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Test nonlinear heat transfer equations with known solutions

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A class of nonlinear 1-D parabolic equations with known solutions are introduced. The computer programs for estimation of the absolute errors of numerical methods are decribed.

1. Introduction

Nonlinear one-dimensional heat transfer equations are now the standard tools in many engineering applications [1]. There are different numerical methods for their analysis: finite differences, finite elements, etc. The errors of the numerical solutions are caused by space and time discretization. The absolute values of the error are as a rule unknown explicitly and only their estimations can be received through the repeated solution of the same problem with decreased space and time steps. The objective of this work is to describe the system of one-dimensional nonstationary nonlinear heat transfer equations with known solutions. The numerical solution of the test equation can be compared with their analytical solution for receiving the values of the computational errors. The computer programs for receiving the exact solutions and the example of their using are briefly described.

2. Test equations

The test heat transfer equations have the form

$$c_0 g_1(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial l}\Big(k_0 g_2(T)\frac{\partial T}{\partial l}\Big) - aG(T), \quad \tau > 0, \ 0 < l < L,$$

$$T\Big|_{\tau=0} = T_0(l), \quad k_0 g_2 \frac{\partial T}{\partial l}\Big|_{l=0} = 0, \quad k_0 g_2 \frac{\partial T}{\partial l}\Big|_{l=L} = h(F(\tau) - G(T)),$$

where g_1, g_2 are dimensionless functions, $F(\tau)$, $G(T)$ have the dimension of the temperature.

The standard transformation to dimensionless variables

$$t = \frac{k_0 \tau}{c_0 L^2}, \quad x = \frac{l}{L}, \quad \theta = \frac{T}{T_{\text{ref}}}$$

gives the equation

$$\begin{aligned} & g_1(\theta)\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\Big(g_2(\theta)\frac{\partial \theta}{\partial x}\Big) - Ag(\theta), \quad t > 0, \ 0 < x < 1, \\ & \theta|_{t=0} = \theta_0(x) = \frac{T_0}{T_{\text{ref}}}, \\ & g_2\frac{\partial \theta}{\partial x}\Big|_{x=0} = 0, \quad g_2\frac{\partial \theta}{\partial x}\Big|_{x=1} = \text{Bi}(f(t) - g(\theta)), \end{aligned} \tag{1}$$

where

$$A = \frac{aL^2}{k_0}, \quad g = \frac{G}{T_{\text{ref}}}, \quad \text{Bi} = \frac{hL}{k_0}, \quad f = \frac{F(\tau)}{T_{\text{ref}}}.$$

In general case, only the numerical technique is applicable for the solution of equation (1). For the receiving the exact solution we must choose the special form of the nonlinear functions

$$g_1(\theta) = g_2(\theta) = g'(\theta).$$

The introduction of the new dependent variable

$$u = g(\theta) \tag{2}$$

transforms the nonlinear problem (1) into the linear one

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - Au, \quad t > 0, \ 0 < x < 1, \tag{3}$$

$$u|_{t=0} = u_0(x) = g(\theta_0), \quad \frac{\partial u}{\partial x}\Big|_{x=0} = 0, \quad \frac{\partial u}{\partial x}\Big|_{x=1} = \text{Bi}(f(t) - u). \tag{4}$$

The solution to problem (3), (4) for

$$u_0(x) = \text{const} = f(0)$$

can be represented in the form

$$u(t,x) = f(t) - \sum_{j=1}^{\infty} U_j(t)\cos(\lambda_j x) \tag{5}$$

where the eigenvalues $\{\lambda_j\}$ are defined as the positive roots of the characteristic equation

$$y \sin(y) = \mathrm{Bi} \cos(y)$$

and

$$U_j(t) = \int_0^t ds H(s) \exp(-\gamma_j(t-s)),$$
$$\gamma_j = \lambda_j^2 + A \geq 0,$$
$$H(s) = Af(s) + f'(s).$$

After calculation of the sum (5) with high relative accuracy we can receive the solution of the nonlinear problem (1) by using the back transformation

$$\theta = g^{-1}(u).$$

The relative accuracy of θ can be connected with the realative accuracy of u:

$$u + \delta u = g(\theta + \delta\theta) \approx g(\theta) + g\prime\delta\theta,$$
$$\varepsilon_\theta = \left|\frac{\delta\theta}{\theta}\right| \leq K_a \left|\frac{\delta u}{u}\right| = K_a \varepsilon_u$$
$$K_a = \max\left|\frac{g(\theta)}{\theta g'(\theta)}\right|. \tag{6}$$

Maximum in (6) is defined over the range $(0 \leq \theta \leq 10)$.

