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Some results of the group approach in the kinematic seismic problem

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On the basis of group bundle of the eikonal equation the new equations, estimates and formulas are obtained which give a new description of the direct and inverse kinematic seismic problem (geometrical optics). The paper given is the summary of these results. They make it possible to reduce the direct and inverse problems to some boundary-value problems for the equations obtained.

Introduction

This paper is the summary of the results in the kinematic seismic (geometrical optics) problem obtained by the author on the basis of the group bundle of the eikonal equation in the space of the variables (t, x, y, τ, n^2). Here t is the source parameter; x, y are the Cartesian coordinates; $\tau = \tau(t, x, y)$ is the ware time field, and $n(x, y) = 1/c(x, y)$, where $c(x, y)$ is the speed of wave propagation. This approach for investigation of inverse and direct problems for differential equations is suggested and applied in [1–7]; ibid., the group bundle of a wide class of differential equations of mathematical physics is constructed. Some of the results given here are stated in details in [1–7]. At this approach, we consider the initial eikonal equation as the equation in two equal dependent variables: $\tau = u^1$ and $n^2 = u^2$. Accordingly, the group which is admitted by this equation is defined as a certain group G of the point transformations of the space t, x, y, u^1, u^2, and the group bundle is constructed relative to G. The properties of the group G are investigated in [1, 2]. The general theory of group bundles is stated in [8]. The group terms are taken as in [8].

In Section 1, we state the quasilinear wave equation as the transformation of the eikonal equation by means of a certain differential change of the independent and dependent variables. This change is generated by the group bundle of the eikonal equation which is constructed in the explicit form in [3]. The obtained quasilinear equation has some interesting properties. In particular, it admits the Lax representation as an L–A-pair. This equation is the resolving system of the group bundle of the eikonal equation relative to the group G. As it is found in [3], for a large family $\{E\}'$ of the differential equations E the corresponding resolving system (of the quasilinear differential equations) RE of group bundle of E relative to the group G admits the Lax representation as an L–A-pair. (The notion of the Lax pair plays an important role in the solution theory and in the theory of nonlinear equations integrable by means of the method of inverse problem [9]). A lot of classic equations of mathematical physics belong to the class $\{E\}'$. The eikonal equation is one element of this family. More detail about this question one can find in [3, 5, 6].

Further, it is occurred that certain simple differential combinations U^1, U^2 of the function $\tau(t, x, y)$ along an arbitrary ray are the solutions of the ordinary differential equations of the classic type, in particular, the Riccati equation and the linear second-order equation

$$U_{\tau\tau} + K(x, y)U = 0.$$

This fact is established in [3]. As it is occurred, the variable $K(x, y)$, contained in these equations of the classic form is the Gauss curvature of certain surface which is defined by the medium characteristic $n(x, y)$ [5]. The variables U^1, U^2, K are the invariants of the group G.

Besides the function $D^* = n(x, y)D^2(t, x, y)$, where $D(t, x, y)$ is the geometric divergence of rays, also satisfies this second-order equation along an arbitrary ray. That is why it is occurred that exactly the function $K(x, y)$ operates by the behavior of the geometric divergence $D(t, x, y)$ (and the function D^*) and by its properties.

Applying the theory of such classic equations, we obtain as the consequences the following results (Sections 2–9):

1. A new method of calculating the geometric divergence of the rays $D(t, x, y)$ in the direct problem.
2. The connection between the $n(x, y)$ ($K(x, y)$) behavior and the ray (and $D(t, x, y)$) properties.
3. The upper and lower estimates for $D(t, x, y)$ and D^* with its derivatives.
4. The comparison estimates for D and D^* in the different points of the same ray.
5. The comparison theorems for the geometrical divergence $D(t, x, y)$ (and D^*) and its derivatives in the different rays of the same medium $n(x, y)$ and in the different media.
6. The connection of the equations obtained with the differential and Riemann geometries.
7. The connection with the Yakobi equation known in the variational calculus and analytical dynamics [10, § 36; 11, § 186].

8. The series of the integral formulas in the local inverse problems. These formulas give the expression of some functionals (the integrals along the ray, along a two-dimensional domain and so on) from the function $n(x,y)$ or from $n(x,y)$, $\tau(t,x,y)$ in terms the data of the inverse kinematic problem (the hodograph of the refracted waves).
9. The explicit formulas in the quadratures for the variables $S = (D^*_\tau)^2$ and τ_{tt} as the functions of the new independent variable $z = z(\tau)$ along the ray and as the solutions of the equation $S_z + K = 0$ and so on.

Also in the one-dimensional case $n = n(y)$ and some explicit formulas are obtained. In particular, they make it possible 1) to reduce the one-dimensional inverse problem to the moment problem which has the explicit solution; 2) to obtain the explicit expressions for the integrals of the form $\int f(n)\, dS$ along a ray for the different functions $f(n)$ in terms of the hodograph function $\tau_0(t,x) = \tau(t,x,0)$.

Further, we state the closed scalar quasilinear differential equations for n and for $(\Delta \ln n)/n^2$ as the functions of new independent ray (group) variables t, τ, $p = \tau_t$ (which are the invariants of the group G). Also the closed system of the nonlinear first-order equations and the closed system of the quasilinear second-order equations in the functions $x = x(t,\tau,p)$, $y = y(t,\tau,p)$ defining the rays are obtained. Unlike the known ray equations they do not contain $n(x,y)$ which is unknown in the inverse problems. Also other systems of the equations are stated (Section 7).

All these equations and formulas obtained give a new description of the two-dimensional (direct and inverse) kinematic seismic problem (geometrical optics) and make it possible to reduce these problems to some problems for these equations.

Further, we shall consider smooth (classic) solutions. Let $\Delta u = u_{xx}+u_{yy}$.

1. The basic variables and operators

Let $c(x,y) = 1/n(x,y)$ be the speed of propagation of some signals in the half-plane $x, y \geq 0$ which kinematics satisfies the Fermat principle. Suppose that the point sources of signals are continuously distributed on the interval $[0,T]$ of the x-axis. Suppose also that $t \equiv J^1$ is the source parameter (its x coordinate), and $\tau = \tau(t,x,y) \equiv J^2$ is a certain solution of the eikonal equation

$$J^7 \equiv \frac{(\tau_x)^2 + (\tau_y)^2}{n^2(x,y)} = 1, \quad n(x,y) \geq n_1 > 0, \tag{1}$$

$p = \tau_t(t,x,y) \equiv J^3 = -n(t,0)\sin\theta_0$, where θ_0 is the angle between the positive direction of the y-axis and the tangent line to the ray (geodesic) in the source point. The function $\tau(t,x,y)$ is the time of the signal (wave) propagation along the ray $\gamma(t,p)$ which connect the points $(t,0)$ and (x,y). The value p is the constant along any ray $\gamma(t,p)$ and is its parameter. We suppose that the equalities $x = x(t,\tau,p)$, $y = y(t,\tau,p)$ give a parameter representation with the parameter τ $(x_\tau^2 + y_\tau^2 = n^{-2})$ of the ray with the vertex in the source $x = t$ and the parameter p.

Let the functions $h(t,\tau,p)$, $v(t,\tau,p)$, $U^1(t,\tau,p)$, $U^2(t,\tau,p)$ be defined by the equalities

$$h(t,\tau,p) = \frac{\Delta\tau}{n^2}(t,x,y) \equiv J^4, \quad v(t,\tau,p) = \tau_{tt}(t,x,y) \equiv J^5, \tag{2}$$

$$U^1(t,\tau,p) = vU^2, \quad U^2(t,\tau,p) = \left\{\frac{\tau_y\tau_{tx} - \tau_x\tau_{ty}}{n^2}\right\}^{-1} \equiv (J^6)^{-1}, \tag{3}$$

where we set $x = x(t,\tau,p)$ and $y = y(t,\tau,p)$ in the right-hand sides. The following equalities hold:

$$\frac{\partial(x,y)}{\partial(\tau,p)} = -\frac{U^2}{n^2}, \quad n^2 = \frac{p^2 + (U^1)^2}{x_t^2 + y_t^2}.$$

Let the operators A_1, A_2, A_3 be defined by the equalities

$$A_1 = \frac{\partial}{\partial t}, \quad A_2 = \frac{1}{n^2}\left(\tau_x\frac{\partial}{\partial x} + \tau_y\frac{\partial}{\partial y}\right), \quad A_3 = \frac{1}{n^2}\left(\tau_y\frac{\partial}{\partial x} - \tau_x\frac{\partial}{\partial y}\right). \tag{4}$$

We have the identities $A_i\Phi = \tilde{A}_i\varphi$ that hold for each smooth function $\Phi(t,x,y) = \varphi(t,\tau,p)$, where

$$\tilde{A}_1 = \frac{\partial}{\partial t} + p\frac{\partial}{\partial\tau} + v\frac{\partial}{\partial p}, \quad \tilde{A}_2 = \frac{\partial}{\partial\tau}, \quad \tilde{A}_3 = \frac{1}{U^2}\frac{\partial}{\partial p} = \frac{1}{n}\frac{\partial}{\partial\nu}, \tag{5}$$

where ν is the normal to the ray $\gamma(t,p)$.

