the elements $[u(t)]_\tau \equiv (u(t + \tau) - u(t))/\tau$ for $\tau \to 0$. Let X_p be a space of defined in Ω functions from $X(\Omega_p)$ in Ω_p which are extended to the outside of Ω_p by zero. Then let us consider the space of vector-functions

$$\hat{X} = X_1 \times X_2$$

with the norm $\|u\|_{\hat{X}}$. For example, $\|u\|_{\hat{H}^1} = (\|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)})^{1/2}$, where u_p, $p = 1,2$, are components of the vector-function u. The scalar product in $\hat{L}_2$ is $(u, v) = (u_1, v_1)_1 + (u_2, v_2)_2$. In the space $H^1(\Omega_p) \times H^1(\Omega_p)$ and $\hat{H}^1(\Omega) \times \hat{H}^1(\Omega)$ one may define families of bilinear forms

$$a_p(t; u, v) = \sum_{k=1}^{m} a_p^{(k)}(t; u, v), \quad p = 1,2, \tag{2.3}$$

$$a(t; u, v) = a_1(t; u, v) + a_2(t; u, v). \tag{2.4}$$

In the space $\hat{H}^1$ we may consider the subspace

$$\hat{H}^{1,0} = \{v \in \hat{H}^1 : \ v_1(\bar{x}) = v_2(\bar{x}),\ \bar{x} \in \Gamma_{1,2}\}.$$

Let us formulate the Neumann parabolic problem in the subspace $\hat{H}^{1,0}$. For $u_0 \in \hat{L}_2$ and $f \in L_2((t_0,t_*);\hat{H}^{-1})$ it is necessary to find the vector-function $u \in L_2((t_0,t_*);\hat{H}^{1,0})$, such that $\frac{du}{dt} \in L_2((t_0,t_*);\hat{H}^{-1})$ and $\forall v \in \hat{H}^{1,0}$ the following equalities are valid:

$$\left(\frac{du}{dt}(t), v\right) + a(t; u(t), v) = (f(t), v), \tag{2.5}$$

$$(u(t_0), v) = (u_0, v). \tag{2.6}$$

It is not difficult to note that (2.5), (2.6) are equivalent to the conventional formulation of the generalized Neumann parabolic problem in the space $H^1(\Omega)$.

In accordance with the paper [3] we formulate the Neumann propblem with a nonideal contact between the neighbouring subdomains. For $u_0 \in \hat{L}_2$ and $f \in L_2((t_0,t_*);\hat{H}^{-1})$. it is necessary to find the vector-function $u^\rho \in L_2((t_0,t_*);\hat{H}^1)$, such that $\frac{du^\rho}{dt} \in L_2((t_0,t_*);\hat{H}^{-1})$ and $\forall v \in \hat{H}^1$ the following equalities are valid:

$$\left(\frac{du^\rho}{dt}(t), v\right) + a(t; u^\rho(t), v) + \frac{1}{\rho}\int_{\Gamma_{1,2}} (u_1^\rho(t) - u_2^\rho(t))(v_1 - v_2)\,d\sigma = (f(t), v), \tag{2.7}$$

$$(u^\rho(t_0), v) = (u_0, v), \tag{2.8}$$

where $\rho > 0$. Now we will rewrite equation (1.7) for every subdomain. Supposing $v = (v_1, 0)$ and $v = (0, v_2)$ we obtain the following form of (2.7)

$$\left(\frac{du_p^\rho}{dt}(t), v_p\right)_p + a_p(t; u_p^\rho(t), v_p) + \frac{1}{\rho}\int_{\Gamma_{1,2}} \left(u_p^\rho(t) - u_q^\rho(t)\right) v_p\,d\sigma = (f_p(t), v_p)_p. \tag{2.9}$$

To these equations the following conditions on the inner bounderies $\Gamma_{1,2}$ for classical parabolic problem with conditions of nonideal contacts correspond

$$\rho\frac{\partial u_1^\rho}{\partial n_1}(t,\bar{x}) + \rho\frac{\partial u_2^\rho}{\partial n_2}(t,\bar{x}) = 0, \quad (t,\bar{x}) \in \Gamma_{t,1,2}, \tag{2.10}$$

$$\rho\frac{\partial u_p^\rho}{\partial n_p}(t,\bar{x}) + u_p^\rho(t,\bar{x}) - u_q^\rho(t,\bar{x}) = 0, \quad (t,\bar{x}) \in \Gamma_{t,1,2}, \tag{2.11}$$

where $p = 1,2,\ q = 2,1$. The normal derivatives may be written in the following form:

$$\frac{\partial u_p^\rho}{\partial n_p} = \sum_{k=1}^{m} \lambda_k \frac{\partial u_p^\rho}{\partial x_k} \cos(\bar{n}_p, \bar{x}^k),$$

where $\bar{n}_p$ and $\bar{x}^k$ are the unit vectors of the external normal to Ω_p on $\Gamma_{1,2}$ and of the coordinate k-axis.

Now we will cite on two inequalities which will give us a theoretical asis for the design of domain decomposition methods. If the solutions to problems (2.5), (2.6) and (2.7), (2.8) of the subdomain are enough smooth, the inequatities are valid

$$\|u^\rho - u\|_{C([t_0,t_*];\hat{L}_2)} \leq c_1\rho\|u\|_{H^1((t_0,t_*);\hat{H}^2)} \tag{2.12}$$

$$\|u^\rho\|_X \leq c_2\|u\|_X, \tag{2.13}$$

where X is some subspace of the space $L_2((t_0,t_*);\hat{H}^1)$ and $\rho \leq \rho_0$, the positive numbers c_1, c_2, ρ_0 do not depend on the parameter ρ and the functions u and u^ρ. The proofs of (2.12) and (2.13) are given in [3].

3. Discretization and some inequalities

Let us introduce some notions for discretization of problem (2.7), (2.8). The terminology, which will be used further, corresponds to the paper [4]. Let $\mathcal{T}_{h,p}$ be a regular set of m-simplexes. Let us note that the set $\mathcal{T}_h = \mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$ is not co-ordinate grid, i.e., in Ω we use a compound grid. For the set $\mathcal{T}_{h,p}$ we will introduce the system of the piecewise linear functions $\{\varphi_{p,i}(\bar{x})\}_{i\in I_p}$, where I_p is a set of numbers of all vertices $\bar{x}_i$ of the set $\mathcal{T}_{h,p}$, $\varphi_{p,i}(\bar{x}_j) = \delta_{ij}$, where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$, further, let us introduce $V_{h,p} = \text{span}\{\varphi_{p,i}(\bar{x})\}_{i\in I_p}$. Then let $\Pi_{h,p} : C(\bar{\Omega}_p) \to V_{h,p}$ be an interpolating operator difined by the formula

$$\Pi_{h,p}u(\bar{x}) = \sum_{i\in I_p} u(x_i)\varphi_{p,i}(\bar{x}), \quad \bar{x} \in \bar{\Omega}_p.$$

