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A qualitative theory to explain the fractal properties of convective turbulence*

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The hydrodynamics statistical model which qualitatively explains the fractal features of the atmospheric convection has been suggested.

1. Introduction

It is well-known from the data of measurements that the turbulent structures of various scales are similar to one another [5]. This phenomenon is clearly exhibited by convective turbulence. Individual convective pulsations (sometimes called convective cells) of the turbulence range in size from millimeter-sized perturbations [1] to thermals, cumulus, thunderstorm clouds, and convective supercells [13]. Such supercells range in size from tens of meters to tens of kilometers. An isolated convective cell is an atmospheric thermal which is by 3 to 4 orders of magnitude smaller, and a thermal is a miniature convective cloud. Also, the quasi-ordered mesoscale convective structures, cloud rolls, and cloud supercells of honeycomb structure surprisingly resemble (on a scale enlarged by several orders of magnitude) the quasi-ordered structures which occur in a viscous liquid during convection between two flat plates with different temperatures.

It is a tradition to use two methods, the probabilistic and hydrodynamic ones, in the simulation of turbulent flows. The comprehensive models based on the hydrodynamic equations describe a complex process which has some probabilistic properties and is close to a stochastic process. The probabilistic models based on the processing of measurements have similar properties. The hydrodynamic simulation of microscale convective turbulent regimes is realized with the use of the Boussinesq equations [2, 4, 6], and the mesoscale regimes are simulated by using the equations of deep [16] and shallow [3] convection. In these equations, the microscale turbulence is parameterized in one way or another. The large eddy simulation approach [3, 16] in which the turbulent pulsations smaller than 100 m are parameterized is widely accepted. Such models can give an accurate description of the be-

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haviour of an ensemble consisting of thermals and convective clouds which interact with one another. There are models in which a part of the entire range of convective pulsations including thermals and convective clouds is parameterized [13]. Such models serve to give a qualitative explanation of the structure of the quasi-ordered mesoscale formations mentioned above. There are also models in which thermals and convective clouds preserving their individual structure in certain conditions form quasiordered clusters shaped as extended lines and irregular hexagons [10].

It is assumed in the model being proposed that an isolated convective pulsation, called a convective cell, is an eigensolution to the thermohydrodynamic equations, and turbulence is generated by a nonlinear interaction of cells. The so-called simplified Boussinesq equations are used to investigate the interaction of several convective cells located above each other [10–12]. A criterion of stability for the cells of an adiabatic atmosphere was obtained in [11]. It was extended to the more general case of a polytropic atmosphere. In this paper, we give a rigorous justification of this hypothesis. The model does not simulate the three-dimensional structure of a convective ensemble. Nevertheless, it was possible to formulate a reasonable hypothesis on the statistical structure of the ensemble, transfer from the hydrodynamic to statistical simulation, and obtain relatively simple spectral relations. Some expressions for the convective fluxes of heat and momentum are obtained by averaging over the entire spectrum. These expressions are used to simulate the interacting convective cells of the next hierarchical level. It is shown that in the framework of the simplifications of the vertical boundary layer, as well as of other assumptions described below, the parameterization of smaller convective pulsations reduces to the multiplication of the coefficients of molecular, turbulent viscosity, and heat conductivity by a constant factor. Thus, the theory proposed gives a justification for the use of the Boussinesq equations in [10–12] to explain theoretically the space and time scales of thermals and cumulus clouds.

2. Simplified hydrodynamic model

Consider the following simplifying assumptions:

1. The convective cells are produced by axially symmetric pulses of heat and kinetic energy that are located above each other.
2. The interaction between only the cells located along the vector of buoyancy forces, i.e., along a vertical line, is taken into account. These two assumptions allow one to consider the process as axially symmetric.
3. It is assumed that the vertical size of cells is larger than their horizontal size, because the viscosity and buoyancy forces affect the vertical

propagation of convective perturbations. This assumption makes it possible to simplify the Boussinesq equations due to the theory of vertical boundary layers [8, 11].