The following statements are established by means of group bundle of the eikonal equation in the variables t, x, y, $u^1 = \tau$, $u^2 = n^2$ (some of them are stated in details in [1–7]).

2. The quasilinear equations and the Lax pair

Theorem 1. *Suppose that in some neighborhood of the point $M(t,x,y)$ the function $\tau(t,x,y)$ satisfies equation (1); the derivatives τ_{ttx}, τ_{tty}, τ_{txx}, τ_{txy}, τ_{tyy}, n_x, n_y are continuous; and $J \equiv \partial(t,\tau,p)/\partial(t,x,y) = \tau_x\tau_{ty} - \tau_y\tau_{tx} \neq 0$ at the point M. Then in a certain neighborhood of the point $\tilde{M}(t,\tau,p)$, where $t = t$, $\tau = \tau(t,x,y)$, $p = \tau_t(t,x,y)$ the function v satisfies the quasilinear wave equation*

$$pv_{\tau\tau} + v_{\tau t} + vv_{\tau p} - 2v_\tau v_p = 0, \tag{6}$$

and the functions U^1, U^2, $v = v_1$, $v_2 = 1/U^2$ *defined by formulas* (2), (3) *satisfy the equivalent systems*

$$pU^2_\tau + U^2_t + U^1_p = 0, \quad U^1U^2_\tau - U^2U^1_\tau = 1, \tag{7}$$

$$v_\tau = -(v_2)^2, \quad pv_2 + vv_{2p} + v_{2t} - v_2v_p = 0. \tag{8}$$

The derivatives entered in (6)–(8) *are continuous.*

The proof of Theorem 1 is given in [3, 5, 6]. Equations (6)–(8) also follow from the system

$$\tilde{A}_1x = 0, \quad \tilde{A}_1y = 0, \tag{9}$$

$$U^2x_\tau + y_p = 0, \quad U^2y_\tau - x_p = 0. \tag{10}$$

The equalities $A_1x = 0$, $A_1y = 0$, $x_\tau = \tau_x/n^2$, $y_\tau = \tau_y/n^2$, $x_p/U^2 = \tau_y/n^2$, $y_p/U^2 = -\tau_x/n^2$ imply (9), (10).

Theorem 1 gives the local transformation of the eikonal equation (1) into equation (6) or into (7), (8). However, equations (6)–(8) hold in whole if the family of rays is regular (see Section 4).

We have

$$\lim_{\tau\to 0} \tau^{k+1}\frac{\partial^k v}{\partial \tau^k} = (-1)^k k!\alpha^2, \quad k = 0,1,2, \tag{11}$$

$$U^1 = -\alpha, \quad U^1_\tau = 0, \quad U^2 = 0, \quad U^2_\tau = -\alpha^{-1} \text{ for } \tau = 0, \tag{12}$$

where $\alpha = \{n^2(t,0) - p^2\}^{1/2} = n(t,0)\cos\theta_0 > 0$.

Equation (6) and systems (7), (8) have some interesting properties:

2.1. For them there exists the so-called L–A-pair or the Lax representation. This means that there exist two linear differential operators L and A acting on the functions of τ, p, with coefficients depending on the solution of equation (6) (or (7), (8)) so that equation (6) (or (7), (8)) is equivalent to the equation

$$\left[L, \frac{\partial}{\partial t} - A\right] = 0 \quad \Longleftrightarrow \quad \frac{\partial L}{\partial t} = [A, L],$$

where [ ] is the commutator. For (6) L and A are as follows:

$$L = \frac{\partial^2}{\partial\tau^2} - v_\tau\frac{\partial^2}{\partial p^2} - \frac{1}{2}\left\{\frac{v_{\tau\tau}}{v_\tau}\frac{\partial}{\partial\tau} + v_{\tau p}\frac{\partial}{\partial p}\right\}, \quad A = -p\frac{\partial}{\partial\tau} - v\frac{\partial}{\partial p}.$$

The operators L and A for (7), (8) are obtained from here after the change $v_\tau = -(v_2)^2$, $v = U^1/U^2$, $v_2 = 1/U^2$. The operator $\hat{L} = U^2L$ is self-adjoint,

$\hat{L}\varphi = (\varphi_\tau U^2)_\tau + (\varphi_p/U^2)_p$, and if $\Phi(t,x,y) = \varphi(t,\tau,p)$, then $L\varphi = \Delta\Phi/n^2$. We also have $L = \tilde{A}_2\tilde{A}_2 + \tilde{A}_3\tilde{A}_3 + h\tilde{A}_2$.

2.2. Equation (6) can be written as the conservation law or in the divergent form

$$\operatorname{div}\{(-v_\tau)^{-1/2}(\vec{t} + p\vec{\tau} + v\vec{p})\},$$

where $\vec{t}$, $\vec{\tau}$, $\vec{p}$ are the orts.

2.3. After the renaming $\tau = t$, $p = x^1$, $t = y$ and the change $x = (x^1)^2/2$, $v(t,\tau,p) = u(t,x,y)$ equation (6) becomes as follows

$$\{u_t + uu_x + (2x)^{-1/2}u_y\}_t - 3u_tu_x = 0,$$

wnere $u_t + uu_x$ is the left-hand side of the known nonlinear equation.

2.4. In the one-dimensional case $n = n(y)$, we have $\tau = \tau(x-t, y)$, $p = -\tau_x$, $v = \tau_{xx}$, $v_t \equiv 0$, $x_t \equiv 1$, $y_t \equiv 0$; and instead of (6) we have the equation $pv_{\tau\tau} + vv_{\tau p} - 2v_\tau v_p = 0$ which has the hyperbolic type everywhere if $v \neq 0$, otherwise it has the parabolic type (if $v = 0$, $y_\tau = 0$). The equivalent system $v_\tau = -(U^2)^{-2}$, $pU_\tau^2 + (vU^2)_p = 0$ has the following conservation laws

$$\frac{\partial\Phi(\tau,p,v,U^2)}{\partial\tau} + \frac{\partial\Psi(\tau,p,v,U^2)}{\partial p} = f(\tau,p,v,U^2),$$

where

$$\Phi = \frac{pF(vU^2)}{v} + \Phi_0, \quad \Psi = F(vU^2),$$

$$f = pf_0(vU^2) - \frac{1}{(U^2)^2}\frac{\partial\Phi_0}{\partial v}, \quad f_0(\xi) = -\left\{\frac{F(\xi)}{\xi}\right\}_\xi,$$

and $F(\xi)$, $\Phi_0(\xi)$ are arbitrary differentiable functions. This means that we have more than a countable set of conservation laws. We also have $U_\tau^1 = \psi(p^2 + (U^1)^2)$, where $\psi(n^2) = [nf_n(n)]^{-1}$, $y = f(n)$ is inverse to $n = n(y)$.

2.5. Equation (6) has the class of solutions of the type of traveling waves $v = V(\mu t + \beta\tau, p)$ ($\mu = \text{const}, \beta = \text{const}$) satisfying the equation $(\mu + \beta p)V_{\xi\xi} + VV_{\xi p} - 2V_\xi V_p = 0$. Thus, the one-dimensional case satisfies the particular class of such solutions with the parameters $\mu = 0$, $\beta = 1$.

3. The Riccati equation and the linear equation

As it is occurred along each ray $\gamma(t,p)$, the functions h, U^1, U^2, v as the functions τ are the solutions of the linear ordinary differential equations of the classic type.