3. Nonlinear transformations

Nonlinear transformatins (2) must satisfy some conditions for preserving the physical sence of the boundary problem (1) :

1) because $u(t, x)$ represents the dimensionless temperature,

$$0 \leq g(\theta) = u < \infty \quad \text{for } 0 \leq \theta < \infty,$$

2) functions $(g_1,\ g_2)$ must be nonnegative, therefore,

$$g'(\theta) \geq 0,$$

3) for reducing the number of the transformation parameters it will be usefull to put

$$g(0) = 0, \quad g(\infty) = \infty, \quad g(1) = 1,$$

4) the change of the relative accuracy, defined by (6), must not be too large,

5) the nonlinear problem (1) must have some similarity with the real situation.

For example, function $g(\theta) = \theta^4$ can simulate the radiative-conduction heat transfer. Seven transformation $g = g_{\rm Itr}(\theta)$ was realized in computer programs which will be described below. These transformations are represented in the table

Standard nonlinear transformations

Itr	$g(\theta)$	$g'(\theta)$	K_a
1	θ	1	1
2	θ^2	2θ	0.5
3	θ^4	$4\theta^3$	0.25
4	$\exp(\theta\ln 2) - 1$	$\ln(2)\exp(\theta\ln 2)$	1.44
5	$\theta^{1/2}$	$0.5\theta^{-1/2}$	2
6	$\theta^{1/4}$	$0.25\theta^{-3/4}$	4
7	$\ln(1+\theta)/\ln 2$	$1/((1+\theta)\ln 2)$	2.53

For linear problem (4) the function $f(t)$ plays the role of the external temperature. If

$$0 \le f_{\min} \le f(t) \le f_{\max} < \infty,$$

then

$$f_{\min} \le u \le f_{\max},$$

therefore,

$$g^{-1}(f_{\min}) \le \theta \le g^{-1}(f_{\max}).$$

The coefficient

$$K_{\rm tr} = \frac{g^{-1}(f_{\max}) - g^{-1}(f_{\min})}{f_{\max} - f_{\min}} \tag{7}$$

estimates the value of the decreasing ($K_{\rm tr} < 1$) or increasing ($K_{\rm tr} > 1$) of the temperature gradients after the nonlinear transformation. The transformations with Itr $= 2, 3, 4$ decrease temperature variation, with Itr $= 5, 6, 7$ – increase.

4. External functions

The external functions $f(t)$ are taken in the form

$$f(t) = \sum_{i=1}^{n} p_i f_{m_i}(t), \tag{8}$$

where $p_i \geq 0$, $\sum p_i = 1$, $m_i \in \{1,2,3,4\}$. Basic external functions f_m represent four main types of temeperature variation [1]

$m = 1:$ Nonresonance relaxation from f_0 to f_{end}

$$f_1 = f_{\text{end}} + (f_0 - f_{\text{end}})\exp(-t/t_{\text{rel}})$$
$$t_{\text{rel}} \neq t_j = 1/\gamma_j, \quad j = 1, \ldots, \infty.$$

Positivity conditions: $f_0 \geq 0$, $f_{\text{end}} \geq 0$.

$m = 2:$ Resonance relaxation from f_0 to f_{end}

$$f_2 = f_{\text{end}} + (f_0 - f_{\text{end}})\exp(-t/t_j)$$
$$t_{\text{rel}} = t_j = 1/\gamma_j, \quad \text{for some } j.$$

Positivity conditions: $f_0 \geq 0$, $f_{\text{end}} \geq 0$.

$m = 3:$ Periodic osillation around f_0

$$f_3 = f_0 + (f_0 - f_{\min})\sin(2\pi t/t_{\text{osc}}).$$

Positivity conditions: $f_0 \geq 0$, $f_{\min} \geq 0$.