4. The temperature and density of the liquid are related by a linear law, $\rho = \rho_0(1 - k\vartheta)$, where ϑ is a temperature deviation from an unperturbed state, $\theta = \theta_0 - \alpha z$ ($\alpha = \text{const}$); ρ and ρ_0 are the density and its average value in the unperturbed liquid; $k = \text{const}$ is the coefficient of linear expansion.
5. The coefficient of viscosity ν and the coefficient of heat conductivity μ are constant and equal: $\nu = \mu$ (i.e., the Prandtl number $\Pr = 1$).

With these simplifications, the Boussinesq equations are [8, 11, 12]:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = \lambda\vartheta + \frac{\nu}{r}\frac{\partial}{\partial r}r\frac{\partial w}{\partial r} + \nu\frac{\partial^2 w}{\partial z^2}, \tag{1}$$

$$\frac{\partial \vartheta}{\partial t} + u\frac{\partial \vartheta}{\partial r} + w\frac{\partial \vartheta}{\partial z} = \alpha w + \frac{\nu}{r}\frac{\partial}{\partial r}r\frac{\partial \vartheta}{\partial r} + \nu\frac{\partial^2 \vartheta}{\partial z^2}, \tag{2}$$

$$\frac{\partial ur}{\partial r} + \frac{\partial wr}{\partial z} = 0, \tag{3}$$

where t is the time; r, z are the radial and vertical cylindrical coordinates (z-axis is directed upwards); u, w are the radial and vertical velocity components; $\lambda = kg$; g is the acceleration due to gravity.

It is known that in the boundary layer equations the terms that take into account turbulent viscosity and heat conductivity along the layer are small [8]. It was, however, shown in [11, 12] that the solutions that do not take into account the last terms in (1), (2) can be hydrodynamically unstable, and the criteria of this instability are determined by solving (1)–(3). The stable solutions with and without vertical viscosity must not differ greatly from each other, otherwise, the above assumptions are not valid.

3. Solutions for several simultaneous heat pulses in an unbounded domain

We take several axially symmetric heat pulses extended vertically as initial conditions for (1)–(3):

$$\text{at } t = 0: \quad \vartheta = \frac{4\nu^2}{\lambda r_0^2}\exp\left(-\frac{r^2}{2r_0^2}\right)f_0(z), \quad w = 0, \tag{4}$$

where $f_0(z)$ determines the vertical distribution of ϑ. This function is nonzero on several disjoint intervals. The attenuation of ϑ with r is determined by the parameter r_0. With heat pulses released at several points with coordinates $r = 0$, $z = z_i$ $(i = 1, 2, \ldots, n)$, $r_0 = 0$ and $f_0 \neq 0$ for $z = z_i$ and $f = 0$ for $z \neq z_i$ at $t = 0$.

We add to these conditions the law of heat variation in the system which is obtained by integrating (2) in the space

$$\int_0^\infty \int_{-\infty}^\infty \left(\frac{d\vartheta}{dt} - \alpha w\right) r\, dz\, dr = 0, \qquad 2\pi \int_0^\infty \int_{-\infty}^\infty \vartheta r\, dz\, dr = \frac{q_e}{c_p \rho} \quad \text{at } t = 0, \quad (5)$$

where q_e is the amount of heat released at the initial time; ρ and c_p are the density and heat capacity of the air at constant pressure.

We solve the Cauchy problem for (1)–(3) with the initial conditions (4). The solution is sought in the following form:

$$w = 4\nu^2 a(t)\varphi(t) \exp(-ar^2/2) f(z,t), \quad (6)$$

$$u = -(4\nu^2 \varphi / r)(1 - \exp(-ar^2/2)) \frac{\partial f}{\partial z}, \quad (7)$$

$$\vartheta = (4\nu^2 a \varphi_1(t)/\lambda) \exp(-ar^2/2) f, \quad (8)$$

where $a(t)$, $\varphi(t)$, $\varphi_1(t)$, $f(z,t)$ are unknown functions.

