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Monte Carlo methods for estimating the time dependence during the process of radiation transfer*

G.Z. Lotova

Let us consider the problem of estimating the reflected light intensity. This problem arises when studying the interference of backward scattering in laser sensing of the ocean from the atmosphere. Let t_a be the time it takes for the intensity to achieve some asymptotical function. In this paper, a new method of calculating the time t_a is proposed.

It is well-known [1] that the whole light intensity $I(t)$ at the point r^* at the time t is

$$I(t) = \int_{V_0} \Phi(r^*, v, t)\, dv.$$

Here $\Phi(r^*, v, t)$ is the intensity, v is the velocity vector, V_0 is the solid angle of the detector. To calculate this integral by a Monte Carlo method, it is necessary to consider the Markov chain with states $x_n = (r_n, v_n, t_n)$ and apply the local estimate [1]:

$$\xi = \sum_{n=0}^{N} Q_n h_0(x_n),$$

where r_n, v_n, and t_n are the coordinate, the velocity, and the time in state with number n,

$$\begin{aligned} h_0(x_n) \; = \; & \frac{\sigma_s(r_n, v_n)}{\sigma(r_n, v_n)} \frac{w_s(v_n \to v_n^*)}{|r_n^* - r_n|^2} \sigma(r(\ell, r_n, \omega), v) \exp(-\tau_{\rm op}(\ell, r_n, \omega), v) \times \\ & \Delta_{V_0}(v_n^*) \delta\Big(t_n + \frac{r_n - r_n^*}{v_n^*} - t\Big), \end{aligned}$$

σ is the reduction coefficient, σ_s is the scattering coefficient, w_s is the indicatrix of scattering, Δ_{V_0} is the indicator of the domain V_0, v_n^* is the velocity with the new direction $\omega_n^* = (r_n^* - r_n)/|r_n^* - r_n|$,

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$$\ell = |r_n^* - r_n|, \qquad \omega = v/|v|, \qquad r(\ell, r_n, \omega) = r_n + \ell\omega,$$

$$\tau_{\text{op}}(\ell, r_n, \omega) = \int\limits_0^{\ell} \sigma(r(s, r_n, \omega), v)\, ds$$

is the optical length of the particle path, Q_n are the special weights [2].

It is known [1] that $I(t)$ asymptotically behaves as follows:

$$I(t) \sim At^{-5/2}\exp(-\lambda t) = I_0(t), \tag{1}$$

where A is a constant. The value λ is equal to $(\sigma - \sigma_s)v$ for the homogeneous medium. From (1) we obtain

$$\frac{I'(t)}{I(t)} \sim -\frac{5}{2t} - \lambda,$$

whence

$$R(t) = \frac{I'(t)}{I(t)} + \frac{5}{2t} \sim -\lambda.$$

Therefore, if we calculate $R(t)$ at a large t, then we can find the unknown parameter λ. The time t_a is the time at which the function $R(t)$ approaches a constant.

In what follows, we propose two ways of estimating the values $I(t)$ and $I'(t)$ by a Monte Carlo method.

1. Assume that $f(r, v, t)$ is distribution density of particle source, $\eta(r, v)$ is collision estimate for the functional

$$J_h^{(0)}(r_0, v_0) = \int\limits_R \int\limits_V \int\limits_0^{\infty} \varphi_0(r, v, \tau; r_0, v_0)|h(r, v)|\, dr\, dv\, d\tau$$

where φ_0 is distribution density of collisions that is a result of one collision at the point (r_0, v_0). Let us use the following theorems [4].

Theorem 1. *Suppose that the point (r_0, v_0) is distributed for $t_0 \equiv 0$ with the density $f_0(r, v)$. Moreover,*

$$\left|\frac{f(r_0, v_0, t)}{f_0(r_0, v_0)}\right| < C < +\infty, \quad f \in L_1(R \times V \times T)$$

and $f_0\mathbf{E}\eta^2 \in L_1(R \times V)$.

Then the relation $I(t) = \mathbf{E}\xi_t$ *holds, where*

$$\xi_t = \sum_{n=0}^{N} Q_n h(r_n, v_n) \frac{f(r_0, v_0, t - t_n)}{f_0(r_0, v_0)}, \qquad Q_0 \equiv 1. \tag{2}$$

In addition, $\mathbf{D}\xi_t < +\infty$.

Theorem 2. *Suppose that the function* $f_t^{(m-1)}(x)$ *is absolutely continuous in* t *on any finite time interval for any* $(r, v) \in R \times V$ *and* $|f_t^{(i)}| \le C f_0(r, v)$ *for almost all* x *and* $i = 0, 1, \dots, m$. *If, moreover, the conditions of Theorem 1 with substitution* $f \to f_t^{(m)}$ *take place, then the relation* $I^{(m)} = \mathbf{E}\xi_t^{(m)}$ *holds. Here*

$$\xi_t^{(m)} = \sum_{n=0}^{N} Q_n h(r_n, v_n) \frac{f^{(m)}(r_0, v_0, t - t_n)}{f_0(r_0, v_0)}. \tag{3}$$

In addition, $\mathbf{D}\xi_t^{(m)} < +\infty$.