Theorem 2. *Suppose that in some neighborhood of the ray $\gamma(t,p) : x = x(t,\tau,p), y = y(t,\tau,p)$ the solution $\tau(t,x,y)$ of equation (1) and a function $n(x,y)$ are defined. Suppose also that the derivatives τ_{xxx}, τ_{xxy}, τ_{xyy}, τ_{yyy}, n_{xx}, n_{yy} are continuous in $\gamma(t,p)$. Then the function $h(t,\tau,p)$ of the form* (2) *satisfies the Riccati equation*

$$h_\tau + h^2 = -K(x,y), \tag{13}$$

and h_τ, K are continuous in $\gamma(t,p)$. Moreover, if the derivatives τ_{txx}, τ_{txy}, τ_{tyy} are continuous, then $U^2_\tau = hU^2$ and the functions $U^1(t,\tau,p)$, $U^2(t,\tau,p)$, of the type (3) *form the fundamental system of solutions of the following linear ordinary (with respect to τ) differential equation of second order*

$$U_{\tau\tau} + K(x,y)U = 0, \tag{14}$$

with the initial conditions (12), *and the function $v(t,\tau,p)$ satisfies the equation*

$$\{v,\tau\} \equiv \frac{v_{\tau\tau\tau}}{v_\tau} - \frac{3}{2}\left(\frac{v_{\tau\tau}}{v_\tau}\right)^2 = 2K(x,y) \tag{15}$$

with the initial conditions (11), *where $\{v,\tau\}$ is the so-called differential Schwarz invariant of equation* (14). *Here the function $K(x,y)$ is defined by the equality*

$$K(x,y) \equiv -\frac{\Delta \ln n(x,y)}{n^2(x,y)} \equiv J^{11}. \tag{16}$$

This statement is formulated in [5, 6]. Note that equations (13), (14), (15) are established in 1988 (see equalities (9.17), (12.1)–(12.3) in [3]), but this statement is not formulated in [3] in the form of the special theorem.

Equation (13) follows from the identity $A_2J^4 + (J^4)^2 = -J^{11}$ which holds for each smooth solution $\tau(t,x,y)$ of equation (1) (see [3, 5]). This identity is a special case of more general formula [1–3, 5]. The equality $U^2_\tau = hU^2$ follows from the identity $A_2J^6 = -J^4J^6$. From here we obtain (14) for $U = U^2$. The equalities $U^2_{\tau\tau} + KU^2 = 0$, and $v_\tau = -(U^2)^{-2}$ imply $U^1_{\tau\tau} + KU^1 = 0$ and (15).

Note that (7) implies $\tilde{A}_1(U^i_{\tau\tau}/U^i) = 0, i = 1,2$ and (6) implies $\tilde{A}_1\{v,\tau\} = 0$. This means that $(U^i_{\tau\tau}/U^i)$ and $\{v,\tau\}$ depend only on x, y.

Remark 1. Equalities (14) for $U = U^1$, $U = U^2$ and (13), (15) also can be obtained from system (9), (10) if we set $n^{-2} = x^2_\tau + y^2_\tau$ (see also Sections 7.5, 7.6).

Theorem 2 implies

Corollary 1. *Let $D(t,x,y)$ be the geometric divergence of the rays with vertices in the source $x = t$. (The function $D(t,x,y)$ is defined by the* *equality* $D^2(t,x,y) = d\sigma/d\theta_0$, *where* $d\sigma$ *is the ray cut with the angle* $d\theta_0$). *Then the function* $D^*(t,\tau,p)$ *of the form*

$$D^*(t,\tau,p) = n(x,y)D^2(t,x,y) \tag{17}$$

is the solution of equation (14) *with the initial conditions*

$$D^* = 0, \quad D^*_\tau = 1 \quad \textit{for } \tau = 0. \tag{18}$$

It follows from the formula

$$D^2(t,x,y) = \alpha|U^2|/n(x,y) \tag{19}$$

and from (12).

Remark 2. Determining D^* as the solution of (14) under conditions (18), we obtain a new method of calculating the geometric divergence of the rays $D(t,x,y)$.

Theorem 2 implies the following statements.

Remark 3. Using the known formula of differential geometry [10, p. 113], we obtain that in equations (13)–(15) the function $K(x,y)$ of the form (16) is the Gauss curvature of the surface in the three-dimensional Euclidean space with the linear element $d\tau^2 = n^2(x,y)(dx^2+dy^2)$. This gives us the connection of equations (13)–(15) with differential and Riemann geometries.

Remark 4. Equation (14) can be associated with the equation of vibration of an inhomogeneous string $\varphi_{zz} + \{k^2V^2(z)\}\varphi = 0$, where the function $V(z) = -v_\tau = (U^2)^{-2}$ is the speed of wave propagation, $z = v(t,\tau,p) +$ const. This equation is transformed into the Sturm–Liouville equation $W_{\tau\tau} + \{k^2 - q(t,\tau,p)\}W = 0$, where $g = -\{v,\tau\}/2 = -K(x,y)$, after the change $\tau = \int_{z_0}^{z}(U^2)^2dz$, $W = U^2\varphi$, and vice versa.

Remark 5. There is the following connection between equation (14) and the Yakobi equation known in the variational calculus *

$$\nabla^2_{\dot{x}}\xi^i + R^i_{jlk}\dot{x}^j\dot{x}^k\xi^l = 0, \quad i = 1,2,\ldots,m,$$

where

$$\nabla_{\dot{x}}\xi^i = \frac{d\xi^i}{d\tau} + \Gamma^i_{jk}\xi^j\dot{x}^k.$$

*The relation of equation (14) to the Yakobi equation was considered by the author after discussion with A.L. Bukhgeim.

Here Γ^i_{jk} are the Christoffel symbols, R^i_{jlk} is the Riemann tensor, $x = x(\tau)$ is the parametric representation of the geodesic (ray) for the metric $g_{ij}(x)$, $x = (x^1, x^2, \ldots, x^m)$, $\dot{x}^i = dx^i/d\tau$ [10, § 36].

In our case, $m = 2$, $x^1 = x$, $x^2 = y$, $g_{ij} = n^2(x,y)\delta_{ij}$, δ_{ij} is the Kronecker symbol, $\dot{x}^1 = x_\tau$, $\dot{x}^2 = y_\tau$,

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = (\ln n)_x, \quad R^1_{212} = R^2_{121} = -\Delta \ln n,$$
$$-\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = (\ln n)_y, \quad R^1_{221} = R^2_{112} = \Delta \ln n,$$

and all the others R^1_{ijk}, R^2_{ijk} are vanishing functions; so the Yakobi equation is the system of the form

$$\xi^1_{\tau\tau} + a_1\xi^1_\tau - a_2\xi^2_\tau + b_1\xi^1 - b_2\xi^2 - \Delta \ln n \cdot y_\tau(y_\tau\xi^1 - x_\tau\xi^2) = 0,$$
$$\xi^2_{\tau\tau} + a_2\xi^1_\tau + a_1\xi^2_\tau + b_2\xi^1 + b_1\xi^2 + \Delta \ln n \cdot x_\tau(y_\tau\xi^1 - x_\tau\xi^2) = 0,$$

where

$$a_1 = 2(\ln n)_\tau, \quad b_1 = (\ln n)_{\tau\tau} + [(\ln n)_\tau]^2 - (A_3 \ln n)^2,$$
$$a_2 = 2A_3 \ln n, \quad b_2 = (A_3 \ln n)_\tau + 2(\ln n)_\tau A_3 \ln n.$$

Here

$$\frac{d}{d\tau} = A_2 = x_\tau \frac{\partial}{\partial x} + y_\tau \frac{\partial}{\partial y}, \quad A_3 = y_\tau \frac{\partial}{\partial x} - x_\tau \frac{\partial}{\partial y} = (U^2)^{-1}\frac{\partial}{\partial p} = \frac{1}{n}\frac{\partial}{\partial \nu},$$

ν is the normal to the ray $\gamma(t,p)$. Suppose that the vector field $\vec{\xi} = (\xi_1, \xi_2)$ solves the Yakobi equation and is orthogonal to the ray $\gamma(t,p)$. As is known [11, § 186], the length $\xi = n(x,y)\{\xi_1^2 + \xi_2^2\}^{1/2}$ of the vector $\vec{\xi}$ in the metric $d\tau^2 = n^2(dx^2 + dy^2)$ satisfies equation (14). Hence it follows that

1) ξ can be represented as $\xi = \sum_{i=1}^2 c_i(t,p)U^i(t,\tau,p)$;
2) if ξ is the Yakobi vector field along the ray $\gamma(t,p)$ with the conjugate points $(t,0)$ and (x_0, y_0), then

$$\xi = c(t,p)U^2(t,\tau,p) = -\frac{1}{\alpha}c(t,p)n(x,y)D^2(t,x,y),$$

where c_i, c are certain functions and $D(t, x_0, y_0) = 0$, i.e., the rays are focused in the point (x_0, y_0).

The other consequences from Theorem 2 are obtained in the subsequent sections.

4. The connection between the $n(x, y)$ behavior and the ray properties

From the known comparison theorem for the equation of the form (14) we obtain the connection between the $n(x, y)$ behavior, i.e., $K(x, y)$ and the ray properties (zeros of the functions $U^2(t, \tau, p)$, $D(t, x, y)$). Suppose that the rays $\gamma(t, p)$ fill some domain Ω in the half-plane $x, y \geq 0$. We consider two examples.