Substituting (6)–(8) into (1)–(5) and equating the terms with equal powers of r, we obtain the following problems:

$$\frac{d\varphi}{dt} = \varphi_1, \quad \frac{d\varphi_1}{dt} = \alpha\lambda\varphi, \quad \frac{da}{dt} = -2\nu a^2, \quad (9)$$

$$\text{at } t = 0: \quad \varphi = 0, \quad \varphi_1 = 1, \quad a = 1/r_0^2; \quad (10)$$

$$\frac{\partial f}{\partial t} + 4\nu^2 a \varphi f \frac{\partial f}{\partial z} = \nu \frac{\partial^2 f}{\partial z^2}, \quad \text{at } t = 0 \quad f = f_0(z); \quad (11)$$

$$2\pi \int_0^\infty \int_{-\infty}^\infty \vartheta r\, dr\, dz = Q, \quad \int_{-\infty}^\infty f\, dz = Q_e, \quad Q_e = \frac{\lambda q_e}{8\pi c_p \rho \nu^2}, \quad (12)$$

$Q = Q_e \varphi_1$ is the buoyancy of a cell (a quantity proportional to the amount of heat contained in the cell).

The problem (9), (10) is solved analytically:

$$\varphi(t) = \begin{cases} t & \text{at } \alpha = 0, \\ \sin(\sqrt{-\alpha\lambda}t)/\sqrt{-\alpha\lambda} & \text{at } \alpha < 0, \\ sh(\sqrt{\alpha\lambda}t)/\sqrt{\alpha\lambda} & \text{at } \alpha > 0; \end{cases} \quad (13)$$

$$\varphi_1(t) = \begin{cases} 1 & \text{at } \alpha = 0, \\ \cos(\sqrt{-\alpha\lambda}t) & \text{at } \alpha < 0, \\ ch(\sqrt{\alpha\lambda}t) & \text{at } \alpha > 0; \end{cases} \quad (14)$$

$$a(t) = 1/(2\nu t + r_0^2). \quad (15)$$

Thus, the two-dimensional nonstationary problem (1)–(4) is reduced to the one-dimensional problem (11)–(15).

The interaction of convective cells caused by a simultaneous release of several point heat pulses at neutral stratification of the atmosphere ($\alpha = 0$, $r_0 = 0$) was investigated in [11, 12]. It was shown that problem (11)–(15) can be solved analytically. It was also shown that at a strong but physically admissible heat release the cells spread in the direction of their motion (as it was assumed in the formulation of the problem) and moved upwards at a speed proportional to their size. In the process, larger cells grew in size owing to their merging with smaller cells. The coagulation of convective cells took place by merging of several smaller cells into a larger one. Cells with the same buoyancy Q were almost identical, no matter how many cells had formed them. Although these effects were obtained theoretically in [11, 12] for the nonlinear interaction of cells in the atmosphere, they were substantiated by some laboratory experimental data for a viscous liquid [19]. Therefore, the theory proposed is universal for cells of different scales. Also, it was shown in [11, 12] that the analytical solutions (6)–(15) at $\alpha = 0$ were in agreement with the results of the following laboratory experiment:

Portions of a light water solution were injected into water at regular intervals. Convective cells of the same buoyancy and of the same trajectory were generated [19]. Thus, a system of three interacting convective cells located above one another was formed. It was shown experimentally that the upper boundary of the first cell rose at a speed $v \sim \sqrt{Q/t}$. The second cell during the interaction with the first one was moving upward at a constant rather than decreasing with time speed. The upper boundary of the third (lowest) thermal during its interaction with the thermal formed by the coalescence of the first two thermals was moving at the same constant speed. As a result, a cell moving at a speed $v \sim \sqrt{3Q/t}$ was formed by the coalescence of the three similar cells. Similar mechanisms were revealed by a theoretical investigation of the interaction of atmospheric thermals and convective clouds and substantiated by observations [14, 15].

4. Instability of convective cells at $\alpha > 0$

A criterion of instability relative to finite-amplitude perturbations was obtained from the solution (11)–(15) at $\alpha = 0$, $r_0 = 0$ [11, 12]. In accordance with this criterion, the stability of a cell decreases very quickly, exponentially, with its buoyancy. A hypothesis that produced a similar criterion valid not only for $\alpha = 0$, but also for $\alpha > 0$ was proposed in [10–14]. Let us try to substantiate the hypothesis for an arbitrary constant value of α. The case of $\alpha = 0$ is the most interesting one for applications, since in this case the convection is developing. In accordance with (14), the buoyancy of a cell

Q_i at $\alpha > 0$ increases with time near-exponentially due to the energy of instability of the medium. The initial thermal pulses play the role of a "trigger mechanism" for the onset of convection. Unfortunately, for $\alpha = 0$, $r_0 = 0$ exact solutions to (11)–(15) have not been obtained. It is difficult to obtain numerical estimates for the instability of convective cells at $\alpha > 0$, since instability at unboundedly increasing Q may take place at external influences that are smaller than numerical errors occurring in problem (11)–(15).

