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Rules of application of algorithms for tsunami waves kinematic computations based on the Huygence principle*

An.G. Marchuk

The process of tsunami waves propagation can be well described by the shallow-water equations [1]. One of the corollaries of this model is the independence of a wave-front velocity from the amplitude of a tsunami wave. This velocity depends only on the depth and can be found by the following expression

$$C = \sqrt{gH}, \tag{1}$$

which is called the Lagrange formula.

Methods of wave-front computation, that are based on a determination of trajectories of wave rays [2] have some certain shortcomings. At first, as the result of divergence (during computations), the distance between the neighboring wave rays becomes too long. It means that we need to insert a new wave rays or wave-front points. In order to make such a procedure correctly (without distortion of an actual segment of a front motion), the complicated enough algorithm is required. It can cause the significant growth of calculations volume. The simplified procedure of an insertion of new points of front (or rays) can consist in "installation" of a new point of a wave-front to the middle of a segment connecting neighboring points that moved away from each other. However, this method decelerates the movement of a wave-front. The same concerns to the wave rays.

The second significant shortcoming of the indicated methods is the practical impossibility of a prolongation of the wave-front kinematic calculation in those places of an area, where the wave passes through narrow straits. In this situation, 1–2 rays can pass through a strait (or 1–2 points of a wave-front set), and, then the continuation of further movement of this part of a front is impossible, using standard algorithm. The minimum three points of a wave-front set is required to find the next position of only one central point [2]. The further movement of two edge points of this set remains undefined. Therefore, adding new points of the wave-front to the edges of safely past through a narrow strait segment is necessary.

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As far as wave rays are concern, in such situation they have a full autonomy (do not depend on neighboring rays) (see [2]), but the sudden reduction of their quantity after passing a narrow strait can cause the situation that in a water area behind this strait we will observe only a narrow bunch of wave rays. In a reality, after passing through a strait, the wave rays due to diffraction will propagate practically in all directions. Thus, during computations of wave-front sets by the indicated methods in areas with complicated topology (narrow straits, long narrow islands) significant areas of "shadow" can remain. In such places at the coastline or in water areas, tsunami travel-time can not be defined. As an example, we want to show a fragment of a tsunami travel-time chart which is produced in the Krasnoyarsk Computing Center under the contract with IOC UNESCO [3].