In accordance with the paper [5] we introduce the lumping operator $P_{h,p}$: $V_{h,p} \to L_{h,p}$, where $L_{h,p} \subset L_2(\Omega_p)$. Let us consider the bilinear form

$$d_{h,p}(v,w) = (P_{h,p}v, P_{h,p}w)_p, \quad v,w \in V_{h,p}, \quad p = 1,2, \tag{3.1}$$

which is continuous in $L_2(\Omega_p)\times L_2(\Omega_p)$, $L_2(\Omega_p)$ is elliptic, and the following estimation is valid:

$$\left|d_{h,p}(\Pi_{h,p}u, v) - (u,v)_p\right| \leq ch\left(|u|^2_{H^1(\Omega_p)} + h^2\|u\|^2_{H^2(\Omega_p)}\right)^{1/2}\|v\|_{L_2(\Omega_p)}. \tag{3.2}$$

The analogous estimation is valid in the space of traces of the grid functions. However, this estimation will be not applied in the case of using the compound grids, because the functions coming in the estimation integrals belong to different finite-dimentional subspace of traces. For this case the corresponding estimation is given in [1]. In the space $H^3(\Omega_q) \times V_{h,p}$ we will consider the bilinear form

$$l_{pq}(u,v) = \int_{\Gamma_{1,2}} \left[(P_{h,q}\Pi_{h,q}u)(P_{h,p}v) - uv\right] d\sigma. \tag{3.3}$$

Then the following estimation is valid:

$$|l_{pq}(u,v)| \le ch\left(\|u\|^2_{H^2(\Omega_q)} + h^2\|u\|^2_{H^3(\Omega_q)}\right)^{1/2}\|v\|_{L_2(\Gamma_{1,2})}, \tag{3.4}$$

where $u \in H^3(\Omega_q)$, $v \in V_{h,p}$. In case of co-ordinate grid the estimation (3.4) may be strengthened

$$|l_{pq}(u,v)| \le ch^{3/2}\left(\|u\|^2_{H^3(\Omega_q)} + h^2\|u\|^2_{H^4(\Omega_q)}\right)^{1/2}\|v\|_{L_2(\Gamma_{1,2})}. \tag{3.5}$$

Finally, let us introduce the vector space

$$\hat{V}_h = V'_{h,1} \times V'_{h,2},$$

where $V'_{h,p}$ is a space of functions from $V_{h,p}$ in Ω_p and are extended to the outside of Ω_p by zero. It is easy to note that $\hat{V}_h \subset \hat{H}^1 \cap \hat{C}$. In the space $\hat{V}_h \times \hat{V}_h$ we will consider the bilinear form

$$d_h(v,w) = d_{h,1}(v_1,w_1) + d_{h,2}(v_2,w_2), \tag{3.6}$$

which is positive definite in $\hat{L}_2$.

4. Formulation of the method

Let Ω_1 be some polytop, Ω_2 be m-parallelepiped. Let us introduce some notions. Let $\bar{n}_{1,2}$ and $\bar{x}^k$ be the unit vectors of the external normal to Ω_p on $\Gamma_{1,2}$ and of the coordinate k-axis. Let

$$\Gamma^{(k)}_{1,2} = \left\{\bar{x} \in \Gamma_{1,2} : \; \cos^2(\bar{n}_{1,2}, \bar{x}^k) = 1\right\}.$$

Along with (2.3) we will denote

$$l^{(k)}_{pq}(u,v) = \int_{\Gamma^{(k)}_{1,2}} \left[(P_{h,q}\Pi_{h,q}u)(P_{h,p}v) - uv\right] d\sigma. \tag{4.1}$$

We will note that $\sum_{k=1}^{m} l^{(k)}_{pq}(u,v) = l_{pq}(u,v)$.

Now let us formulate the domain decomposition method with using the explicit scheme in Ω_1 and splitting scheme in Ω_2 for solving the Neuman problem. Let N be some integer number, $\tau = (t_* - t_0)/N$ and $t_n = t_0 + n\tau$, $n = 1, \dots, N$. We introduce the sequence of numbers $\{\tau_l\}_{l=1}^{s}$ such that $\tau_l > 0$ and $\sum_{l=1}^{s} \tau_l = \tau$. Hereafter we will use the notion $\Delta_l = \sum_{r=1}^{l} \tau_r$. It is required to find the functions $u_1^{n+l/s} \in V_{h,1}$ and $u_2^{n+k/m} \in V_{h,2}$ such that

$$
\begin{gathered}
d_{h,1}\Big(\frac{u_1^{n+l/s} - u_1^{n+(l-1)/s}}{\tau_l}, v_1^l\Big) + a_1(t_n + \Delta_{l-1}; u_1^{n+(l-1)/s}, v_1^l) \\
+ \frac{1}{\rho} \int_{\Gamma_{1,2}} P_{h,1} u_1^{n+(l-1)/m} P_{h,2} v_1^l d\sigma = \qquad (4.2) \\
\frac{1}{\rho} \sum_{k=1}^{m} \int_{\Gamma_{1,2}^{(k)}} P_{h,2} u_2^{n-1+k/m} P_{h,1} v_1^l d\sigma + (f_1(t_n + \Delta_{l-1}), v_1^l)_1, \quad l = 1, \dots, s,
\end{gathered}
$$

$$
\begin{gathered}
d_{h,2}\Big(\frac{u_2^{n+k/m} - u_2^{n+(k-1)/m}}{\tau}, v_2^k\Big) + a_2^{(k)}(t_{n+1}; u_2^{n+k/m}, v_2^k) \\
+ \frac{1}{\rho} \int_{\Gamma_{1,2}^{(k)}} P_{h,2} u_2^{n+k/m} P_{h,2} v_2^k d\sigma = \qquad (4.3) \\
\frac{1}{\rho} \sum_{l=1}^{s} \frac{\tau_l}{\tau} \int_{\Gamma_{1,2}^{(k)}} P_{h,1} u_1^{n+(l-1)/s} P_{h,2} v_2^k d\sigma + (f_2^{n,k}, v_2^k)_2, \quad k = 1, \dots, m,
\end{gathered}
$$

$$
u_p^0 = \Pi_{h,p} u_{0,p}, \quad p = 1, 2, \qquad (4.4)
$$

where $f_2^{n,k} = 0$, $k \le m-1$, and $f_2^{n,m} = f_2(t_{n+1})$, $u_2^{k/m-1} = u_2^0$. It is supposed that $u_0 \in \hat{H}^2$ and according to the concluding theorem $u_0 \in \hat{C}$ and $u_{0,p}$ belong to a set of definition of the operator $\Pi_{h,p}$.