4.1. If $K(x, y) \leq 0$, i.e.,

$$\Delta \ln n(x, y) \geq 0 \tag{20}$$

in each ray $\gamma(t, p) \subset \Omega$ then each non-trivial solution of (14) has not more than one zero in each ray $\gamma(t, p)$. Therefore, according to (12), the function $U^2(t, \tau, p)$ for each $t =$ const $p =$ const has the zero only for $\tau = 0$ and $U^2 < 0$ for $\tau > 0$. (Also, we have $U^2_\tau < 0$, $U^2_{\tau\tau} \leq 0$ for $\tau > 0$). Thus we obtain the following statement as the corollary of equation (14) and formula (19).

Rule 1. *If condition* (20) *holds in the ray* $\gamma(t, p)$ *or in the domain* Ω*, then the geometric divergence of the rays* $D(t, x, y) > 0$ *everywhere in the ray* $\gamma(t, p)$ *or in* Ω *except the source points* $x = t$, $y = 0$ (*where* $D = 0$).

The condition $D(t, x, y) > 0$ is necessary and sufficient for regularity of the ray family for each t. Inequality (20) is the known sufficient condition for regularity.

The theorem about alternation of zeros of linearly independent solutions of (14) and (12) imply the following

Rule 2. *If the condition* (20) *holds in the ray* $\gamma(t, p)$ *or in the domain* Ω *then the functions* U^1, v *have no zeros and* $U^1 <$ const < 0 (*also* $U^1_\tau \leq 0$, $U^1_{\tau\tau} \leq 0$), $v(t, \tau, p) = \tau_{tt}(t, x, y) >$ const > 0 *in the ray* $\gamma(t, p)$ *or in* Ω.

More exact valuations for D,U^1, U^2 and its derivatives are obtained in Section 5.

4.2. If $K(x, y) \leq (\pi/\bar{\tau})^2$ in the interval $[0, \bar{\tau}]$ of the ray $\gamma(t, p)$, then each non-trivial solution of (14) has no more than one zero in this interval. So $U^2(t, \tau, p)$ has zero only for $\tau = 0$, $U^2 < 0$ for $\tau \in (0, \bar{\tau}]$ and $D(t, x, y) > 0$ everywhere in the interval $\tau \in (0, \bar{\tau}]$ of the ray $\gamma(t, p)$. If we consider the whole ray $\gamma(t, p)$, then $\bar{\tau} = \varphi(t, p)$ is the hodograph function in τ, t, p. (We have $\varphi(t, p) = \tau_0(t, x)$, where $\tau = \tau_0(t, x) \equiv \tau_0(t, x, 0)$ is the hodograph function in τ, t, x; $p = \tau_{0t}(t, x)$.) Therefore we have

Rule 3. *If*

$$\frac{\Delta \ln n(x,y)}{n^2(x,y)} > -\{\pi/\varphi(t,p)\}^2 \tag{21}$$

(where $\tau = \varphi(t,p)$ is the hodograph function) in the ray $\gamma(t,p)$ or in the domain Ω, then the geometric divergence of the rays $D(t,x,y) > 0$ everywhere in $\gamma(t,p)$ or in Ω except the source points $x = t,\ y = 0$ $(D(t,t,0) = 0)$.

Condition (21) is more general than (20) since the values of the function $\Delta \ln n(x,y)$ may be negative.

Rule 4. *If condition* (20) *holds in the ray $\gamma(t,p)$, then the functions $D^* = n(x,y)D^2(t,x,y)$, $|U^2|$, $\{\tau_{tt}(t,x,y)\}^{-1} = \{v(t,\tau,p)\}^{-1} = U^2/U^1$ are increasing and D^*_τ, $|U^2_\tau|$, $|U^1|$, $|U^1_\tau|$ are non-decreasing functions of τ for $\tau \geq 0$ in $\gamma(t,p)$ and*

$$\frac{U^2(\tau,t,p)}{U^2(\xi,t,p)} > \frac{U^1(\tau,t,p)}{U^1(\xi,t,p)} \quad \textit{for} \quad \tau > \xi > 0.$$

5. The estimates for the geometric divergence $D(t,x,y)$ and the functions U^1, U^2

Let the function $V^1(t,\tau,p)$, $V^2(t,\tau,p)$ be defined by the equalities

$$V^1 = -\alpha^{-1}U^1, \quad V^2 = -\alpha U^1.$$

Then V^1, V^2 form the fundamental system of solutions of (14) with the initial conditions

$$V^1 = 1, \quad V^1_\tau = 0, \quad V^2 = 0, \quad V^2_\tau = 1 \quad \text{for } \tau = 0.$$

The equality (19) implies

$$V^2 = D^* = nD^2.$$

5.1. Using the comparison theorem for (14) and V^1, V^2 [12, Sections 24.3, 25.5] we obtain the first set of estimates in the form of

Rule 5. *Suppose that the ray $\gamma(t,p)$ fill some domain Ω in half-plane $x, y \geq 0$. If condition* (20) *holds in the ray $\gamma(t,p)$ or in the domain Ω, then in each points (x,y) of $\gamma(t,p)$ or of Ω we have the following estimates for the geometric divergence $D(t,x,y)$ and its derivatives:*

$$\begin{aligned}\tau(x,y) \le n(x,y)D^2(t,x,y) \le k^{-1}\,\mathrm{sh}\{k\tau(x,y)\},\\ 1 \le (nD^2)_\tau \le \mathrm{ch}\{k\tau(x,y)\},\\ k_0\tau(x,y) \le |K(x,y)|\tau(x,y) \le (nD^2)_{\tau\tau}\\ \le k^{-1}|K(x,y)|\,\mathrm{sh}\{k\tau(x,y)\} \le k\,\mathrm{sh}\{k\tau(x,y)\},\end{aligned} \tag{22}$$

where $k > 0$, $k^2 = \sup|K(x,y)|$, $k_0 = \inf|K(x,y)|$ *and the operations* sup, inf *are making in the ray* $\gamma(t,p)$ *or in the domain* Ω *respectively.*

The estimates for U^2, U^2_τ, $U^2_{\tau\tau}$ can be obtained from (22) by meas of division into α since $U^2 = -\alpha^{-1}nD^2$. Also, we have

$$1 \le V^1 \le \mathrm{ch}(k\tau), \quad 0 \le V^1_\tau(t,\tau,p) \le k\,\mathrm{sh}(k\tau)$$

and the analogous estimates for $|U^1|$, $|U^1_\tau|$.

5.2. Representing equation (14) in the form of the first order system $U_\tau = \tilde{U}$, $\tilde{U}_\tau = -KU$ and using the estimates theorem for its solutions U, $\tilde{U}$ [12, Section 8.4] we obtain the second set of estimates for the functions $D(t,x,y)$, V^1, V^2, U^1, U^2 in arbitrary ray $\gamma(t,p)$.

Rule 6. *Denote*

$$\begin{aligned}\Sigma_j^{(i)}(\tau,t,p) &= |V^i(\tau,t,p)|^j + |V^i_\tau(\tau,t,p)|^j,\\ S_j(t,x,y) &= \{n(x,y)D^2(t,x,y)\}^j + \{[n(x,y)D^2(t,x,y)]_\tau\}^j,\\ M_1 &= \max\{1,\sup_\gamma|K(x,y)|\}, \quad M_2 = 1+\sup_\gamma|K(x,y)|,\end{aligned}$$

where $i,j = 1,2$. *Then for each two points* (x,y) *and* (x_0,y_0) *of the same ray* $\gamma(t,p)$ *and for each two values* $\tau = \tau(t,x,y)$ *and* $\xi = \tau(t,x_0,y_0)$ *of the variable* τ *in the ray* $\gamma(t,p)$ *the following estimates hold true:*

$$\begin{aligned}\Sigma_j^{(i)}(\tau,t,p) &\le \Sigma_j^{(i)}(\xi,t,p)e^{M_j|\tau-\xi|},\\ S_j(t,x,y) &\le S_j(t,x_0,y_0)e^{M_j|\tau(t,x,y)-\tau(t,x_0,y_0)|}\end{aligned}$$

for $i,j = 1,2$. *In particular, for* $\xi = 0$, *i.e.,* $x_0 = t$, $y_0 = 0$ *we obtain the estimates*

$$\Sigma_j^{(i)}(\tau,t,p) \le e^{M_j\tau}, \quad S_j(t,x,y) \le e^{M_j\tau(t,x,y)}, \quad i,j = 1,2.$$