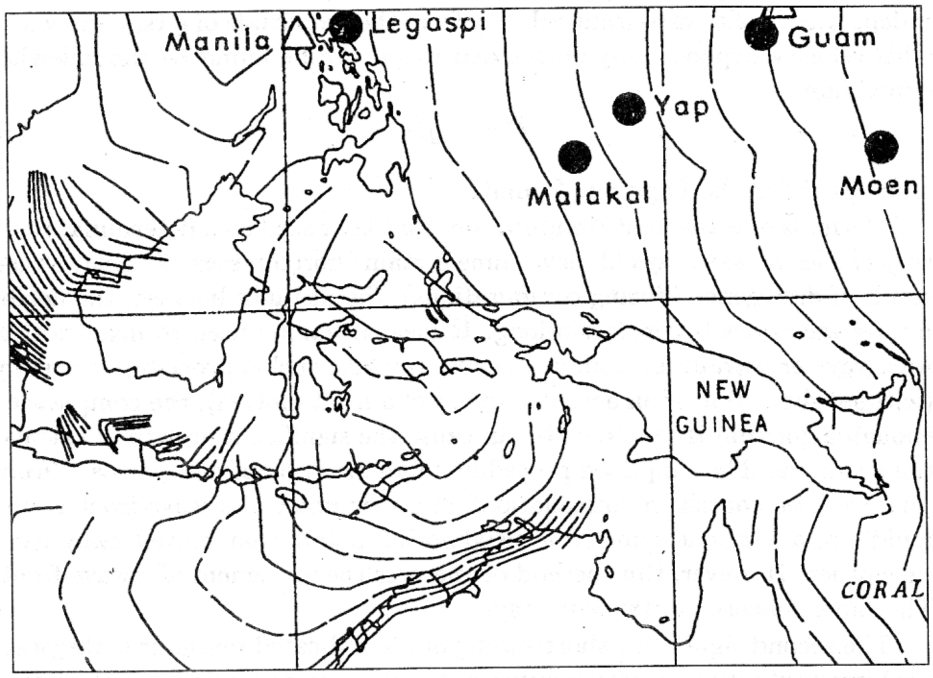

Figure 1. The fragment of the travel-time chart which was calculated with the help of wave rays

Methods for tsunami travel-time calculations that are based on a Huygence principle are free of the indicated shortcomings. The essence of this principle is, that all points of water area, where the wave perturbation has already arrived, become the sources of wave radiation and radiate wave energy in all directions. Thus the algorithm of computation is based on a looking through all neighboring to a wave-front grid-points (in which perturbation has not come yet) and calculation of travel-times there, as minimum among all possible travel-times from nearest points. The sum of travel-times from the initial source up to each wave-front points and from these points to the regarded grid-point is minimized among all possible travel ways. Let us explain this by Figure 1. Let the travel-times calculations be carried out in the area with a rectangular computational grid. It means, that in all points of this grid the values of depths are known and it is required to find tsunami travel-times from a specified source (one or several grid-points) up to all the other grid-points.

Let us schematically represent in Figure 2 the fragment of computational area. Here black small squares designate those points of the grid, in which the perturbation from the initial tsunami source at this time moment has already come, and travel-times in these grid-points are known to us. We need to find the travel-time from the source up to the point A (see Figure 2). Relatively to the point A the neighboring points, where the travel-times are known, there will be the points B, C, D and E. Let the travel-times up to them be equal correspondingly to $\boldsymbol{T_B}$, $\boldsymbol{T_C}$, $\boldsymbol{T_D}$ and $\boldsymbol{T_E}$. Between adjacent grid-pints the depth varying under the linear law. Let us find tsunami travel-time between two neighboring grid-points, when the distance between them is equal to $\boldsymbol{L}$, and the depth varies from value H_1 up to H_2. Let us introduce auxiliary value – the angle of declination of the bottom $\alpha = (H_2 - H_1)/L$. Then the travel-time will be expressed as

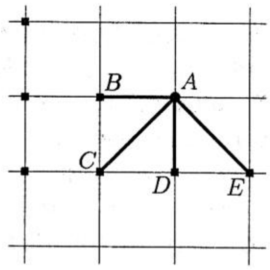

Figure 2. The scheme of travel-time definition using the Huygence principle

$$
\begin{aligned}
\boldsymbol{T} &= \int_0^T \frac{dl}{\sqrt{g(H_1 + l\,\mathrm{tg}\,\alpha)}} \\
&= \frac{1}{\sqrt{g\,\mathrm{tg}\,\alpha}} \int_0^T \left(l + \frac{H_1}{\mathrm{tg}\,\alpha}\right)^{-1/2} d\left(l + \frac{H_1}{\mathrm{tg}\,\alpha}\right) \\
&= \frac{2}{\sqrt{g\,\mathrm{tg}\,\alpha}} \left(l + \frac{H_1}{\mathrm{tg}\,\alpha}\right)^{1/2} \Bigg|_0^L = \frac{2}{\sqrt{g\,\mathrm{tg}\,\alpha}} \frac{\sqrt{H_2} - \sqrt{H_1}}{\mathrm{tg}\,\alpha} \\
&= \frac{2}{\sqrt{g}\,\mathrm{tg}\,\alpha} \frac{H_2 - H_1}{\sqrt{H_2} + \sqrt{H_1}} = \frac{2L}{\sqrt{gH_2} + \sqrt{gH_1}}.
\end{aligned} \tag{2}
$$