We make an attempt to obtain a stability criterion at $\alpha > 0$ by analytical methods. Divide the entire time interval into sufficiently small discrete intervals: $(\Delta t(i-1), \Delta ti)$, $i = 1, \ldots, n$. In each of these intervals, the coefficient $b = 2\nu a\varphi$ at the nonlinear term in (11) can be considered constant: $b = 2\nu a\varphi = b_i = \text{const}$ at $\Delta ti \leq t \leq \Delta t(i+1)$. We assume that at $t = 0$ a thermal pulse affects an unstable atmosphere and at $t = t_i = \Delta ti$ a sufficiently strong thermal with buoyancy Q is formed. Also, at $t = t_i$ let the atmosphere be weakly affected with $Q_2 = \varepsilon \ll 1$. In this case, we have for $\Delta ti \leq t \leq \Delta t(i+1)$ instead of (11) the following:

$$\frac{\partial f}{\partial t} + 2\nu b_i f \frac{\partial f}{\partial z} = \nu \frac{\partial^2 f}{\partial z^2}; \quad \text{at } t = t_i \quad f = f_1(z) + f_2(z), \tag{16}$$

f_1 is the solution of (16) at $t = t_i$; f_2 is the weak effect at $t = t_i$.

Taking into account that for $\Delta ti < t < \Delta t(i+1)$ (16) is a Burgers equation, we solve, instead of (16), the following problem:

$$f = \frac{\exp Q(F_1 + \varepsilon F_2)}{b_i(1 + \exp Q(\int_z^\infty (F_1 + \varepsilon F_2)\, dz)}; \tag{17}$$

where F_1, F_2 satisfy the following relations:

$$\int_{-\infty}^{\infty} F_1\, dz = \int_{-\infty}^{\infty} F_2\, dz = 1; \quad \frac{\partial F}{\partial t} = \nu \frac{\partial^2 F}{\partial z^2}, \quad F = F_1 + F_2.$$

In (17), the denominator vanishes, and the solution has no sense at

$$\varepsilon = -\Big(\exp(Q_1) + \int_z^\infty F_1\, dz\Big) \Big/ \int_z^\infty F_2\, dz. \tag{18}$$

Thus, ε depends on t and z. However, since fluctuations in the temperature field can occur at any time and at any point of the convective layer, the minimum absolute value of ε from (18) should be used. It corresponds to the minimal strength of a thermal fluctuation destructing a thermal; it is reached at $\int_\infty^\infty F_1\, dz = 0$ and $\int_{-\infty}^\infty F_2\, dz = 1$. Substituting these values into (18), we finally obtain:

$$\varepsilon = \varepsilon_{\text{cr}} = -\exp(-Q_1). \tag{19}$$

Weaker fluctuations do not destruct the thermal. It can be easily shown that at certain space points the numerator in (17) tends to zero faster than the denominator. Thus, the destruction of a cell is due to its instability with respect to external finite-amplitude perturbations. The stronger a rising cell, the weaker the external influence that is necessary to destruct it. The same relation for ε in the solution for two simultaneous point thermal pulses given at $t = 0$ was obtained in [11, 12]. That the solution, however, has an essential shortcoming: when (19) is fulfilled, it is not valid at $t > 0$, i.e., immediately after the injection of thermal pulses, when there are still no motions, and all heat is concentrated at the two space points. This result has no physical interpretation. It is evident that relations (16)–(19) do not have this shortcoming. The reason for the fact that (19) is valid not only at $b =$ const, but also at $b = b(t)$ is as follows: at $\varepsilon \geq \varepsilon_{cr}$ a cell is destructed instantly, at the time of injection of a weak fluctuation. The continuity and smoothness of $b(t)$ at this time are sufficient conditions for the validity of (19). Nevertheless, the relations do not describe the cell destruction process itself: in this case there is no solution to this problem. A numerical solution to a similar problem was realized in the same conditions, but for the Boussinesq equations without the simplifications of the vertical boundary layer theory. It has been shown that the destruction of a cell in the more comprehensive model may be due to a process of the "wave reversal" type initiated by a collision between rising and descending cells. The process is accompanied by an entrainment of the surrounding air into the cell which causes its rapid dissipation. In the simplified theory described in this paper this dissipation is considered instantaneous and called "cell destruction". A comparison between the theory and the calculations carried out by using a model without the simplifications of the vertical boundary layer has shown that the life cycle of each cell consists of two stages: the laminar (for microscale pulsations), or quasilaminar (for thermals and convective clouds), stage and the turbulent one. At the first stage the cell spontaneously grows. At the second stage, it instantly collapses (in the simplified model) or gradually dissipates (in the more comprehensive model). The second stage can begin (with different probability) at any instant. This is due to the instability of convective cells that determines the probabilistic properties of the simplified model proposed.