Let $u(t)$ be a solution to problem (2.5), (2.6). We will assume that

$$
\begin{gathered}
\lambda_k \in C\big([t_0, t_*]; \hat{C}^3\big), \quad u \in C\big([t_0, t_*]; \hat{H}^4\big), \\
\frac{du}{dt} \in L_2\big((t_0, t_*); \hat{H}^2\big), \quad \frac{d^2u}{dt^2} \in \mathrm{L}_2\big((t_0, t_*); \hat{L}_2\big).
\end{gathered} \qquad (4.5)
$$

According to inequality (2.14) the vector-function u^ρ holds the same smoothness and, therefore, the functions $\Pi_{h,1} u_1^\rho(t_n + \Delta_l)$ and $\Pi_{h,2} u_2^\rho(t_n)$ exist. Then we introduce the sequences:

$$
\xi_1^{n+l/s} = u_1^{n+l/s} - \Pi_{h,1} u_1^\rho(t_n + \Delta_l), \quad l = 1, \dots, s, \qquad (4.6)
$$

$$
\xi_2^{n+k/m} = u_2^{n+k/m} - \Pi_{h,2} u_2^\rho(t_{n+1}) + \tau r_2^{n+k/m}, \quad k = 1, \dots, m, \qquad (4.7)
$$

where $r_2^{n+k/m} \in V_{h,2}$ and $r_2^n = 0$, $n = 0, \dots, N$. Then $\xi_p^n = u_p^n - \Pi_{h,p} u_p^\rho(t_n)$ and according to (4.4), (4.6) and (4.7) $\xi_p^0 = 0$. Moreover, we will assume $\xi_2^{k/m-1} = 0$, such that $u_2^{k/m-1} = u_2^0$ from (4.7) it follows $r_2^{k/m-1} = 0$. According to (4.2), (4.3) let us write the scheme for the functions $\xi_1^{n+l/s}$ and $\xi_2^{n+k/m}$. We have

$$
\begin{aligned}
&d_{h,1}\Big(\frac{\xi_1^{n+l/s} - \xi_1^{n+(l-1)/s}}{\tau_l}, v_1^l\Big) + a_1(t_n + \Delta_{l-1}; \xi_1^{n+(l-1)/s}, v_1^l) \\
&\qquad + \frac{1}{\rho}\int_{\Gamma_{1,2}} P_{h,1}\xi_1^{n+(l-1)/m} P_{h,2} v_1^l d\sigma = \qquad (4.8)\\
&\frac{1}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}} P_{h,2}\xi_2^{n-1+k/m} P_{h,1} v_1^l d\sigma + g_1^{n,l}(v_1^l),
\end{aligned}
$$

$$
\begin{aligned}
&d_{h,2}\Big(\frac{\xi_2^{n+k/m} - \xi_2^{n+(k-1)/m}}{\tau}, v_2^k\Big) + a_2^{(k)}(t_{n+1}; \xi_2^{n+k/m}, v_2^k) \\
&\qquad + \frac{1}{\rho}\int_{\Gamma_{1,2}^{(k)}} P_{h,2}\xi_2^{n+k/m} P_{h,2} v_2^k d\sigma = \qquad (4.9)\\
&\frac{1}{\rho}\sum_{l=1}^{s}\frac{\tau_l}{\tau}\int_{\Gamma_{1,2}^{(k)}} P_{h,1}\xi_1^{n+(l-1)/s} P_{h,2} v_2^k d\sigma + g_2^{n,k}(v_2^k),
\end{aligned}
$$

Using the initial equation (2.9), after simple transformations we obtain the formula for the functionals $g_1^{n,l}(v_1)$:

$$
g_1^{n,l}(v_1) = \alpha_1^{n,l}(v_1) + \beta_1^{n,l}(v_1) + \tau\gamma_1^{n,l}(v_1) + \tau\delta_1^n(v_1), \quad l = 1, \dots, s, \qquad (4.10)
$$

where the functionals $\alpha_1^{n,l}(v_1)$, $\beta_1^{n,l}(v_1)$, $\gamma_1^{n,l}(v_1)$ and $\delta_1^n(v_1)$ are given by the equalities:

$$
\alpha_1^{n,l}(v_1) = \alpha_{1,1}^{n,l}(v_1) + \alpha_{1,2}^{n,l}(v_1), \quad l = 1, \dots, s, \qquad (4.11)
$$

$$
\alpha_{1,1}^{n,l}(v_1) = \Big(\frac{du_1^\rho}{dt}(t_n + \Delta_{l-1}) - [u_1^\rho(t_n + \Delta_{l-1})]_{\tau_l}, v_1\Big)_1, \qquad (4.12)
$$

$$
\alpha_{1,2}^{n,l} = ([u_1^\rho(t_n + \Delta_{l-1})]_{\tau_l}, v_1)_1 - d_{h,1}(\Pi_{h,1}[u_1^\rho(t_n + \Delta_{l-1})]_{\tau_l}, v_1), \qquad (4.13)
$$

$$
\beta_1^{n,l}(v_1) = \beta_{1,1}^{n,l}(v_1) + \beta_{1,2}^{n,l}(v_1), \quad l = 1, \dots, s, \qquad (4.14)
$$

$$
\beta_{1,1}^{n,l}(v_1) = a_1\big(t_n + \Delta_{l-1}; u_1^\rho(t_n + \Delta_{l-1}) - \Pi_{h,1} u_1^\rho(t_n + \Delta_{l-1}), v_1\big), \qquad (4.15)
$$

$$
\beta_{1,2}^{n,l}(v_1) = \frac{1}{\rho} l_{2,1}\big(u_2^\rho(t_n, v_1) - \frac{1}{\rho} l_{1,1}(u_1^\rho(t_n + \Delta_{l-1}), v_1\big), \qquad (4.16)
$$

$$\gamma_1^{n,1}(v_1) = 0,$$
$$\gamma_1^{n,l}(v_1) = -\frac{\Delta_{l-1}}{\rho\tau}\int_{\Gamma_{1,2}} [u_2(t_n)]_{\Delta_{l-1}} v_1 d\sigma, \quad l = 2,\dots,s, \tag{4.17}$$

$$\delta_1^n(v_1) = -\frac{1}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}} P_{h,2} r_2^{n-1+k/m} P_{h,1} v_1 d\sigma. \tag{4.18}$$

Bilinear forms $l_{pq}(u_q, v_p)$ are difined by formula (3.3).