$m = 4:$ Relaxation from f_0 to f_{end} with oscillation

$$f_4 = f_{\text{end}} + (f_0 - f_{\text{end}})\cos(2\pi t/t_{\text{osc}})\exp(-t/t_{\text{rel}}).$$

Positivity conditions: $0 \leq f_0 < 2f_{\text{end}}$, or
$2f_{\text{end}} < f_0 = f_{\text{end}}(2 + R)$, $R > 0$,
$(1 + Z^2)^{1/2} \leq (1 + R)\exp(-ZY_{\min})$,
where $Z = t_{\text{osc}}/(2\pi t_{\text{rel}})$, $Y_{\min} = \pi - \text{arctg(Z)}$.

The nonnegative weights p_i in (8) are defined by their relations. For example, for $n = 4$ the relations $2:1:1:1$ define $p_1 = 0.4 = 2/(2+1+1+1)$.

5. Computer realization

Calculation of the exact solution of eqauation (1) was realized on Borland's Turbo Pascal 6.0 for IBM PC. The computer code consists of the Pascal unit **TestTpu1.pas** and two programs, **Step1.pas** and **Step2.pas**.

The program **Step1.pas** prepares the data for next steps of caclculation. The program **Step2.pas** demonstrates the example of the estimation the accuracy of the numerical solution of the test nonlinear heat transfer equation by the simplest finite-difference scheme. The file **readme.pas** has the short instruction for users, the files **tstres1.pas, tstres2.pas, tstres3.pas** contains the results of running the test problem.

The input data for the test example are listed below in the left column of the table:

1.36	Bi = 1.36, this value Bi gives $\lambda_1^2 = 0.91663$,
-0.27	$A = -0.27$ ($A + \lambda_1^2 > 0$) the response time $t_{sys} = 1.54648$,
3.0	t_{max}, the limiting time for drawing f(t),
4	$n = 4$, the number of the basic external function in (8),
1 1.0	$m_1 = 1$, $p_1 \sim 1.0$,
0.51 2.356 1.37	f_0, f_{end} and t_{rel} for f_1,
2 1.0	$m_2 = 1$, $p_2 \sim 1.0$,
0.51 2.356 1	f_0, f_{end} and $j = 1$ for f_2,
3 0.13	$m_3 = 3$, $p_3 \sim 0.13$,
0.51 0.1 0.38	f_0, f_{min} and t_{osc} for f_3,
4 1.0	$m_4 = 1$, $p_1 \sim 1.0$,
0.51 2.356 0.5 2.0	f_0, f_{end}, t_{rel} and t_{osc} for f_4.

The program calculates the values f_{min} and f_{max}

f_{min}=0.51, f_{max}=2.121337.

On the screen the user sees the external function $f(t)$ for $0 \leq t \leq t_{max} = 3.0$. This picture can be printed with any screen-printing program which works in the graphic regime. Then the user can see the table of the standard transformations with computed values of K_{tr} for every transformation.

5 Itr = 5, $g = \theta^{1/2}$, $\theta = u^2$.

The results of the first test is demonstrated on the screen: the values of the stationary part of the sum (5) and the steady-state solution of (3) in the knots of the uniform grid. The maximum relative error defines the number of the correct digits for computed values of $u(t, x)$:

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MaxRelError=3.634845e-7, Ndig=6.
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For nonstationary state the arruracy will be not worse than for stationary one.

After the arruracy determination, the user must define the time for which he whishes to calculate the sum (5). If problem (3) has the steady-state regime, the sum (5) for $t \gg t_{sys}$ will be very close to stationary solution. In the test example the external function contains the periodic component f_3 and $t = 2.0$ is not larger sufficiently $t_{sys} = 1.55$. The results of the running of the program **Step1.pas** are saved in the file **tstres1.pas**.

The program **Test2.pas** demonstrates the application of the test programs. Nonlinear problem (1) is solved on the uniform grid with $\delta x = 1/\mathrm{mX}$, ($\mathrm{mX} = 5$) by explicit finite-difference scheme with constant time steps

$$\delta t < 0.5(\delta x)^2 = 0.02.$$

Results of the computations are saved in the file **tstres2.pas** for $\delta t = 0.01$ for times $t = 0.1k$, $k = 1, \ldots, 10$. The file **tstres3.pas** contains solution of the same problem with $\delta t = 0.005$.

References

[1] Y. Yaluria, K.E. Torrance, *Computational Heat Transfer*, Hemisphere Publishing Company, Washington, 1986, 118–147.