Therefore, the tsunami travel-time between neighboring grid-points is equal to a distance between them divided by arithmetically average velocity of a tsunami in these grid-points. Thus, in order to find the travel-time from a source up to a point A, it is necessary to find a minimum of four time values

$$
\begin{aligned}
\boldsymbol{T}_1 &= \boldsymbol{T}_B + \frac{2\Delta x}{\sqrt{gH_B} + \sqrt{gH_A}},\\
\boldsymbol{T}_2 &= \boldsymbol{T}_C + \frac{2\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\sqrt{gH_C} + \sqrt{gH_A}},\\
\boldsymbol{T}_3 &= \boldsymbol{T}_D + \frac{2\Delta y}{\sqrt{gH_D} + \sqrt{gH_A}},\\
\boldsymbol{T}_4 &= \boldsymbol{T}_E + \frac{2\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\sqrt{gH_E} + \sqrt{gH_A}},
\end{aligned}
\tag{3}
$$

where Δx, Δy are the steps of a grid in horizontal and vertical directions and H_A, H_B, H_C, H_D, H_E are the depth values in corresponding points. Minimum of the values $\boldsymbol{T}_i$ $(i = 1, 2, 3, 4)$ will give us the tsunami travel-time from the source up to a point A. In such a way, it is possible point by point to find travel-times to all points of a computational grid.

Some words about meaning of neighboring points. Such grid-points, relatively to considered one, are determined by so-called templates. The most simple one is the eight-dot template (Figure 3).

In such a template, the following eight grid-points: $(i-1, j+1)$, $(i, j+1)$, $(i+1, j+1)$, $(i+1, j)$, $(i+1, j-1)$, $(i, j-1)$, $(i-1, j-1)$, and $(i-1, j)$ will be the neighboring to a point A with the grid coordinates (i, j). The travel-times from each of them to central point are calculated along eight straight rays (segments) using formula (2). Therefore, this template sometimes is named eight-radial. However, this template is too simplified, and, as a result, the wave front computations from a point source using this template give above bottom of constant depth octagonal front line instead of the circle.

More precise results can be received, if for computations we will use the sixteen-dot template (Figure 4).

In this template, the neighbors to a point A, that has a grid coordinates (i, j), are the grid-points which indexes differ from grid coordinates of a point A not only by one unit, but also some grid-points, whose coordinates differ by 2 units (see Figure 4). In this case, if in any of these 16 points of the tsunami travel-time is already known, it is possible to find travel-time to the point A (the center of the template).

Let us dwell on some details of application of sixteen-dot algorithm for the tsunami travel-time calculations. It is possible to use the described algorithm "blindly", i.e., in a case, when at least in one of sixteen points of the template travel-time is known, then using formula (2), the user calculates travel-time in a considered grid-point (in a central point of the template). But then the results of calculations in all area will hardly depend on a direction of exhaustive search of grid-points. For example, the index i can be changed as from minimum value up to maximum, and on the contrary. The same concerns to the index j (vertical position of points).

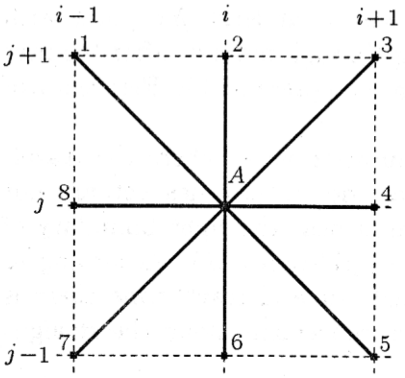

Figure 3. The eight-dot template for computation of tsunami isochrones

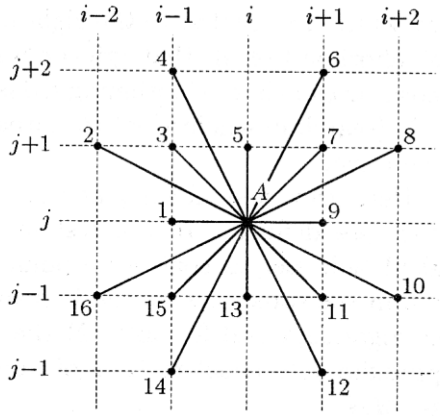

Figure 4. The sixteen-dot template for tsunami travel-time calculations on a rectangular computational grid