Let us integrate by parts the bilinear form (2.2) for $p = 2$, taking into account conditions (2.10), (2.11) and the Neuman conditions on Γ. Besides, we will use the fact that Ω_2 is m-parallelepiped. As a cosequence we obtain

$$a_2^{(k)}(t; u_2^\rho(t), v_2) = (z_2^{(k)}(t), v_2)_2 - \frac{1}{\rho}\int_{\Gamma_{1,2}} (u_2^\rho(t) - u_1^\rho(t)) v_2 d\sigma, \tag{4.19}$$

where

$$z_1^{(k)}(t) = -\frac{\partial}{\partial x_k}\Big(\lambda_k \frac{\partial u_2^\rho}{\partial x_k}\Big)(t).$$

Now we introduce the functions $r_2^{n+k/m} \in V_{h,2}$

$$r_2^{n+k/m} = \Pi_{h,2}\Big\{[u_2^\rho(t_n)]_\tau + \sum_{r=1}^{k} z_2^{(r)}(t_{n+1})\Big\}, \quad k = 1,\dots,m-1. \tag{4.20}$$

Using the initial equality (2.9), equality (4.19) and expression for the functions $r_2^{n+k/m}$, we obtain the formula forthe functionals $g_2^{n,k}(v_2)$

$$g_2^{n,k}(v_2) = \alpha_2^{n,k}(v_2) + \beta_2^{n,k}(v_2) + \tau\gamma_2^{n,k}(v_2) + \tau\delta_2^{n,k}(v_2), \quad k \le m-1, \tag{4.21}$$

$$g_2^{n,m}(v_2) = \alpha_2^{n,m}(v_2) - \sum_{k=1}^{m-1}\alpha_2^{n,k}(v_2) + \beta_2^{n,m}(v_2) + \tau\gamma_2^{n,m}(v_2), \tag{4.22}$$

where functionals $\alpha_2^{n,k}(v_2)$, $\beta_2^{n,k}(v_2)$, $\gamma_2^{n,k}(v_2)$ and $\delta_2^{n,k}(v_2)$ have the form

$$\alpha_2^{n,k}(v_2) = d_{h,2}\big(\Pi_{h,2} z_2^{(k)}(t_{n+1}), v_2\big) - \big(z_2^{(k)}(t_{n+1}), v_2\big)_2, \quad k \le m-1, \tag{4.23}$$

$$\alpha_2^{n,m}(v_2) = \alpha_{2,1}^{n,m}(v_2) + \alpha_{2,2}^{n,m}(v_2), \tag{4.24}$$

$$\alpha_{2,1}^{n,m}(v_2) = \Big(\frac{du_2^\rho}{dt}(t_{n+1}) - [u_2^\rho(t_n)]_\tau, v_2\Big)_2, \tag{4.25}$$

$$\alpha_{2,2}^{n,m} = \big([u_2^\rho(t_n)]_\tau, v_2)_2 - d_{h,2}(\Pi_{h,2}[u_2^\rho(t_n)]_\tau, v_2\big), \tag{4.26}$$

$$\beta_2^{n,k}(v_2) = \beta_{2,1}^{n,k}(v_2) + \beta_{2,2}^{n,k}(v_2), \quad k = 1,\dots,m, \tag{4.27}$$

$$\beta_{2,1}^{n,k}(v_2) = a_2^{(k)}\big(t_{n+1}; u_2^\rho(t_{n+1}) - \Pi_{h,2}u_2^\rho(t_{n+1}), v_2\big), \tag{4.28}$$

$$\beta_{2,2}^{n,k}(v_2) = \frac{1}{\rho}l_{1,2}^{(k)}(u_1^\rho(t_n + \Delta_{l-1}), v_2) - \frac{1}{\rho}l_{2,2}^{(k)}(u_2^\rho(t_{n+1}), v_2), \tag{4.29}$$

$$\gamma_2^{n,k}(v_2) = -\frac{1}{\rho}\sum_{l=1}^{s}\frac{\tau_l(\tau - \Delta_{l-1})}{\tau^2}\int_{\Gamma_{1,2}^{(k)}}[u_1^\rho(t_n + \Delta_{l-1})]_{\tau-\Delta_{l-1}} v_2 d\sigma, \tag{4.30}$$

$$k = 1, \ldots, m,$$

$$\delta_2^{n,k}(v_2) = \delta_{2,1}^{n,k}(v_2) + \delta_{2,2}^{n,k}(v_2), \quad k \le m-1, \tag{4.31}$$

$$\delta_{2,1}^{n,k}(v_2) = a_2^{(k)}(t_{n+1}; r_2^{n+k/m}, v_2), \tag{4.32}$$

$$\delta_{2,2}^{n,k}(v_2) = \frac{1}{\rho}\int_{\Gamma_{1,2}^{(k)}} P_{h,2} r_2^{n+k/m} P_{h,2} v_2 d\sigma. \tag{4.33}$$

Bilinear forms $l_{p,q}^{(k)}(u_q, v_p)$ are defined by formula (4.1)

The equalities (4.8)–(4.18), (4.21)–(4.32) completely define the form of the error equations of method (4.2)–(4.4).

5. Error analysis

We carry out the error analysis similarly to the paper [3]. We begin the error analysis from the obtaining the integral identity, which is a foundation of the stability analysis in the norm of space L_2. Let us assume $v_1^l = 2\tau_l\xi_1^{n+l/s}$ in (4.8), $v_2^k = 2\tau\xi_2^{n+k/m}$ in (4.9), and then we will sum equations (4.8) over l from 1 to s, and (4.9) over k from 1 to m. It is an easy matter to notice that the inequality is valid

$$\begin{aligned}
&2\Big|\sum_{l=1}^{s}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}} P_{h,2}\xi_2^{n+k/m-1} P_{h,1}\xi_1^{n+l/s} d\sigma\Big| \le \\
&\qquad \sum_{l=1}^{s}\tau_l\int_{\Gamma_{1,2}}(P_{h,1}\xi_1^{n+l/s})^2 d\sigma + \tau\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n-1+k/m})^2 d\sigma, \\
&2\Big|\sum_{l=1}^{s}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}} P_{h,2}\xi_2^{n+k/m} P_{h,1}\xi_1^{n+(l-1)/s} d\sigma\Big| \le \\
&\qquad \sum_{l=1}^{s}\tau_l\int_{\Gamma_{1,2}}(P_{h,1}\xi_1^{n+(l-1)/s})^2 d\sigma + \tau\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n+k/m})^2 d\sigma.
\end{aligned} \tag{5.1}$$

Using (5.1) it is obtained

$$d_{h,1}(\xi_1^{n+1},\xi_1^{n+1})+\frac{1}{\rho}\sum_{l=1}^{s}\tau_l\int_{\Gamma_{1,2}}(P_{h,1}\xi_1^{n+(l-1)/s})^2d\sigma+$$

$$\frac{1}{2}\sum_{l=1}^{s}\tau_l a_1\big(t_n+\Delta_{l-1};\xi_1^{n+(l-1)/s}+\xi_1^{n+l/s},\xi_1^{n+(l-1)/s}+\xi_1^{n+l/s}\big)+$$

$$\sum_{l=1}^{s}\{d_{h,1}(\eta_1^l,\eta_1^l)-\frac{\tau_l}{2}a_1(t_n+\Delta_{l-1};\eta_1^l,\eta_1^l)-\frac{\tau_l}{\rho}\int_{\Gamma_{1,2}}(P_{h,1}\eta_1^l)^2d\sigma\}\le$$

$$d_{h,1}(\xi_1^n,\xi_1^n)+\frac{\tau}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n-1+k/m})^2d\sigma+2\sum_{l=1}^{s}\tau_l g_1^{n,l}(\xi_1^{n+l/s}), \tag{5.2}$$