5. Theorem on a critical value of an external effect on a convective cell

Theorem. *If a convective cell described by equations* (1)–(3) *with initial conditions* (4) *and determined by their solution* (5)–(15) *at* $t = t_1 \geq 0$ *is affected by a perturbation*

$$\begin{aligned} w &= 4\nu^2 a_1 \varphi(t_1)\exp(-a(t_1)r^2/2)f_\varepsilon(z),\\ u &= -(4\nu^2\varphi(t_1/r)(1-\exp(-a(t_1)r^2/2))\frac{\partial f_\varepsilon}{\partial z},\\ \vartheta &= (4\nu^2 a\varphi_1(t_1)/\lambda)\exp(-a(t_1)r^2/2)f_\varepsilon, \end{aligned}$$

there is no solution at $t > t_1$ *when* $\varepsilon \geq \varepsilon_{\rm cr} = -\exp(-Q_1)$,

$$Q_1 = 2\pi\int_0^\infty\int_{-\infty}^\infty \vartheta_{t=t_1-\varepsilon} r\,dz\,dr = \frac{q_1}{c_p\rho},$$

$$\varepsilon = 2\pi\int_0^\infty\int_{-\infty}^\infty \vartheta_\varepsilon r\,dz\,dr = \frac{q_\varepsilon}{c_p\rho}.$$

Here $a(t_1)$, $\varphi(t_1)$, $\varphi_1(t_1)$ are determined from (13)–(15); $f_\varepsilon(z)$ is a limited function which is nonzero only on one or several intervals; q_1 is the amount of heat in the system before the effect on the convective cell; $\varepsilon \to 0$; $q_\varepsilon < 0$ is the amount of heat released from the system due to the effect on the convective cell; $\varepsilon > 0$; $\varepsilon < 0$. The problem formulated is the Cauchy problem for $t > t_1$, $0 \leq r < \infty$, $-\infty \leq z \leq \infty$, in which the solution at $t = t_1 - \varepsilon$ plus the additional perturbation given at $t = t_1$ is the initial condition.

6. A statistical model of an ensemble of convective cells

It has been observed that the sizes of convective cells in a viscous liquid differ but slightly. The result obtained in the previous sections indicates that this can happen when smaller cells are absorbed by larger cells, and very large cells are destructed by neighbouring cells. We assume that the small cells with buoyancy $Q < Q_m$ are instantly absorbed by larger cells, and very large cells are destructed by neighbouring cells. Then, in accordance with (19), we assume that for the probability density of cell distribution $P(Q)$ the relations

$$P(Q) = \exp(Q_m - Q), \qquad \int_{Q_m}^{Q} P(x)\,dx = 1, \tag{20}$$

are valid.

However, (20) cannot be verified by measurements. Let us try to express Q in terms of parameters which can be estimated by measurements. Consider the most interesting case of a convective layer of unstable stratification. Substituting (13), (15) at $\alpha > 0$ into (11), we obtain

$$\frac{\partial f}{\partial t} + \frac{4\nu^2 \sin h(\sqrt{\alpha\lambda}t)}{\sqrt{\alpha\lambda}(r_0 + 2\nu t)} f\frac{\partial f}{\partial z} = \nu\frac{\partial^2 f}{\partial z^2} \tag{21}$$