Let us consider an example. In rectangular area of computations $ABCD$, the bottom is like a sloping plane and the depth linearly increases from the side AD to the side BC (Figure 5). Let the dot tsunami source $M(i_0, j_0)$ be located in the center of area. Let the search direction be from left to the right and row by row downwards. It means that the index i varies from 1 on the left boundary (AB) up to maximum value (on the boundary CD), and the index j varies from 1 on a upper boundary (BC) to its maximum value on lower boundary (AD).

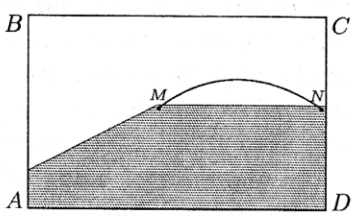

Figure 5. A state of computational area after the first cycle (search through all points). The grey color filled that part of area, where the travel-times are found

In the beginning of the calculating process, there is only one grid-point with known travel-time in it – the point M with the coordinates (i_0, j_0), where travel-time is equal to zero. After the start of searching through all grid-points in upper part of area, where the index j is less than $j_0 - 2$, no new tsunami travel-times will be calculated, because in neighborhoods of all these grid-points (see template in Figure 5) has not appeared any, where the travel-time is known. During the looking through the points of $j_0 - 2$ row in a neighborhood of a point $(i_0 - 1, j_0 - 2)$ there will be a source of a tsunami – M with the coordinates (i_0, j_0), and the possibility to calculate travel-time in this point appeared. According to formula (2) we have

$$\boldsymbol{T}_{i_0-1,j_0-2} = \frac{2\sqrt{(\Delta x)^2 + (2\Delta y)^2}}{\sqrt{gH_{i_0,j_0}} + \sqrt{gH_{i_0-1,j_0-2}}}. \tag{4}$$

Then it is possible to define the tsunami travel-time to all points of this row which are located to the right of a point $(i_0 - 1, j_0 - 2)$, because in the neighborhood of all that grid-points there will be at least one point with known travel-time. Continuing these looking-through process, after the first cycle (search in whole area) the situation that is represented in Figure 5 will be received.

Here grey color filled that part of computation area, where the travel-times are defined. In the rest of the area, the travel-times yet are not calculated. Let us consider a point N, located near the right boundary of area in the same row, as the tsunami source (grid-point M). According to the algorithm and formula (2) the calculated tsunami travel-time there is approximately equal to time of perturbation movement along the straight line MN.

$$T \approx \frac{l}{\sqrt{gH_N}}, \tag{5}$$

where l is the distance between the points M and N, H_N is the depth at the point N. However, as it was shown in publication [4], the fastest trajectory of wave perturbation movement from the point M to the point N is the cycloid which is schematically represented in Figure 5. Therefore, practically in all grid-points of the row with the index $(j + 2)$, which are located to of a source, the travel-times will be defined incorrectly! But as far as the algorithm does not assume recalculation of the defined travel-times, then after completing computations in all area (after repeated cycles) we will have an incorrect map of tsunami isochrones.