The analogous estimates we have for the functions $U^1 = -\alpha V^1$ and $U^2 = -\alpha^{-1}V^2 = -\alpha^{-1}nD^2$. We remind that (see (4), (5))

$$\frac{\partial}{\partial\tau} = \frac{1}{n^2}\left\{\tau_x\frac{\partial}{\partial x} + \tau_y\frac{\partial}{\partial y}\right\} \quad (= A_2).$$

6. The comparison theorem for the geometric divergence $D(t, x, y)$ and for the functions U^1, U^2

6.1. The comparison of two rays in the same medium. Suppose that $\gamma_1 = \gamma(t, p_1)$ and $\gamma_2 = \gamma(t, p_2)$ are two any rays in the same medium $n(x, y)$ with the parameters p_1, p_2 and with the vertices in the source point $x = t$, $y = 0$. We consider the same interval $0 \le \tau \le \tilde{\tau}$ in the rays γ_1, γ_2, and denote $(i = 1, 2)$

$$\begin{aligned} K_i(\tau) &= K(t, \tau, p_i) = K(x(t, \tau, p_i), y(t, \tau, p_i)), \\ D_i^*(\tau) &= D^*(t, \tau, p_i) = n(x(t, \tau, p_i), y(t, \tau, p_i))D^2(t, x(t, \tau, p_i), y(t, \tau, p_i)), \end{aligned}$$

where $x = x(t, \tau, p)$, $y = y(t, \tau, p)$ are the equations of the ray $\gamma(t, p)$, $D(t, x, y)$ is the geometric divergence of the rays with vertices in t.

Rule 7. *Suppose that* $\Delta \ln n(x, y) \ge 0$ *and* $|K_1(\tau)| \le |K_2(\tau)|$ *for* $0 \le \tau \le \tilde{\tau}$. *Then*

$$\frac{d^m D_1^*}{d\tau^m}(\tau) \le \frac{d^m D_2^*}{d\tau^m}(\tau) \ \textit{for}\ 0 \le \tau \le \tilde{\tau}, \quad m = 0, 1, 2. \tag{23}$$

Note the points with the value $\tau =$ const in γ_1 and γ_2 are in the wave front $\tau =$ const. Analogously we can consider two rays with different vertices t_1 and t_2.

6.2. The comparison of two rays in the different media. Suppose that $\gamma_1 = \gamma_1(t_1, p_1)$ and $\gamma_2 = \gamma_2(t_2, p_2)$ are two any rays in the medium $n_1(x, y)$ and $n_2(x, y)$ respectively. The source coordinates t_1, t_2 and parameters p_1, p_2 may be arbitrary (the same or different). We consider the same interval $0 \le \tau \le \tilde{\tau}$ of the rays γ_1, γ_2. Denote $(i = 1, 2)$

$$\begin{aligned} K_i(\tau) &= K_i(t_i, \tau, p_i) = K_i(x_i(t_i, \tau, p_i), y_i(t_i, \tau, p_i)), \\ D_i^*(\tau) &= D_i^*(t_i, \tau, p_i) \\ &= n_i(x_i(t_i, \tau, p_i), y_i(t_i, \tau, p_i))D_i^2(t_i, x_i(t_i, \tau, p_i), y_i(t_i, \tau, p_i)), \end{aligned}$$

where $x = x_i(t, \tau, p)$, $y = y_i(t, \tau, p)$ are the equations of the ray $\gamma_i(t, p)$, $D_i(t, x, y)$ is the geometric divergence of the rays with vertexes in t in the medium with the parameter $n_i(x, y)$,

$$K_i(x, y) = -\frac{\Delta \ln n_i(x, y)}{n_i^2(x, y)}.$$

Rule 8. *Suppose that* $\Delta \ln n_i(x, y) \ge 0$ *and* $|K_1(\tau)| \le |K_2(\tau)|$ *for* $0 \le \tau \le \tilde{\tau}$. *Then the inequality of the form* (23) *holds true for* $0 \le \tau \le \tilde{\tau}$, $m = 0, 1, 2$.

6.3. The comparison of two intersecting rays in the different media. Suppose that $\gamma_1 = \gamma_1(t, p_1)$ and $\gamma_2 = \gamma_2(t, p_2)$ are two rays with the same vertex t in the medium $n_1(x, y)$ and $n_2(x, y)$ respectively and γ_1, γ_2 have the intersection point (x_0, y_0). Denote $(i = 1, 2)$

$$\begin{aligned} K_i(\tau) &= K_i(t, \tau, p_i) = K_i(x_i(t, \tau, p_i), y_i(t, \tau, p_i)), \\ D_i^*(\tau) &= D_i^*(t, \tau, p_i) \\ &= n_i(x_i(t, \tau, p_i), y_i(t, \tau, p_i)) D_i^2(t, x_i(t, \tau, p_i), y_i(t, \tau, p_i)), \end{aligned}$$

where $K_i(x, y)$, $D_i(t, x, y)$, x_i, y_i are defined in Section 6.2. Using Rule 4 (Section 4) we obtain

Rule 9. *Suppose that* $\Delta \ln n_i(x, y) \geq 0$ *and* $\tau_1 = \tau_1(t, x_0, y_0) \leq \tau_2(t, x_0, y_0)$, *where* $\tau_i(t, x, y)$ *is the time field* $\tau(t, x, y)$ *for* $n_i(x, y)$, $i = 1, 2$. *If* $|K_1(\tau)| \leq |K_2(\tau)|$ *for* $0 \leq \tau \leq \tau_1$, *then*

$$n_1(x_0, y_0) D_1^2(t, x_0, y_0) \leq n_2(x_0, y_0) D_2^2(t, x_0, y_0)$$

and in the point $x = x_0$, $y = y_0$

$$\{n_1(x, y) D_1^2(t, x, y)\}_1^{(k)} \leq \{n_2(x, y) D_2^2(t, x, y)\}_2^{(k)}, \quad k = 1, 2, \tag{24}$$

where we denote by $f_i^{(k)}$ *the derivative of the* k*-th order of the function* $f(t, x, y)$ *along the ray* γ_i *(with respect to* τ*). We have*

$$f_i^{(1)} = \frac{1}{n_i^2(x, y)} \left\{ \tau_{ix}(t, x, y) \frac{\partial}{\partial x} + \tau_{iy}(t, x, y) \frac{\partial}{\partial y} \right\} f, \quad i = 1, 2. \tag{25}$$

6.4. The "global" comparison theorem. Suppose that for some source $t = \text{const}$ the rays $\gamma_i(t, p)$ in the medium $n = n_i(x, y)$ fill the domain Ω in the half-plane $x, y \geq 0$ and the functions $K_i(x, y)$, $D_i(t, x, y)$, $\tau_i(t, x, y)$ are defined in Ω_i according to Section 6.2, 6.3, $i = 1, 2$. Rule 9 implies

Theorem 3. *Suppose that:*

1) $\Delta \ln n_i(x, y) \geq 0$ *in* Ω_i, $i = 1, 2$;
2) *for each point* (x, y) *of the intersection domain* $\Omega = \Omega_1 \cap \Omega_2$
$$\tau_1(t, x, y) \leq \tau_2(t, x, y);$$
3) *for each pair of points* (x, y) *and* (x_0, y_0) *of the sum domain* $\Omega^+ = \Omega_1 \cup \Omega_2$
$$|K_1(x, y)| \leq |K_2(x_0, y_0)|.$$

Then in every point (x, y) *of the domain* Ω *we have the inequality*

$$n_1(x,y)D_1^2(t,x,y) \le n_2(x,y)D_2^2(t,x,y)$$

and the inequalities of the form (24), (25).

Note the inequality $n_1(x,y) \le n_2(x,y)$ implies the second condition of Theorem 3.

The analogous statements we obtain for U^1, U^2, V^1, V^2.