In accordance with (12), the total amount of heat in a convective cell increases with time

$$Q = Q_e \cos h(\sqrt{\alpha\lambda}t). \qquad (22)$$

If the solution to problem (21)–(22) at $Q \gg 1$ exists, it must be close to the solution of the problem for the case of zero vertical viscosity in (28). The solution to (21)–(22) without vertical viscosity has the following form:

$$w = p_1 zR, \quad \vartheta = p_2 zR, \quad u = p_3(1 - R) \quad \text{at} \quad 0 \le z \le h, \qquad (23)$$

$$w = \vartheta = u = 0 \quad \text{at} \quad z < 0 \quad \text{or} \quad z > h. \qquad (24)$$

Here $R = \exp(-ar^2/2)$, $h = 2\nu\sqrt{2Q/p}$, $p = 1/\int_0^t a\varphi dt$, $p_1 = a\varphi p$, $p_2 = (a\varphi/\lambda)p$, $p_3 = -(\varphi/r)p$. The relation between Q and the other parameters is complex in this form. Find an approximate form of the solution (23), (24) late in the development of cells, when the probability of their collapse is high. We assume that in this case $t \gg 1/\sqrt{\alpha\lambda}$, $t \gg r_0^2/(2\nu)$. Then $\sin h(\sqrt{\alpha\lambda}t) \approx \cos h(\sqrt{\alpha\lambda}t) \approx \exp(\sqrt{\alpha\lambda}t)$, $p \approx 2\nu\alpha\lambda t/\exp(\sqrt{\alpha\lambda}t)$, $a^2 \approx 1/(4\nu t)$. Substituting the approximate values into (23), (24), we obtain

$$w = \sqrt{\alpha\lambda}zR, \quad \vartheta = \alpha zR \quad \text{at} \quad 0 \le z \le h, \qquad (25)$$

$$w = \vartheta = 0 \quad \text{at} \quad z < 0 \quad \text{or} \quad z > h, \quad h = 2(Q\nu)^{1/2}(\alpha\lambda)^{-1/4}. \qquad (26)$$

It is easily seen that

$$w_{\max} = 2(Q\nu)^{1/2}(\alpha\lambda)^{1/4}, \quad \vartheta_{\max} = 2(Q\nu)^{1/2}\alpha^{3/4}\lambda^{-1/4}, \qquad (27)$$

$$H = w_{\max}\vartheta_{\max} = 4\nu\alpha Q, \quad E = w_{\max}w_{\max} = 4\nu(\alpha\lambda)^{1/2}Q, \qquad (28)$$

$$Q^2 = \alpha\lambda h^4/(16\nu^2) = gk\Delta\theta h^3/(16\nu^2) = Ra_q/16, \qquad (29)$$

where $w_{\max}$, $\vartheta_{\max}$ are equal to w, ϑ at $z = h$ and at $r = 0$; $\Delta\theta = \theta_{z=h} - \theta_{z=0} = \alpha h$; Ra_q is the Rayleigh number for a cell with buoyancy Q.

Thus, we obtained a simple relation between Q and h, $w_{\max}$, $\vartheta_{\max}$, H, which makes it possible to obtain the following expressions for the densities of distributions of convective cells:

$$P(Y_i)dY_i = 2(Y_i - X_i)/D_i^2 \exp(-((Y_i - X_i)^2/D_i)^2)dY_i, \quad i = 1, 2, 3;$$

$$P(H)dH = 1/H_0 \exp(-(H - H_m)/H_0)dH,$$

$$P(E)dE = 1/E_0 \exp(-(E - E_m)/E_0)dE,$$

where $Y_1 = h$, $Y_2 = w_{\max}$, $Y_3 = \vartheta_{\max}$; D_i are variances of Y_i: $D_1 = h_0 = 2(\nu)^{1/2}(\alpha\lambda)^{-1/4}$, $D_2 = w_0 = 2(\nu)^{1/2}(\alpha\lambda)^{1/4}$, $D_3 = \vartheta_0 = 2(\nu)^{1/2}\alpha^{3/4}\lambda^{-1/4}$; $X_1 = h_m = Q_m^{1/2}h_0$, $X_2 = w_m = Q_m^{1/2}w_0$, $X_3 = \vartheta_m = Q_m^{1/2}\vartheta_0$; $H_0 = 4\nu\alpha$, $H_m = Q_m H_0$, $E_0 = 4\nu(\alpha\lambda)^{1/2}$, $E_m = Q_m E_0$.