In the case of using other directions of checking out the grid-points, the incorrect values of tsunami travel-times can appear in other parts of computational area. In order to avoid this, the multiple cycles are necessary, and during each a cycle it is necessary to calculate travel-times only in those points, that are adjoined to ones, where travel-times are already defined. This can be realized by the following operations: before the start of calculations some conventional time step Δt is selected, whose value is approximately determined by the expression

$$\Delta t \approx \frac{\min\{\Delta x, \Delta y\}}{\sqrt{gH_{\max}}}, \tag{6}$$

where $H_{\max}$ is the maximum depth value in considered area, and Δx and Δy are the distances between grid-points in horizontal and vertical directions. During the first cycle (looking through all grid-points of computation area) the values of travel-times in new points are recorded into memory only in that case, when the travel-time, calculated there using the described algorithm, will be less, than the value Δt, or is equal to it. During the second cycle this time limit becomes equal to $2\Delta t$, and during the cycle number n, new travel-times that are calculated in new points will be limited by the value $n\Delta t$.

Thus repeated exhaustive searches of grid-points in the computational area are necessary for correct definition of travel-times in all points. But during some cycles no new point with defined travel-times can be added.

Such algorithm which is based on repeated exhaustive search of points of computation area needs large enough computing resources (for calculations in large areas). Therefore, then arises the desire how to optimize the process of computations. The optimization can be realized by two ways: the reduction of an amount of points that we need to look through during each cycle, or the cutting down the number of arithmetical operations for tsunami travel-time calculation between points of the template. It is possible easily to shorten the volume of calculations for each point as follows. In the computational formulas (3), the depths values themselves are not required anywhere. A velocity of tsunami wave propagation in the given point, determined by the depth value (1), is used in travel-times computations. Therefore, it is possible before the start of calculations, to prepare the array of propagation velocities in all points of computation area and put it into the memory of the computer. As far as the looking through the area reduction is concern, it is possible to watch over the current sizes of area, where the wave was propagated, and to carry out exhaustive search of grid-points only in a rectangle circumscribed around this area. For example, if in some moment the indexes of all points, where the wave has arrived, are contained in the intervals

$$i_{\min} \leq i \leq i_{\max}, \quad j_{\min} \leq j \leq j_{\max},$$

then during the next cycle it is not necessary to look through all points of computation area, and it is possible to limit searching area with that grid-points, where the grid coordinates i, j are contained in the intervals

$$(i_{\min} - 2) \leq i \leq (i_{\max} + 2), \quad (j_{\min} - 2) \leq j \leq (j_{\max} + 2).$$

Obviously, that such a trick can give significant acceleration of calculations only on an initial stage of process, when the perturbed area takes a small part of all computation area. When the wave front will come close to all boundaries of the area of computations, this optimization trick will lose a sense. However, for tsunami kinematics calculations in large areas such method of optimization can significantly reduce the duration of computations.

Some words should be said about visualization of results of such calculations. Two approaches to this procedure are possible. First consists of waiting for a moment, when computation algorithm will calculate travel-times to all possible points of computation area. Then on a base of the obtained field of travel-time values with the help of program for isoline construction it is possible to draw on a screen (or to store in the digital vector form) the required travel-time isolines (isochrones). As the illustration of such an approach the inverse isochrones of tsunami waves for Burevestnik village (located on the coast of the Iturup island) are shown in Figure 6. Here for isolines drawing the program IZOLIN from a graphic system SMOG [5] was used. Here the corresponded values of travel-time from an isochrone up to a source (in this case to the observation point) is marking the isolines.

The second approach of visualization procedure is the simultaneous with numerical computations plotting the received positions of a wave-front after specified time intervals. Most simple way to do this is highlighting by different colors on the screen of the computer monitor the grid-points according to calculated travel-times in these points. The change of colors happens, when the travel-time values begin to exceed time value on next in turn isochrone. An example of such a visualization can be seen in Figure 7.