6.5. The comparison of U^1, U^2, V^1, V^2. Suppose that $U_i^j(\tau)$, $V_i^j(\tau)$ are the solutions of the equation $U_{\tau\tau} + K_i(\tau)U = 0$ $(i,j = 1,2)$ with the initial conditions for $\tau = 0$ of the form

$$U_i^1 = -\alpha_i, \quad U_{i\tau}^1 = 0, \quad U_i^2 = 0, \quad U_{i\tau}^2 = -\alpha_i^{-1}, \quad \alpha_i > 0,$$
$$V_i^1 = 1, \quad V_{i\tau}^1 = 0, \quad V_i^2 = 0, \quad V_{i\tau}^2 = 1.$$

Suppose that $K_i \le 0$ $(i = 1,2)$ and $|K_2| \ge |K_1|$ in some interval $0 \le \tau \le \bar{\tau}$. Then for $0 \le \tau \le \bar{\tau}$

$$V_2^j \ge V_1^j, \quad V_{2\tau}^j \ge V_{1\tau}^j, \quad j = 1,2;$$
$$|U_2^1| \ge \frac{\alpha_2}{\alpha_1}|U_1^1|, \quad |U_{2\tau}^1| \ge \frac{\alpha_2}{\alpha_1}|U_{1\tau}^1|, \quad |U_2^2| \ge \frac{\alpha_1}{\alpha_2}|U_1^2|, \quad |U_{2\tau}^2| \ge \frac{\alpha_1}{\alpha_2}|U_{1\tau}^2|;$$
$$\frac{U_{2\tau}^1}{U_2^1} \ge \frac{U_{1\tau}^1}{U_1^1}, \quad \text{i.e.,} \quad \left(\ln\frac{U_2^1}{U_1^1}\right)_\tau \ge 0;$$
$$\{U_2^2(\tau)\}^2\left(\ln\frac{U_2^2(\tau)}{U_1^2(\tau)}\right)_\tau \ge \{U_2^2(\xi)\}^2\left(\ln\frac{U_2^2(\xi)}{U_1^2(\xi)}\right)_\xi$$

for $\tau > \xi > 0$. In two latter inequalities we may substitute V_i^j for U_i^j and $D_i^* = n_i D_i^2$ for U_i^2, $i = 1,2$. If $|K_2| > |K_1|$ at least in one point $\tau = \tau_0 \in [0,\bar{\tau}]$, then we everywhere have strict inequality for $\tau \ge \tau_0$ (for $\tau \ge \tau_0 \ge \xi > 0$, $\tau > \xi$ in the last inequality) and

$$h_2 = \frac{U_{2\tau}^2}{U_2^2} > \frac{U_{1\tau}^2}{U_1^2} = h_1, \quad \text{i.e.,} \quad \left(\ln\frac{U_2^2}{U_1^2}\right)_\tau > 0 \quad \text{for} \quad \tau \ge \tau_0 > 0.$$

In the last inequality, we may substitute $D_i^* = n_i D_i^2$ or V_i^2 for U_i^2, $i = 1,2$.

7. The equations for $x(t,\tau,p)$, $y(t,\tau,p)$, $n(t,\tau,p)$ and others

7.1. The known system of differential equations of the ray for the functions x, y is not closed since it contains the third function $n(x,y)$ which is unknown in the inverse problems. The closed nonlinear system of first order for $x(t,\tau,p)$, $y(t,\tau,p)$ of the form

$$px_\tau = -y_p \frac{x_t y_p - y_t x_p}{x_p^2 + y_p^2}, \tag{26}$$

$$py_\tau = x_p \frac{x_t y_p - y_t x_p}{x_p^2 + y_p^2} \tag{27}$$

is obtained as the consequence of (9), (10). The closed quasilinear system of second order for x and y of the form

$$A_x x_\tau - (y_\tau)^2 = 0, \quad A_y y_\tau - (x_\tau)^2 = 0 \tag{28}$$

is also obtain. Here

$$A_g \equiv g_p\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial \tau}\right) - (g_t + pg_\tau)\frac{\partial}{\partial p}.$$

It follows from (9), (10) or from (26), (27) if we take into account the equality $v_\tau = -(U^2)^{-2}$ (see also [7, Section 1.1]). Besides each of the functions x, y solves the scalar nonlinear equation of third order

$$S_g g \equiv B_g A_g g_\tau - 2A_g g_\tau (g_\tau g_p)_\tau = 0.$$

Here $B_g = g_\tau g_p \partial/\partial\tau - A_g g_\tau \partial/\partial p$. This equation can be represented in the divergent form

$$\left\{\frac{g_\tau g_p}{Q}\right\}_\tau + Q_p = 0, \quad Q = \{A_g g_\tau\}^{1/2}.$$

7.2. The functions v, U^1, U^2 can be expressed in terms of x, y as follows

$$v = -\frac{x_t x_p + y_t y_p}{x_p^2 + y_p^2}, \quad U^1 = vU^2, \quad U^2 = \frac{p(x_p^2 + y_p^2)}{x_t y_p - y_t x_p}, \tag{29}$$

$$v^2 = \frac{x_t^2 + y_t^2 - p^2(x_\tau^2 + y_\tau^2)}{x_p^2 + y_p^2}, \quad (U^1)^2 = \frac{x_t^2 + y_t^2}{x_\tau^2 + y_\tau^2} - p^2, \quad (U^2)^2 = \frac{x_p^2 + y_p^2}{x_\tau^2 + y_\tau^2} \tag{30}$$

and

$$n^2 = (x_\tau^2 + y_\tau^2)^{-1} = \frac{(U^2)^2}{x_p^2 + y_p^2} = \frac{p^2 + (U^1)^2}{x_t^2 + y_t^2}. \tag{31}$$

The closed system of second order for x and y of the form

$$\begin{aligned} &pU^2 x_{\tau\tau} - n^2 x_\tau\{y_\tau L_1 + p^{-1}U^2 x_\tau L_2\}x - \\ &\qquad n^2 y_\tau\{y_\tau L_1 + p^{-1}U^2 x_\tau L_2\}y - p^{-1}U^1 x_\tau - y_\tau = 0, \\ &pU^2 y_{\tau\tau} - n^2 y_\tau\{-x_\tau L_1 + p^{-1}U^2 y_\tau L_2\}y - \\ &\qquad n^2 x_\tau\{-x_\tau L_1 + p^{-1}U^2 y_\tau L_2\}x - p^{-1}U^1 y_\tau + x_\tau = 0, \end{aligned} \tag{32}$$

and the equivalent system

$$
\begin{aligned}
&p^2 x_{\tau\tau} - L_2 x + v x_\tau - p(U^2)^{-2} x_p - x_p(v v_p + v_t) = 0, \\
&p^2 y_{\tau\tau} - L_2 y + v y_\tau - p(U^2)^{-2} y_p - y_p(v v_p + v_t) = 0, \\
&v_\tau = -(U^2)^{-2}
\end{aligned} \tag{33}
$$

does not include the mixed derivatives $x_{p\tau}$, $x_{t\tau}$, $y_{p\tau}$, $y_{t\tau}$ (unlike (28)). Here

$$
L_1 = v\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial p \partial t}, \quad L_2 = v^2\frac{\partial^2}{\partial p^2} + 2v\frac{\partial^2}{\partial p \partial t} + \frac{\partial^2}{\partial t^2}
$$

and v, U^1, U^2, n^2 can be excluded by means of formulas (29), (30), (31).

7.3. The function $n(x, y)$ is arbitrary, so, in terms of the variables x, y the closed differential equation for it is absent (besides, $n_t = 0$). However, in the independent variables t, τ, p, there exists a closed scalar differential equation for $n(t, \tau, p) = n(x, y)$ which has no other functions: the function $g = \ln n(t, \tau, p)$ satisfies the nonlinear equation of fourth order

$$
g_p M_g g - 2A_g g_\tau S_g g = 0,
$$

where

$$
M_g g \equiv 2A_g g_\tau \{3(A_g g_{\tau\tau})_\tau - 2A_g g_{\tau\tau\tau}\} - 3(A_g g_{\tau\tau})^2
$$

which is represented in the divergent form

$$
\left\{ \frac{g_\tau g_p}{Q} - \left(\frac{g_p}{Q} \right)_\tau \right\}_\tau + Q_p = 0, \quad Q = (A_g g_\tau)^{1/2}.
$$

Also, the function

$$
f(t, \tau, p) = -K(x, y) \equiv \frac{\Delta \ln n}{n^2} = L \ln n(t, \tau, p)
$$

solves the fourth-order equation

$$
M_f f = -4(A_f f_\tau)^2 f
$$

which is represented in the form

$$
\left\{ \frac{f_p}{F} \right\}_{\tau\tau} = \frac{f f_p}{F}, \quad F = (A_f f_\tau)^{1/2}.
$$

7.4. All the functions x, y, $\ln n$, f satisfy the third-order equation

$$
A_g\{A_g g_\tau + (g_\tau)^2\} = 0.
$$

Also, the closed scalar nonlinear equations of third order for U^1, U^2 are obtained.

7.5. The following system is obtained

$$U^1_{\tau\tau} + K(x,y)U^1 = 0, \; K(x,y) \equiv -\frac{\Delta \ln n(x,y)}{n^2(x,y)} = -L \ln n(t,\tau,p), \qquad (34)$$

$$\{p^2 + (U^1)^2\}y_\tau = -py_t - U^1 x_t, \quad \{p^2 + (U^1)^2\}x_\tau = -px_t + U^1 y_t, \qquad (35)$$

$$n^2(x,y) = \frac{p^2 + (U^1)^2}{x_t^2 + y_t^2}, \qquad (36)$$

where we differentiate only with respect to two variables t, τ (unlike (6)–(8) and other equations of this section), and the third variable p is the parameter.