By averaging over the ensemble,

$$\bar{x} = \int_{X_i}^{\infty} Y_i P(Y_i) dY_i, \quad \bar{H} = \int_{H_m}^{\infty} HP(H)dH, \quad \bar{E} = \int_{E_m}^{\infty} EP(E)dE, \tag{30}$$

we obtain $\bar{h} = h_0(Q_m^{1/2} + \pi^{1/2}/2)$, $\bar{w}_{\max} = w_0(Q_m^{1/2} + \pi^{1/2}/2)$, $\bar{\vartheta}_{\max} = \vartheta_0(Q_m^{1/2} + \pi^{1/2}/2)$, $\bar{H} = H_0(Q_m + 1)$, $\bar{E} = E_0(Q_m + 1)$. Now it is evident that Q_m can be determined in terms of the average values and their variances:

$$Q_m + 1 = \bar{H}/H_0 = \bar{E}/E_0; \qquad Q_m + \pi^{1/2}/2 = (Y_i/D_i)^2. \tag{31}$$

It is clear that $\bar{H}$ and $\bar{E}$ are proportional to the convective fluxes of heat and momentum through the upper boundary of a convective cell into the surrounding atmosphere. With a horizontal distribution of cells we average over a horizontal line and obtain

$$\hat{\bar{H}} = 4C\nu\alpha(Q_m + 1), \qquad \hat{\bar{E}} = 4C\nu(\alpha\lambda)^{1/2}(Q_m + 1), \tag{32}$$

where C is inversely proportional to the average distance between the cells. If the cells are sufficiently close to each other, $C \approx 0.7$.

If a convective perturbation occurs not in an atmosphere at rest, but inside a larger convective cell, then, instead of $\alpha = d\theta/dz$, we take $\alpha_2 = \partial\theta_2/\partial z$, where θ_2 is the temperature in this larger cell (the subscript 2 denotes the distributions of a convective ensemble of a second hierarchical level).

We study the behaviour of several interacting cells of the second hierarchical level located above each other. The heat and momentum fluxes due to the microscale convection in this ensemble are parameterized as

$$\hat{\bar{H}} = 4C\nu(Q_m + 1)\partial\theta_2/\partial z, \tag{33}$$

$$\hat{\bar{E}} = 4C\nu(Q_m + 1)\partial w_2/\partial z, \tag{34}$$

$$\hat{\bar{\vartheta u}} = 4C\nu(Q_m + 1)\partial\theta_2/\partial x, \tag{35}$$

$$\hat{\bar{wu}} = 4C\nu(Q_m + 1)\partial w_2/\partial x. \tag{36}$$

Relation (33) was obtained without the use of additional hypotheses. Since this model does not take into account the lateral interaction of cells, the horizontal fluxes of heat and momentum can be calculated only roughly. Therefore, we take (35), (36) as a hypothesis supplementing the hypotheses used in the statement of the problem and the construction of the statistical model. The form of (35), (36) is similar to that of (33), (34). For an ensemble of interacting cells of the second hierarchical level located above each other one should use equations similar to (1)–(3), in which u, w, ϑ, ν, α are replaced by u_2, w_2, ϑ_2, $\nu_2 = 4C\nu(Q_m + 1) + \nu$, α_2. Here α_2 is responsible for temperature stratification with allowance for the averaged convection of the first hierarchical level. Clearly, since in the equations of vertical boundary layer $\hat{\bar{E}} = (\alpha/\lambda)^{1/2}\hat{\bar{H}}$, relation (34), as well as relation (33), is obtained without any additional hypotheses. Relations (33)–(36) are valid for the subsequent hierarchical levels. Thus,

$$\frac{h_{i+1}}{h_i} = LS_i^{-1/4}, \quad \frac{w_{i+1}}{w_i} = LS_i^{1/4}, \quad \frac{u_{i+1}}{u_i} = L, \quad \frac{\vartheta_{i+1}}{\vartheta_i} = LS_i^{3/4}, \qquad (37)$$

where $S_i = \alpha_{i+1}/\alpha_i$, $L = 2(C(Q_m + 1))^{1/2}$.