$$d_{h,2}(\xi_2^{n+1},\xi_2^{n+1})+\frac{\tau}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n+k/m})^2d\sigma+$$

$$2\tau\sum_{k=1}^{m}a_2^{(k)}(t_{n+1};\xi_2^{n+k/m},\xi_2^{n+k/m})+d_{h,2}(\eta_2^k,\eta_2^k)\le$$

$$d_{h,2}(\xi_2^n,\xi_2^n)+\frac{1}{\rho}\sum_{l=1}^{s}\tau_l\int_{\Gamma_{1,2}}(P_{h,1}\xi_2^{n+(l-1)/s})^2d\sigma+2\tau\sum_{k=1}^{m}g_2^{n,k}(\xi_2^{n+k/m}), \tag{5.3}$$

where $\eta_1^l=\xi_1^{n+l/s}-\xi_1^{n+(l-1)/s}$ and $\eta_2^k=\xi_2^{n+k/m}-\xi_2^{n+(k-1)/m}$. Our further actions are connected with estimation of the functionals $g_1^{n,l}(\xi_1^{n+l/s})$. (Estimation $g_2^{n,k}(\xi_2^{n+k/m})$ is executed similarly to [1].) For this aim we take the standard technique based on the use of the Cauchy-Buniakovsky inequalities, the ε-inequalities, estimations (3.2) and

$$\|[u(t)]_\tau\|_{\hat{H}^l}\le\tau^{-1/2}\Big\|\frac{du}{dt}\Big\|_{L_2((t,t+\tau);\hat{H}^l)}. \tag{5.4}$$

Let us introduce the notion for norms of vector-functions components

$$\begin{aligned}\|w_p\|_{p,(l)}&=\big(\|w_p\|^2_{C([t_0,t_*];H^l(\Omega_p))}+h^2\|w_p\|^2_{C([t_0,t_*];H^{l+1}(\Omega_p))}\big)^{1/2},\\ \|w_p\|_{p,(t',t'')}&=\big(\|w_p\|^2_{L_2((t',t'');H^1(\Omega_p))}+h^2\|w_p\|^2_{L_2((t',t'');H^2(\Omega_p))}\big)^{1/2},\end{aligned} \tag{5.5}$$

and for norms of vector-functions

$$\begin{aligned}\|w\|_{(l)}&=\big(\|w\|^2_{C([t_0,t_*];\hat{H}^l)}+h^2\|w\|^2_{C([t_0,t_*];\hat{H}^{l+1})}\big)^{1/2},\\ \|w\|_{(t',t'')}&=\big(\|w\|^2_{L_2((t',t'');\hat{H}^1)}+h^2\|w\|^2_{L_2((t',t'');\hat{H}^2)}\big)^{1/2}.\end{aligned} \tag{5.6}$$

We notice that

$$\xi_1^{n+l/s} = \xi_1^n + \sum_{r=1}^{l} \eta_1^r. \tag{5.7}$$

It is not difficult to obtain the following estimation:

$$|\alpha_{1,1}^{n,l}(\xi_1^n)| \le \frac{\varepsilon_{1,1}}{2} d_{h,1}(\xi_1^n, \xi_1^n) + \frac{c_{1,1}}{\varepsilon_{1,1}} \tau_l \left\| \frac{d^2 u_1^\rho}{dt^2} \right\|^2_{L_2((t_n+\Delta_{l-1}, t_n+\Delta_l); L_2(\Omega_1))}. \tag{5.8}$$

Using inequality (3.2) and estimation (5.4) we get

$$|\alpha_{1,2}^{n,l}(\xi_1^n)| \le \frac{\varepsilon_{1,2}}{2} d_{h,1}(\xi_1^n, \xi_1^n) + \frac{c_{1,2}}{\varepsilon_{1,2}} h^2 \tau_l^{-1} \left\| \frac{du_1^\rho}{dt} \right\|^2_{1,(t_n+\Delta_{l-1}, t_n+\Delta_l)}. \tag{5.9}$$

Similarly we obtain the inequalities

$$|\alpha_{1,1}^{n,l}(\eta_1^r)| \le \frac{\varepsilon_{1,3}}{2} d_{h,1}(\eta_1^r, \eta_1^r) + \frac{c_{1,3}}{\varepsilon_{1,3}} \tau_l \left\| \frac{d^2 u_1^\rho}{dt^2} \right\|^2_{L_2((t_n+\Delta_{l-1}, t_n+\Delta_l); L_2(\Omega_1))}, \tag{5.10}$$

$$|\alpha_{1,2}^{n,l}(\eta_1^r)| \le \frac{\varepsilon_{1,4}}{2} d_{h,1}(\eta_1^r, \eta_1^r) + \frac{c_{1,4}}{\varepsilon_{1,4}} h^2 \tau_l^{-1} \left\| \frac{du_1^\rho}{dt} \right\|^2_{1,(t_n+\Delta_{l-1}, t_n+\Delta_l)}, \tag{5.11}$$

further

$$\beta_{1,1}^{n,l}(\xi_1^{n+l/s}) = \frac{1}{2} \beta_{1,1}^{n,l}(\xi_1^{n+(l-1)/s} + \xi_1^{n+l/s}) + \frac{1}{2} \beta_{1,1}^{n,l}(\eta_1^l). \tag{5.12}$$

The estimation of the functional $\beta_{1,1}^{n,l}(v_1)$ follows from the continuity of bilinear form (2.2) and the standard estimation of interpolation in the norm of space $H^1(\Omega_1)$ [4]

$$\begin{aligned} &|\beta_{1,1}^{n,l}(\xi_1^{n+(l-1)/s} + \xi_1^{n+l/s})| \\ &\qquad \le \frac{\varepsilon_{1,5}}{2} a_1\big(t_n + \Delta_{l-1}; \xi_1^{n+(l-1)/s} + \xi_1^{n+l/s}, \xi_1^{n+(l-1)/s} + \xi_1^{n+l/s}\big) + \\ &\qquad \frac{c_{1,5}}{\varepsilon_{1,5}} h^2 \|u_p^\rho\|^2_{C([t_0,t_*]; H^2(\Omega_1))}, \end{aligned} \tag{5.13}$$

$$|\beta_{1,1}^{n,l}(\eta_1^l)| \le \frac{\varepsilon_{1,6}}{2} a_1(t_n + \Delta_{l-1}; \eta_1^l, \eta_1^l) + \frac{c_{1,6}}{\varepsilon_{1,6}} h^2 \|u_p^\rho\|^2_{C([t_0,t_*]; H^2(\Omega_1))}. \tag{5.14}$$

From inequality (3.4) follows that

$$|\beta_{1,2}^{n,l}(v_1)| \le \frac{\varepsilon_{1,7}}{2} \|v_1\|^2_{L_2(\Gamma_{1,2})} + \frac{c_{1,7}}{\varepsilon_{1,7}} \rho^{-2} h^2 \|u^\rho\|^2_{(2)}. \tag{5.15}$$