This is the system in the functions $U^1(t,\tau,p)$, $x = x(t,\tau,p)$, $y = y(t,\tau,p)$. For $n = n(y)$ we have $y_t \equiv 0$, $x_t \equiv 1$, and we obtain the known system

$$y_\tau = \pm\frac{(n^2 - p^2)^{1/2}}{n^2}, \quad x_\tau = -\frac{p}{n^2}$$

which is generalized by (35), (36).

7.6. Also, we obtain the system of the form

$$U^2_{\tau\tau} + K(x,y)U^2 = 0, \qquad (37)$$

$$U^2 x_\tau + y_p = 0, \quad U^2 y_\tau - x_p = 0, \qquad (38)$$

$$n^2(x,y) = (x_\tau^2 + y_\tau^2)^{-1} \qquad (39)$$

in the functions U^2, x, y.

We can prove that (34) follows from (35), (36) and (37) follows from (38), (39). Both system (35), (36) and (38), (39) implies the known equations of the ray

$$x_{\tau\tau} + (x_\tau^2 - y_\tau^2)(\ln n)_x + 2x_\tau y_\tau (\ln n)_y = 0,$$
$$y_{\tau\tau} + 2x_\tau y_\tau (\ln n)_x - (x_\tau^2 - y_\tau^2)(\ln n)_y = 0.$$

7.7. Joining equations (34), (37), and $\tilde{A}_1 K = 0$, we obtain the system

$$\begin{aligned} &U^1_{\tau\tau} + KU^1 = 0, \quad U^2_{\tau\tau} + KU^2 = 0, \\ &pU^2 K_\tau + U^1 K_p + U^2 K_t = 0, \end{aligned} \qquad (40)$$

in the functions U^1, U^2, $K = K(t,\tau,p) = K(x,y)$.

8. The transformation of equations (14), (15) to the linear equations of first order

For each solution U of equation (14) the function $S(z,t,p)$ of the form $S(z,t,p)=\{U_\tau(t,\tau,p)\}^2$ where $z=\{U(t,\tau,p)\}^2$ satisfies the equation

$$S_z+\tilde{K}(z,t,p)=0,\quad \tilde{K}(z,t,p)=K(x(t,\tau,p),y(t,\tau,p)).$$

Suppose that $\Delta\ln n(x,y)\geq 0$. We consider two cases.

8.1. Set $U=D^*(t,\tau,p)=n(x,y)D^2(t,x,y)$, where $D(t,x,y)$ is the geometric divergence of the rays which vertices in $x=t$. Then

$$z=(D^*)^2=(nD^2)^2,\quad S(z,t,p)=(D^*_\tau)^2=\{(nD^2)_\tau\}^2.$$

We have $z=0$ for $\tau=0$ and $z>0$, $z_\tau>0$ for $\tau>0$ according to the properties of the function D^* (see Rules 4, 5 and the estimates (22) in Sections 4, 5) and obtain the explicit formula in the quadratures

$$S(z,t,p)=S(z_0,t,p)-\int\limits_{z_0}^{z}\tilde{K}(\bar{z},t,p)\,d\bar{z},\quad z\geq z_0\geq 0. \tag{41}$$

8.2. Set $U=U^2(t,\tau,p)$. Then

$$z=-1/v_\tau=\{U^2(t,\tau,p)\}^2,\quad S(z,t,p)=\{U^2_\tau(t,\tau,p)\}^2=-\frac{1}{4}\frac{(v_{\tau\tau})^2}{(v_\tau)^3}$$

and formula (41) for S holds true. Besides for each solution $v(t,\tau,p)$ of equation (15) functions $V(z,t,p)$, $V^*(z,t,p)$ defined by

$$V(z,t,p)=v(t,\tau,p),\quad V^*(z,t,p)=(V_z)^{-2}$$

are the solutions of the equations

$$V_{zz}+\frac{3}{2}z^{-1}V_z-2z^3\tilde{K}(z,t,p)(V_z)^3=0, \tag{42}$$

$$(V_z)^2=(4z^3S)^{-1},\quad S_z=-\tilde{K}, \tag{43}$$

$$V^*_z-\frac{3}{2}V^*+4z^3\tilde{K}(z,t,p)=0. \tag{44}$$

Equation (42) has the second order unlike the initial equation (15) of the third order and (44) is the linear equation of the first order. Equalities (41)–(44) imply the explicit formulas

$$\tau_{tt}=V(z,t,p)=\frac{1}{2}\int\limits_{z_0}^{z}\bar{z}^{-3/2}\{S(\bar{z},t,p)\}^{-1/2}\,d\bar{z}+V(z_0,t,p),$$

$$V^*(z,t,p)=4z^3S(z,t,p),$$

where S is defined by (41).

9. The inverse problem

In this problem the function $n(x,y)$ is unknown and should be determined. The hodograph function $\Gamma : \tau = \tau(t,x,0) \equiv \tau_0(t,x)$ is given for $t \in [0,T]$, $x \in [t,T]$. Suppose that $\gamma(t,p)$ is the ray connecting the points t and x of the line $y = 0$ so that $p = \tau_{0t}(t,x)$ and $D_{t,x}$ is a domain in the plane x, y bounded by the interval $[t,x]$ of the line $y = 0$, and the ray $\gamma(t,p)$. We suppose that the set of rays is regular in $D_{t,x}$, i.e., through each point of the domain $D_{t,x}$ only one ray from the source t is passing. In this case, we have one-to-one correspondence between the points t, x, y and t, τ, p except for the points of the source $x = t$, $y = 0$. The point $(t,0)$ develops into the interval $[-n(t,0),p]$ of the line $t = \text{const}$, $\tau = 0$. Let $\tilde{\gamma}(t,p)$ (this is the interval of the line $t = \text{const}$, $p = \text{const}$), $\tilde{\Gamma} : \tau = \varphi(t,p)$ and $\Delta_{t,p}$ be the images of the ray $\gamma(t,p)$, the hodograph Γ, and the domain $D_{t,x}$ respectively in the space t, τ, p. The rays $\tilde{\gamma}(t,p)$, the surface $\tilde{\Gamma}$ and the boundary of $\Delta_{t,p}$ are uniquely determined by the inverse problem data.

9.1. The integral formulas and the local inverse problems. The regularity condition in $D_{t,x}$ implies $D^2(t,x,y) > 0$ for $\tau > 0$ in each ray $\gamma(t,p)$. Hence $U^2(t,\tau,p) \neq 0$ for $\tau > 0$ and $U^2 = -|U^2| = -nD^2/\alpha$. Besides $dx\,dy = |U^2|\,d\tau\,dp/n^2 = -U^2\,d\tau\,dp/n^2$. Using these equalities and integrating term-by-term (14) for $U = U^2$ along arbitrary ray $\gamma(t,p)$ or along arbitrary domain $D_{t,x}(\Delta_{t,p})$) we obtain the following equalities for each twice continuously differentiable function $W(t,x,y) = w(t,\tau,p)$

$$-\int_{\gamma(t,p)} P_n w D^2(t,x,y)\,ds = \alpha \int_0^{\varphi(t,p)} Q_n w U^2\,d\tau = \alpha\Phi(t,p), \tag{45}$$

$$-\iint_{D_{t,x}} P_n w\,dx\,dy = \iint_{\Delta_{t,p}} Q_n w U^2\,d\tau\,dp = -\int_{-n(t,0)}^{p} \Phi(t,p^1)dp^1, \tag{46}$$

where

$$\begin{aligned} P_n w &= \Delta \ln n(x,y) w - n^2(x,y) w_{\tau\tau}, \\ Q_n w &= \frac{1}{n^2} P_n w = \frac{\Delta \ln n}{n^2} w - w_{\tau\tau}, \\ \Phi(t,p) &= \{U_\tau^2 w - U^2 w_\tau\}|_{\tau=\varphi(t,p)} + \frac{w(t,0,p)}{\alpha}. \end{aligned}$$

Here ds is the length element of the ray $\gamma(t,p)$. The right-hand side in the equality with the integral along $\gamma(t,p)$ is uniquely determined by the function $\tau_0(t,x)$ given locally in the intervals $[t,t+\varepsilon]$, $[x-\varepsilon,x]$. The right-hand side in the equality with the integral along $D_{t,x}$ is uniquely determined by the function $\tau_0(t,x)$ given in the intervals $[t, t+\varepsilon]$, $[t, x]$. Here $\varepsilon > 0$ is arbitrary small. In such local inverse problems, using these equalities, we can determine certain functionals from $n(x,y)$ or from $n(x,y)$, $\tau(t,x,y)$ setting various W or w despite the fact that the rays $\gamma(t,p)$ are unknown. For example, we may set $w = 1$, $w = \tau$, $W = w(\tau(t,x,y), \tau_t(t,x,y))$, $W = f(K(x,y))$ and so on. (The value $n(t,0)$ necessary for calculation of the right-hand side in (45), (46) we can define by means of the formula: $n(t,0) = -\lim \tau_{0t}(t,x)$ for $x \to t$. If $w(t, \varphi(t,p), p) \neq 0$, we must give $n_y(x,0)$ also).