However, such method of visualization of isochrones not always satisfies the user, because the colored grid-points completely cover the background map. For example, if isochrones is drawn in such a way on a bottom relief map, then in those places, where tsunami isochrones are visualized, it is impossible to see a relief of ocean bottom. But sometimes it is of great importance to see simultaneously bottom relief and tsunami isochrones. So, in this case we must visualize only isochrone lines. Such new algorithm of visualization is a little bit more complicated. Let us describe in two words the essence of this method of visualization. If during numerical calculations it is required to draw a computed tsunami isochrone corresponding, for example to 10 minutes, then we will visualize (highlight on a screen by the specified colors) those grid-points, where the travel-times will exceed 10 minutes. But only those where the travel-times are calculated with the use of those points of the template, in which the travel-time values do not reach yet 10 minutes. Using such a technique the thickness of drawn isochrone in some places can be as two pixels. If we want the isochrone line width everywhere to be of one pixel wide, the additional procedure is required. But here we shall not stop on its description. As an example of visualization of isochrones in such a way the map of isochrones of the tsunami occurred in July 17, 1998 near to the coast of Papua and New Guinea island is presented in Figure 8. Here the time interval between isochrones is equal to 30 minutes.

Here the white colors fills the water areas of the Pacific Ocean. The grey circle indicates the tsunami source and the dark fills the land. In the figure, the deceleration of the tsunami front in shallow zones of ocean is clearly visible.

Let us make some notes concerning computations in large areas of ocean (with expansion of several geographical degrees by latitude). According to decreasing of the longitudinal degree length from equator to poles (proportionally to cosine of the latitude) during computations on grids, that linked to geographical coordinates, it is necessary to take into account a modification of length of the grid step in the West-East direction. Technically it can

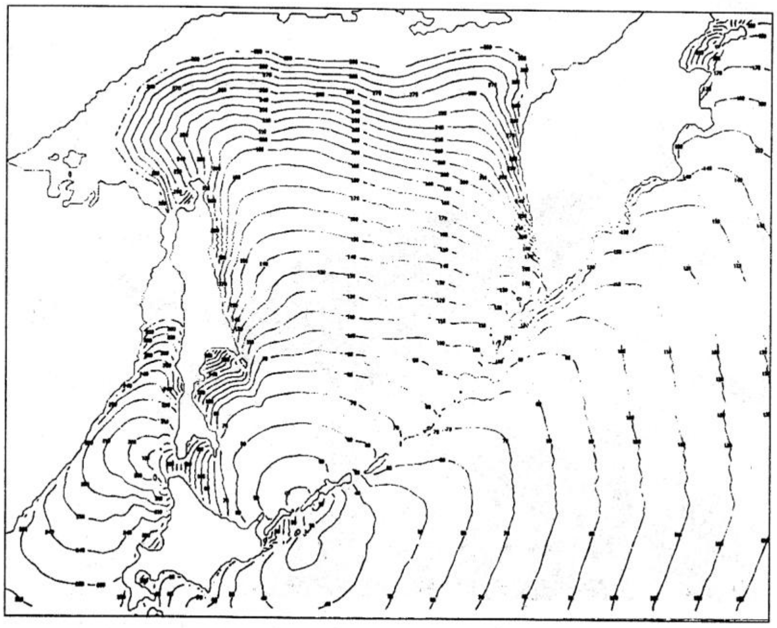

Figure 6. Visualization of calculated travel-times up to Burevestnik village (Iturup island) with the help of isolines drawing program