Thus, we can find the values $I(t)$ and $I'(t)$ for sufficiently smooth distribution function of the source by formulae (2) and (3).

Let us consider model problem of estimating of reflected light intensity [2]. Simplified system "atmosphere-ocean" is defined as follows. Medium that scattered the radiation ("ocean") fills a half-space $z \le 0$. Whole reduction coefficient is $\sigma = 0.216m^{-1}$, scattering coefficient is $\sigma_s = 0.175m^{-1}$, so that survival probability for "quantum" of radiation at interaction is $q = 0.81$. Cosine of scattering angle μ for the "quantum" is simulated by formula [3]

$$\mu = \frac{1}{2\mu_0}\left(1 + \mu_0^2 - \left(\frac{1 - \mu_0^2}{2\mu_0\alpha + 1 - \mu_0}\right)^2\right), \qquad \mu_0 = 0.9,$$

that corresponds to the known indicatrix of Henyey–Greenstein

$$w_s(\mu) = \frac{1}{2}\frac{1 - \mu_0^2}{(1 + \mu_0^2 - \mu_0\mu)^{3/2}}.$$

Here α is a random value uniformly distributed in the interval $(0, 1)$. Upper half-space $z > 0$ is filled by vacuum. Instantaneous source (with the dependence $\delta(t)$) is located at the point with the coordinates $(0, 0, H)$ and radiate the "quantums" in the direction $\omega = (0, 0, -1)$ along negative semi-axis of z. The radiation detector registers distribution (with respect to time) of whole particle flux at the same point $(0, 0, H)$.

The time t_a is calculated by Monte Carlo method. In the problem mentioned above the parameter λ is known and is equal to $\sigma_c v = (\sigma - \sigma_s)v = -0.009225$. The calculations were made for the model with the following parameters $\sigma^{(c)} = 0$, $\sigma^{(1)} = \sigma_s$, $\sigma_s^{(1)} = \sigma_s$, $H = 5$ m, $v = 0.225$ m/nsec. The source angle is $\gamma = 0.001$ rad. Corresponding weights have an exponential type $Q_n = \exp(-\sigma_c L_n)$, where L_n is particle length of path from entrance to the "ocean". Since $\sigma_c^{(1)} = 0$, the trajectory can be terminated, when the inequality $L_n + \ell > vT_0$ is achieved for some T_0. For t_a estimating we use artificial model source with $f_0(t) = t\exp(-\sigma vt)$. It is clear that this way gives overestimated value of t_a that is satisfied for the practice.

Numerical experiment show us that t_a is greater than $t_0 = 370$ nsec that corresponds to dimensionless time $\sigma vt \approx 17$.

The ratio of $I(t)$ intensity to asymptotical function $I_0(t)$:

t	364.4	377.0	389.6	402.1
$I(t)/I_0(t)$	0.9569	0.9808	1.0032	1.0000

2. The method considered above is not applicable for estimating the function $I(t)$ when the particle source is impulsive over time. This source is practically important. Suppose that the function $\sigma_t^{(n-1)}(r_0 + \omega t)$ is totally continuous and $\sigma(r_0 + \omega t) = 0$ for $t < 0$. We take advantage of direct Fourier transform and then inverse one for the local estimate of Monte Carlo method [4]. For the inverse Fourier transform we bound the interval of integration by the segment $[-S, S]$ for comparatively big value of S. We obtain that

$$2\pi \int_{V_0} \Phi(r^*, v, t)\, dv \approx \mathbf{E} \sum_{n=0}^{N} Q_n h_S(x_n) \tag{4}$$

where

$$h_S(x_n) = h_0(x_n) \frac{\sin(S(t_n^* - t))}{t_n^* - t}.$$

Since the segment $[-S, S]$ is bounded, the estimate of intensity obtained by formula (4) have fluctuations. Therefore, it is important to calculate integrals of the function $I(t)$ over time in the intervals with the length $2\pi/S$ to the given limit T_0. Note that these integrals of the function $\sin(S(t_n - t))/(t_n - t)$ can be expressed in terms of integral sine

$$\int_{t^{(1)}}^{t^{(2)}} \frac{\sin(S(t_n - t))}{t_n - t}\, dt = \mathrm{si}(S(t^{(2)} - t_n)) - \mathrm{si}(S(t^{(1)} - t_n)).$$

In order to calculate the function $I'(t)$, it is enough to differentiate the estimate of Monte Carlo method for $I(t)$. For the model problem described above the factor A in asymptotic (1) is calculated by this relation

$$J(t) = \int_{t}^{t+\Delta t} I(\hat{t})\, d\hat{t} \approx \int_{t}^{t+\Delta t} \exp(-\sigma_c v\hat{t})\hat{t}^{-5/2}\, d\hat{t} = J_0(t).$$

In calculations, we use the following constants $S = 2$, $\Delta t = 12.56637$. Computational experiments shows that t_a is close to $t_0 = 380$ nsec that corresponds to dimensionless time $\sigma v t_0 \approx 18$. The obtained results stay in the agreement with results in [1].

The ratio of integral of $J(t)$ intensity to asymptotical function $J_0(t)$:

t	364.4	377.0	389.6	402.1
$J(t)/J_0(t)$	0.9569	0.9808	1.0032	1.0000

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