Using inequality (5.4) and the trace theorem it yields

$$|\gamma_1^{n,l}(v_1)| \le \frac{\varepsilon_{1,8}}{2}\|v_1\|^2_{L_2(\Gamma_{1,2})} + \frac{c_{1,8}}{\varepsilon_{1,8}}\rho^{-2}\tau^{-1}\Big\|\frac{du_2^\rho}{dt}\Big\|^2_{L_2((t_n,t_{n+1});H^1(\Omega_2))}. \quad (5.16)$$

And finally, similarly to estimation (5.16), it is not difficult to obtain

$$|\delta_1^n(v_1)| \le \frac{\varepsilon_{1,9}}{2}\|v_1\|^2_{L_2(\Gamma_{1,2})} + \frac{c_{1,9}}{\varepsilon_{1,9}}\rho^{-2}\Big\{\tau^{-1}\Big\|\frac{du_2^\rho}{dt}\Big\|^2_{2,(t_n,t_{n+1})} + \|u_2^\rho\|^2_{2,(3)}\Big\}. \quad (5.17)$$

Besides the obtaining inequalities we will use the estimation of the norm $\|v_p\|_{L_2(\Gamma_{1,2})}$ [1]:

$$\|v_p\|^2_{L_2(\Gamma_{1,2})} \le c_0\Big(\frac{1}{\delta}d_{h,p}(v_p,v_p) + \delta a_p(t;v_p,v_p)\Big). \quad (5.18)$$

We notice that

$$d_{h,1}(\xi_1^{n+l/s},\xi_1^{n+l/s}) \le 2d_{h,1}(\xi_1^n,\xi_1^n) + 2l\sum_{r=1}^{l} d_{h,1}(\eta_1^r,\eta_1^r), \quad (5.19)$$

$$a_1(t_n+\Delta_{l-1};\xi_1^{n+l/s},\xi_1^{n+l/s}) \le \frac{1}{2}a_1(t_n+\Delta_{l-1};\eta_1^l,\eta_1^l) +$$
$$\frac{1}{2}a_1\big(t_n+\Delta_{l-1};\xi_1^{n+(l-1)/s}+\xi_1^{n+l/s},\xi_1^{n+(l-1)/s}+\xi_1^{n+l/s}\big). \quad (5.20)$$

At last, we have

$$\sum_{l=1}^{s}\tau_l\sum_{r=1}^{l} d_{h,1}(\eta_1^r,\eta_1^r) = \sum_{l=1}^{s}(\tau-\Delta_{l-1})d_{h,1}(\eta_1^l,\eta_1^l) \le \tau\sum_{l=1}^{s} d_{h,1}(\eta_1^l,\eta_1^l), \quad (5.21)$$

$$\sum_{l=1}^{s} l\tau_l\sum_{r=1}^{l} d_{h,1}(\eta_1^r,\eta_1^r) = \sum_{l=1}^{s}\Big(\sum_{r=1}^{l} r\tau_r\Big)d_{h,1}(\eta_1^l,\eta_1^l) \le \tau s\sum_{l=1}^{s} d_{h,1}(\eta_1^l,\eta_1^l). \quad (5.22)$$

Considering the presentations (5.7), (5.12) and inequalities (5.18)–(5.22) we will substitute the estimations (5.8)–(5.11), (5.13)–(5.17) into inequality (5.2). As a result it is obtained

$$d_{h,1}(\xi_1^{n+1},\xi_1^{n+1}) + \frac{1}{\rho}\sum_{l=1}^{s}\tau_l\int_{\Gamma_{1,2}}(P_{h,1}\xi_1^{n+(l-1)/s})^2 d\sigma +$$

$$\frac{1}{2}(1-\varepsilon_{1,5}-c_0\delta\varepsilon')\sum_{l=1}^{s}\tau_l a_1(t_n+\Delta_{l-1};\xi_1^{n+(l-1)/s}+\xi_1^{n+l/s},\xi_1^{n+(l-1)/s}+\xi_1^{n+l/s}) +$$

$$\sum_{l=1}^{s}\{(1-\tau\varepsilon_{1,3}-\tau\varepsilon_{1,4}-\frac{2c_0}{\delta}\tau s\varepsilon')d_{h,1}(\eta_1^l,\eta_1^l) +$$

$$\frac{1}{2}(1+\varepsilon_{1,6}+c_0\delta\varepsilon')\tau_l a_1(t_n+\Delta_{l-1};\eta_1^l,\eta_1^l)-\frac{\tau_l}{\rho}\int_{\Gamma_{1,2}}(P_{h,1}\eta_1^l)^2d\sigma\}\leq$$

$$(1+\tau(\varepsilon_{1,1}+\varepsilon_{1,2}+\frac{2c_0}{\delta}\varepsilon'))(d_{h,1}(\xi_1^n,\xi_1^n)+$$

$$\frac{\tau}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n-1+k/m})^2d\sigma+\psi_1^n, \tag{5.23}$$

where $\varepsilon'=\varepsilon_{1,7}+\tau\varepsilon_{1,8}+\tau\varepsilon_{1,9}$. Let us assume

$$\begin{gathered}\tau s\leq\alpha,\ \ \varepsilon_{1,1}=\varepsilon_{1,2}=\frac{3}{8},\ \ \varepsilon_{1,3}=\varepsilon_{1,4}=\frac{3}{8\tau}\alpha,\ \ \varepsilon_{1,5}=\varepsilon_{1,6}=\frac{1}{2},\\ \varepsilon_{1,7}=\frac{1}{12c_0},\ \ \varepsilon_{1,8}=\varepsilon_{1,9}=\frac{1}{12c_0\tau},\ \ \delta=2.\end{gathered} \tag{5.24}$$

Moreover, let $\tau\leq 1$. Then (5.23) takes the form

$$d_{h,1}(\xi_1^{n+1},\xi_1^{n+1})+\frac{1}{\rho}\sum_{l=1}^{s}\tau_l\int_{\Gamma_{1,2}}(P_{h,1}\xi_1^{n+(l-1)/s})^2d\sigma+\sum_{l=1}^{s}\mathcal{F}_{l,\alpha}(\eta_1^l)\leq$$

$$(1+\tau)d_{h,1}(\xi_1^n,\xi_1^n)+\frac{\tau}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n-1+k/m})^2d\sigma+\psi_1^n, \tag{5.25}$$

where

$$\mathcal{F}_{l,\alpha}(v_1)=(1-\alpha)d_{h,1}(v_1,v_1)-\tau_l(a_1(t;v_1,v_1)+\frac{1}{\rho}\int_{\Gamma_{1,2}}(P_{h,1}v_1)^2d\sigma). \tag{5.26}$$