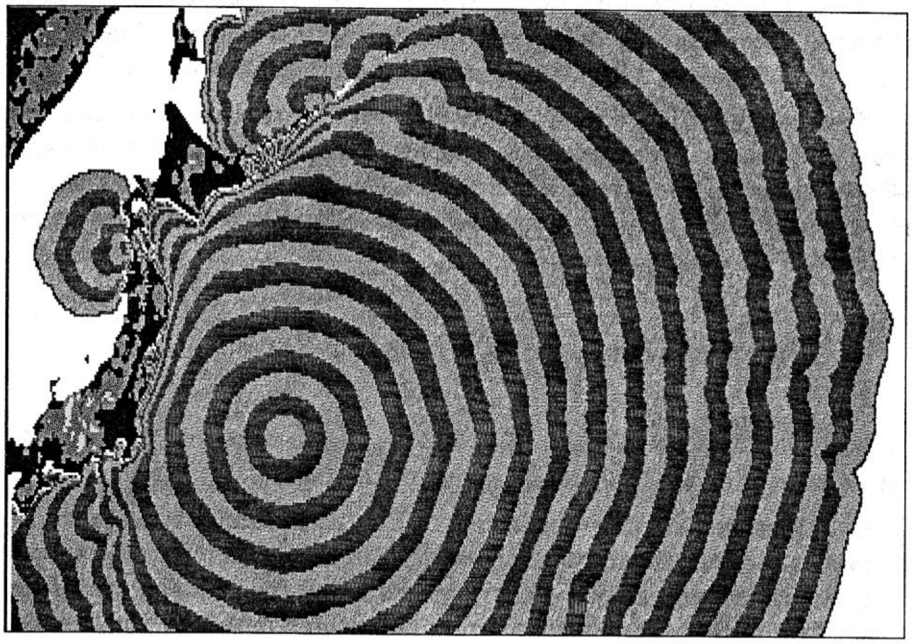

Figure 7. An example of the most simple travel-time chart visualization

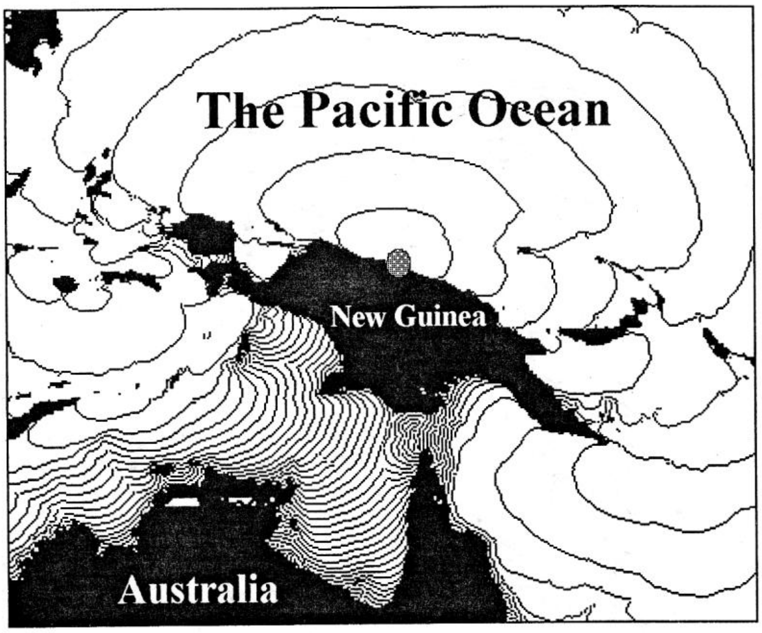

Figure 8. The map of isochrones of the tsunami in July 17, 1998

be made very simply. Let introduce the one-dimensional array of grid-step lengths in longitudinal direction – $\Delta x(j)$, where the length of grid-step is calculated according to geographical degree length is varying. The length of grid-steps – Δy along Meridian remains constant in all computation area.

When rough grids are used for calculation of travel-times between neighboring grid-points it is necessary to take into account that a shortest distance between them on the Globe is not a straight line on the rectangular map projection, but the arc of a large circle, which length is connected to polar coordinates by the expression

$$L = R \arccos\left(\sin\varphi_A \sin\varphi_B + \cos\varphi_A \cos\varphi_B \cos(\lambda_A - \lambda_B)\right), \qquad (7)$$

where φ_A and φ_B are the latitudes, and λ_A, λ_B are the longitudes of these grid-points (in radians), R is the radius of the Globe.

So, let us make some conclusions. The algorithms of the wave-front kinematics computation incorrectly work in areas with complicated topology (narrow straits long islands etc.). In such areas the tsunami travel-times calculations must be carried out using algorithms based on the Huygence principle. This approach for its correct application requires more computing resources as compared with the methods that use the ray theory.

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