Now let us define Q_m. Unfortunately, the convective turbulence of the first hierarchical level in the atmosphere does not differ from the dynamic turbulence. Therefore we can use the data of an experiment, in which convection of water in a wide saucepan was studied [7]. The saucepan was heated by water up to 100°C from below. The heating was done in such a way that the amount of heat coming from below was equal to its outflow through the upper and side walls of the saucepan. At a maximal thickness of the convective layer of 10 cm the effect of the bottom and side walls on the convection was minimal. Then the average velocity of the very small particles suspended in water was 1.8 bk s^{-1}, and its variance constituted 20%. Assuming that $D_2/\bar{w} = 0,2$ and substituting it into (30), we obtain

$$Q_m \approx 25. \qquad (38)$$

Assuming that $\nu = 10^{-2}\mathrm{bk}^2\mathrm{s}^{-1}$, $\alpha = 1°\mathrm{C\,s}^{-1}$, $\lambda = gk = 3$ bk $\mathrm{b}^{-1}(°\mathrm{C})^{-1}$, we have $\bar{w} \approx 2(Q_m\nu)^{1/2}(\alpha\lambda)^{1/4} \approx 1.8$ bk s^{-1}, which is in agreement with the measurements [7].

7. Conclusion

The results obtained make it possible to conclude that some processes of nonlinear interaction, namely, the coagulation of cells and the process of their instability due to external factors are responsible for the disordered structure of convective ensembles. Such external factors can be neighbouring cells and numerical errors. At an unlimited spontaneous growth of cells their sensitivity to perturbing factors increases and, finally, they collapse under the effect of the random factors. Computer simulation of the turbulent regimes by using the Navier–Stokes equations [6], the Boussinesq equations [2, 4, 6], and the equations of deep [16] and shallow [3] convection make it possible to assert that an isolated pulsation is an eigensolution of the corresponding equations, and turbulence is generated by the nonlinear interaction between pulsations. The proposed simple analytical, rather than finite-difference, model confirms this.

The theory, as well as the calculations carried out by using a more comprehensive model without the simplifications of the vertical boundary layer have shown that the life cycle of each cell consists of two stages: the laminar (for microscale pulsations) or quasilaminar (for thermals and convective clouds) stage and the turbulent one.

At the first stage the cell spontaneously grows. At the second stage it instantly collapses (in the simplified model) or gradually dissipates (in the more comprehensive model). The second stage can begin (with different probability) at any instant. This is due to the instability of convective cells that determines the probabilistic properties of the simplified model proposed.

The linear dimensions of convective cells of the first hierarchical level, in accordance with (30), are approximately 10 cm. These cells produce the cells of the second hierarchical level with dimensions which are, in accordance with (31), greater by approximately an order of magnitude. This rule is valid also for the cells of the next hierarchical levels.

In accordance with the theory and the observational data, atmospheric convection develops in the following way. First, a convective ensemble is formed over the underlying surface heated by the Sun. This ensemble consists of cells which range in size from several centimeters to several decimeters. Such an ensemble often has the form of a haze over an arable or asphalt surface. As the lower layer is heated, the cells of the next hierarchical levels are produced. Thus, in accordance with the theory, large thermals are convective cells of the fourth and fifth hierarchical levels. If a moist layer with unstable stratification is over a dry unstable layer, convective clouds of various types are formed. They often merge with large thermals of the lower layer. These convective clouds are cells of the fifth and sixth hierarchical levels. This result is in agreement with indirect evidence of numerous observations. Thus, in accordance with [14, 15], the convective clouds consist of fairly large cells. This causes an increased danger for an aircraft that finds its way into such a cloud.

The two local maxima in the spectra given in [10] in Figures 4, 5, and 8 correspond to these two hierarchical levels of cells. The comments given to the figures confirm the conclusions of the theory. The process of increasing of cells continues until they fill the entire unstable layer. The total effect of the convective ensemble is in the following: the unstable stratification of the convective layer becomes dry and moist- adiabatic. The lateral interaction of cells as well as the background large-scale wind affect the formation of larger cells. Some mechanisms of this effect leading to the formation of quasiordered cells with a horizontal scale larger than 10 km are considered theoretically in [9].

The theory can be used in the construction of simplified three-dimensional models of convective ensembles [9] to parameterize atmospheric convection [11] in models of general atmospheric circulation.

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