The estimation of the functional $g_2^{n,k}$ brings (5.3) to the form

$$d_{h,2}(\xi_2^{n+1},\xi_2^{n+1})+\frac{\tau}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n+k/m})^2d\sigma\leq$$

$$(1+\tau)d_{h,2}(\xi_2^n,\xi_2^n)+\frac{1}{\rho}\sum_{i=1}^{s}\tau_l\int_{\Gamma_{1,2}}(P_{h,1}\xi_1^{n+(l-1)/s})^2d\sigma+\psi_2^n. \tag{5.27}$$

Let us denote

$$\theta^n=d_h(\xi^n,\xi^n)+\frac{\tau}{\rho}\sum_{k=1}^{m}\int_{\Gamma_{1,2}^{(k)}}(P_{h,2}\xi_2^{n-1+k/m})^2d\sigma. \tag{5.28}$$

Using additive presentation (3.6) and notion (5.28) the inequality (5.27) is summed with (5.26). As a result it is obtained the energetic inequality

$$\theta^{n+1} + \sum_{l=1}^{s} \mathcal{F}_{l,\alpha}(\eta_1^l) \leq (1+\tau)\theta^n + \psi^n, \tag{5.29}$$

where the function $\psi^n = \psi_1^n + \psi_2^n$ depends on h, τ, ρ and on Sobolev's norms of the function u^ρ. The form ψ^n will be defined later. As one can see from (5.29) the inequality

$$\sum_{l=1}^{s} \mathcal{F}_{l,\alpha}(\eta_1^l) \geq 0$$

is a sufficient condition for stability of method (4.2)–(4.4).

6. Stability

In this paper we will consider the scheme with constant step $\tau_l = \tau/s$. In this case $\mathcal{F}_{l,\alpha}(v_1) = \mathcal{F}_\alpha(v_1)$. According to the condition of stability $\sum_{l=1}^{s} \mathcal{F}_{l,\alpha}(\eta_1^l) \geq 0$ we have a sufficient condition

$$\mathcal{F}_\alpha(v_1) \geq 0, \quad \forall v_1 \in V_{h,1}. \tag{6.1}$$

Henceforward, we will use vector-matrix form of condition (6.1). Let $\bar{v}_1$ be a vector with components $\rho_{1,i}v_1(\bar{x}_1)$, where $\rho_{1,i}^2 = d_{h,1}(\varphi_{1,i}, \varphi_{1,i})$. The matrix A with elements $\frac{1}{\rho_{1,i}\rho_{1,j}} a_1(t; \varphi_{1,i}, \varphi_{1,j})$ and the diagonal matrix D with elements $\frac{1}{\rho_{1,i}^2} \int_{\Gamma_{1,2}} (P_{h,1}\varphi_{1,i})^2 d\sigma$ are also defined. Let us assume $A_\rho = A + \frac{1}{\rho} D$, then,

$$\mathcal{F}_\alpha(v_1) = (1-\alpha)\|\bar{v}_1\|_2^2 - \frac{\tau}{s}\|\bar{v}_1\|_{A_\rho}^2. \tag{6.2}$$

If the symmetric matrix $M_\alpha = (1-\alpha)E - \frac{\tau}{s}A_\rho$ is introduced, then (6.2) takes the form

$$\mathcal{F}_\alpha(v_1) = \langle M_\alpha \bar{v}_1, \bar{v}_1 \rangle_2. \tag{6.3}$$

Thus the condition of stability is a nonnegativity of the matrix M_α. Let

$$\|A_\rho\| \leq \lambda_\rho, \tag{6.4}$$

and it is not difficult to note that $\lambda_\rho = \lambda_0 \frac{1}{h}(\frac{1}{h} + \frac{1}{\rho})$, where positive number λ_0 does not depend on h and ρ. It is defined

$$\bar{\lambda}_{\rho,0} = \min\left\{\lambda_0 : \ \lambda_\rho \geq \|A_\rho\|, \ \frac{1}{1-\alpha}\tau\lambda_\rho \text{ is a whole number}\right\},$$

$$\bar{\lambda}_\rho = \bar{\lambda}_{\rho,0}\frac{1}{h}\Big(\frac{1}{h}+\frac{1}{\rho}\Big). \tag{6.5}$$

Let us assume

$$s = \frac{1}{1-\alpha}\tau\bar{\lambda}_\rho, \tag{6.6}$$

then,

$$\lambda(M_\alpha) = 1-\alpha-\frac{\tau\lambda(A_\rho)}{s} = \frac{1-\alpha}{\bar{\lambda}_\rho}(\bar{\lambda}_\rho-\lambda(A_\rho)) \geq \frac{1-\alpha}{\bar{\lambda}_\rho}(\bar{\lambda}_\rho-\|A_\rho\|) \geq 0.$$

Therefore, by virtue of (6.6) M_α is a non-negative matrix. Using the equality (6.6) and the condition $\tau s \leq \alpha$ we may obtain the connection between parameters τ, s, h and ρ. There is $\tau^2\bar{\lambda}_\rho \leq \alpha(1-\alpha)$. Therefore, taking into account (6.5), we have

$$\tau^2 \leq \alpha(1-\alpha)\frac{h^2}{\bar{\lambda}_{\rho,0}(1+\frac{h}{\rho})}.$$

Since $\alpha(1-\alpha)$ takes the maximum meaning for $\alpha = 1/2$, then,

$$\tau \leq \frac{h}{2}\sqrt{\frac{1}{\bar{\lambda}_{\rho,0}(1+\frac{h}{\rho})}}. \tag{6.7}$$

From condition (6.6) ($\alpha = 1/2$) it is obtained

$$s = 2\bar{\lambda}_{\rho,0}\Big(1+\frac{h}{\rho}\Big)\frac{\tau}{h^2}. \tag{6.8}$$

The formulas (6.7) and (6.8) are the conditions of stability of the considering method.

7. Error analysis (completion)

From inequality (5.26) with condition $\sum_{l=1}^{s}\mathcal{F}_{l,\alpha}(\eta_1) \geq 0$ we obtain

$$\theta^{n+1} \leq (1+\tau)\theta^n + \psi^n, \tag{7.1}$$

where

$$\begin{aligned}\psi^n = c'\Big\{&h^2\Big(\tau^2\|u^\rho\|^2_{(3)} + \Big\|\frac{du^\rho}{dt}\Big\|^2_{(t_n,t_{n+1})}\Big) + \tau^2\Big\|\frac{d^2u^\rho}{dt^2}\Big\|^2_{\hat{L}_2((t_n,t_{n+1});\hat{L}_2)} \\ &+ \rho^{-2}\tau\Big(h^2\|u^\rho\|^2_{(2)} + \tau^2\|u^\rho\|^2_{(3)} + \tau\Big\|\frac{du^\rho}{dt}\Big\|^2_{(t_n,t_{n+1})}\Big)\Big\}.\end{aligned} \tag{7.2}$$

The use of the grid Gronwall lemma ([6], p. 311) for the inequality (7.1) considering the notion (5.25) for θ^n and the positive definiteness in $\hat{L}^2$ of bilinear form $d_h(v,w)$ gives the following estimation:

$$\|\xi^n\|_{\hat{L}_2} \leq c''\big[M_h h + M_\tau \tau + \rho^{-1}(M_{\rho,h}h + M_{\rho,\tau}\tau)\big], \tag{7.3}$$

where according to expression (5.5) and inequality (2.13) we introduce the following notions:

$$\begin{aligned} M_h &= \|u\|_{(3)} + \left\|\frac{du}{dt}\right\|_{(t_0,t_*)}, & M_\tau &= \left\|\frac{d^2u}{dt^2}\right\|_{L_2(t_0,t_*);\hat{L}_2)}, \\ M_{\rho,h} &= \|u\|_{(2)}, & M_{\rho,\tau} &= M_h. \end{aligned} \tag{7.4}$$

and vector-function u is the solution to problem (2.5), (2.6). Finally from (7.3), the triangle inequality and the estimation of the interpolation in the norm of space $\hat{L}_2$ the resulting estimation follows.

Theorem 7.1. *Let condition* (4.4) *for problem* (2.5), (2.6) *hold. Then from conditions* (6.7), (6.8) *for method* (4.1)–(4.3) *with* $\tau \leq 1$ *and* $\rho \leq \rho_0$ *the following error estimation is valid*

$$\max_{1\leq n\leq N} \|u^n - u^\rho(t_n)\|_{\hat{L}_2} \leq c\big[M_h h + M_\tau \tau + \rho^{-1}(M_{\rho,h}h + M_{\rho,\tau}\tau)\big],$$

where the positive numbers c *and* ρ_0 *do not depend on* h, τ, ρ *and vector-function* u, *the numbers* M_h, M_τ, $M_{\rho,h}$ *and* $M_{\rho,\tau}$ *are defined by formulas* (7.4).

Thus the method (4.1)–(4.3), which is used for solving the problem with nonideal contact (parameter ρ is fixed), has the error estimation $O(h^{1/2} + \tau^{1/2})$. Similarly [6], the error estimation of method (4.1)–(4.3) for solving problem (2.5), (2.6) follows from triangle inequality, the estimation (2.12), Theorem 7.1 and the optimisation of the right-hand side of obtaining inequality by parameter ρ.

Theorem 7.2. *Let the conditions of Theorem 7.1 be valid. Then for method* (4.1)–(4.3) *with* $\rho = c'(h+\tau)^{1/2}$ *the following error estimation is valid*

$$\max_{1\leq n\leq N} \|u^n - u(t_n)\|_{\hat{L}_2} \leq c(M_h' h^{1/2} + M_\tau' \tau^{1/2}),$$

where $M_h' = M_{\rho,h} + M_h h^{1/2}$, $M_\tau' = M_{\rho,\tau} + M_\tau \tau^{1/2}$.

Thus, the method (4.1)–(4.3) for solving problem (2.5), (2.6) has the error estimation $O(h^{1/2} + \tau^{1/2})$. Fall of the degree of h as compared with [3] is the result of using lumping operators on compound grid. If we consider the co-ordinate grid in the domain Ω, then using the estimation (3.5) instead of (3.4) we may obtain the error estimation $O(h^{3/4} + \tau^{1/2})$ with $\rho = c'(h^{3/2} + \tau)^{1/2}$.

8. Numerical experements

Let us introduce some test calculations by the suggested method. The prime purpose is the confirmation of asymptotics of parameter s and $\hat{L}_2$-norm of error estimation, which are given in Theorem 7.2. It is possible to draw a conclusion that the receiving estimations are not improved.

In the square $\Omega = (0,1)\times(0,1)$ we consider the simplest two-dimensional parabolic problem with the Dirichlet conditions

$$\frac{du}{dt} = \lambda_0 \Delta u, \quad (t, x_1, x_2) \in (0,1) \times \Omega,$$
$$u(t, x_1, x_2) = 0, \quad (t, x_1, x_2) \in (0,1) \times \Gamma,$$
$$u(0, x_1, x_2) = \sin \pi x_1 \sin \pi x_2, \quad (x_1, x_2) \in \Omega.$$

The exact solution to this problem is the function

$$u(t, x_1, x_2) = e^{-2\lambda_0 \pi^2 t} \sin \pi x_1 \sin \pi x_2.$$

In calculations we suppose $\lambda_0 = 0.05$. For this value of parameter λ_0 $\hat{L}_2$-norm of the solution within time $t = 1$ reduces approximately in e^{-1} times. We consider the following partition of the domain

$$\Omega_{1,1} = (0, 5/8) \times (0, 5/8), \quad \Omega_{1,2} = (5/8, 1) \times (5/8, 1), \quad \Omega_1 = \Omega_{1,1} \cup \Omega_{1,2},$$
$$\Omega_{2,1} = (5/8, 1) \times (0, 5/8), \quad \Omega_{2,2} = (0, 5/8) \times (5/8, 1), \quad \Omega_2 = \Omega_{2,1} \cup \Omega_{2,2}.$$

All calculations presented below, have been carried out for square grid with constant step h_{pk} in the subdomain $\Omega_{p,k}$. For the grid analog of $L_2(\Omega_{p,k})$-norm estimation in time $t = 1$ we use the notion

$$\varepsilon_{pk} = \Big\{ \sum_{i \in I_{pk}} h_{pk}^2 \big[u^N(x_{1,i}, x_{2,i}) - u(1, x_{1,i}, x_{2,i})\big]^2 \Big\}^{1/2},$$

where $N\tau = 1$. $\hat{L}_2$-norm estimation is a set of equality

$$\varepsilon = \left(\varepsilon_{11}^2 + \varepsilon_{12}^2 + \varepsilon_{21}^2 + \varepsilon_{22}^2\right)^{1/2}.$$

Two series of calculations for co-ordinate square grid with constant space step were executed. In the first case we suppose the step $h_1 = h_{pk} = 2^{-4}$, in the second case we suppose the step $h_2 = h_{pk} = 2^{-5}$. Parameter ρ is defined by the formula from remark in Section 6

$$\rho = \frac{1}{2}(h^{3/2} + \tau)^{1/2}.$$

As one can see from Table the parameter s with reduction τ in 2 times also reduces in 2 times. Besides with reduction h parameter s increased approximately in 4 times. Thus formula (6.8) is confirmed.

$h=2^{-4}$			$h=2^{-5}$		
τ	s	ε	τ	s	ε
	6	0.80140		25	1.4056
2^{-3}	7	0.045848	2^{-3}	26	0.041997
	8	0.045851		27	0.041997
	3	10.290		12	$5.5024 \cdot 10^{7}$
2^{-4}	4	0.032915	2^{-4}	13	0.029793
	5	0.032920		14	0.029793
	1	$4.3852 \cdot 10^{9}$		6	$6.0708 \cdot 10^{9}$
2^{-5}	2	0.023903	2^{-5}	7	0.021249
	3	0.023913		8	0.021249
				3	$6.4076 \cdot 10^{12}$
2^{-6}	1	0.017712	2^{-6}	4	0.015347
	2	0.017732		5	0.015347

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