For example, setting $w = 1$ and $w = \tau$ we define for arbitrary ray $\gamma(t,p)$ or for arbitrary domain $D_{t,x}$ the functionals of the form

$$\iint\limits_{D_{t,x}} \Delta \ln n(x,y)\, dx\, dy, \qquad \iint\limits_{D_{t,x}} \Delta \ln n(x,y) \tau(t,x,y)\, dx\, dy,$$

$$\int\limits_{\gamma(t,p)} \Delta \ln n(x,y) D^2(t,x,y)\, ds, \qquad \int\limits_{\gamma(t,p)} \Delta \ln n(x,y) D^2(t,x,y) \tau(t,x,y)\, ds.$$

Using the Green formula for the Laplace operator we obtain in particular that in the local inverse problem with given hodograph function $\tau_0(t,x)$ in intervals $[t, t+\varepsilon]$, $[t, x]$ the values of the functional

$$\int\limits_{\gamma(t,p)} \frac{\partial \ln n}{\partial \nu}\, ds$$

are uniquely determined for all rays $\gamma(t,p)$ filling $D_{t,x}$. Here ν is the normal to the ray $\gamma(t,p)$.

The other formula for $\iint \Delta \ln n\, dx\, dy$ was obtained earlier by Anikonov [13] who used another method. The analogs of the equalities (45), (46) are obtained if we change D^2 by $\tau_{tt} D^2$, $P_n w$ by $\tau_{tt} P_n w$ and Φ by $\Psi(t,p) = (U^1_\tau w - U^1 w_\tau)|_{\tau = \varphi(t,p)} - \{U^1_\tau w - \alpha w_\tau\}|_{\tau=0}$ and also the three-dimensional analogs with the integrals along x, y, t.

In local inverse problems with given hodograph function $\tau = \varphi(t,p)$, we also can define the functionals

$$-\iint\limits_{D_{t,x}} \Delta \ln n \left(\frac{\Delta \tau D^2}{\alpha n}\right)^k dx\, dy = (-1)^k \iint\limits_{\Delta_{t,p}} K U^2 (h U^2)^k\, d\tau\, dp,$$

$$\iint\limits_{D_{t,x}} \frac{\alpha n}{D^2} \left\{ \Psi - \left[\left(\frac{n D^2}{\alpha}\right)^2 \Psi \right]_\tau \tau_{tt} \right\} dx\, dy = \iint\limits_{\Delta_{t,p}} \left\{ \psi - \left[\left(U^2\right)^2 \psi \right]_\tau v \right\} d\tau\, dp,$$

where $k = 0, \pm 1, \pm 2, \ldots$, $\Psi(t,x,y) = \psi(t,\tau,p)$ is each differentiable function. Also we obtain analogous integrals along the ray $\gamma(t,p)$ and the other functionals.

9.2. The reduction to the direct problem. We suppose that the set of rays $\gamma(t,p)$ is regular in $D_{t,T}$ for each $t\in[0,T]$. Denote the union of all flat domains $D_{t,T}$ and $\Delta_{t,P}((P=\tau_{0t}(t,T))$ for $t\in[0,T]$ by D_T and Δ_T. The inverse problem to determine $n(x,y)$ in D_T is reduced to one of the following direct boundary-value problems in the domain Δ_T of the variables t, τ, p with the known boundary: 1) to find solutions v_1, v_2 or U^1, U^2 of system (8) or (7) with the given values in $\tilde{\Gamma}$ and conditions (11) or (12); 2) to find the solution v of equation (6) with conditions (11) and with data in $\tilde{\Gamma}$; 3) to find the solution x, y of system (26), (27) or (28) (and of each other system from Section 7 for x, y) with the data in $\tilde{\Gamma}$ and for $\tau=0$ ($x=t$, $y=0$ for $\tau=0$); 4) to find the solutions $g=\ln n(t,\tau,p)$, $f(t,\tau,p)=\Delta\ln n/n^2$ and so on equations described in Section 7 with data in $\tilde{\Gamma}$ and for $\tau=0$.

These scheme can be inverted. For example, we can consider the quasilinear equation (6) as initial equation and each boundary-value problem for this equation we may reduce to the inverse (or direct) problem for the eikonal equation.

9.3. The one-dimensional inverse problem. In the one-dimensional inverse problem ($n=n(y)$), the hodograph function $\tau=\tau(0,x,0)=\tau_0(x)=\varphi(p)$ ($p=-\tau_{0x}(x)$) is given for $x\in[0,T]$.

Setting in the conservation law (see Section 2.4) $\Phi_0=1$, $F(\xi)=\xi^k$, $k=1,2,\dots$ and applying the Ostrogradsky formula in the domain $\Delta_{0,p}$ and the mathematical induction, we obtain the explicit formula to determine the integrals from degrees of the function $n(y)$ taken over an arbitrary ray $\gamma(0,p)=\gamma(p)$

$$\int\limits_0^{\varphi(p)}[n^2(\tau,p)]^k\,d\tau=\int\limits_0^{l(p)}[n(s,p)]^{2k+1}\,ds=\int\limits_{\gamma(p)}[n(y)]^{2k+1}\,ds=\Psi_k(p),\quad k=0,1,\dots,\tag{47}$$

where

$$\Psi_k(p)=\varphi(p)p^{2k}+\sum_{j=1}^{k}(-1)^j\beta_jC_k^jp^{2(k-j)}\psi_j(p),\quad k=1,2,\dots,$$

$$\psi_j(p)=\int\limits_{-n_0}^{p}p_1\int\limits_{-n_0}^{p_1}p_2\dots\int\limits_{-n_0}^{p_{j-1}}p_j\varphi(p_j)\,dp_j\dots dp_2\,dp_1,\quad j=1,2,\dots,$$

$\Psi_0(p)=\varphi(p)$, $n_0=n(0)$, $C_k^j=k!/[j!(k-j)!]$ are the binomial coefficients, $\beta_j=2^{1-j}(2j-1)!/(j-1)!=1\cdot3\cdot5\cdot\ldots\cdot(2j-1)$. Here $n(\tau,p)=n(y(\tau,p))=n(s,p)$, where $x=x(\tau,p)$, $y=y(\tau,p)$ and $x=x(s,p)$, $y=y(s,p)$ are the parametric equations of the ray $\gamma(p)$ with the parameter τ (the time of the signal propagation along the ray $\gamma(p)$) and s (the length of the ray from the source $(0,0)$), $l(p)$ is the complete length of the ray $\gamma(p)$.

The known formula of the mathematical analysis implies

$$\psi_j(p) = \frac{(-1)^{j-1}}{2^{j-1}(j-1)!}\int\limits_{-n_0}^{p}(t^2-p^2)^{j-1}t\varphi(t)\,dt, \quad j=1,2,\dots\,.$$

Hence

$$\Psi_k(p) = p^{2(k-1)}\Big\{p^2\varphi(p) - \int\limits_{-n_0}^{p}\Big\{\sum_{j=1}^{k}\alpha_k^j\Big(\frac{t^2}{p^2}-1\Big)^{j-1}\Big\}t\varphi(t)\,dt,$$

$$\alpha_k^j = C_k^j\beta_j 2^{1-j}/(j-1)!.$$

Also we have

$$\int\limits_0^{\varphi(p)}\{U^1(\tau,p)\}^{2k} = (-1)^k\beta_k\psi_k(p), \quad k=1,2,\dots\,. \tag{48}$$

Taking the equality $y_\tau = f_n(n)n_\tau = [n^2-p^2]^{1/2}/n^2$ ($y=f(n)$ is the inverse function to $n(y)$) into account, we reduce the inverse problem to the known problem of moments which has the explicit solution. More detail exposition of these results is given in [3, 6].

Multiplying both parts of equality (47) or (48) by the coefficients of the known expansions in a power series for some functions f and then summing, we obtain the expressions for the integrals along the ray $\gamma(p)$ from the corresponding function $f(U^1)$ or $f(n)$ in terms of the hodograph function $\varphi(p)$. For example, multiplying (48) by $\{(-1)^{k+1}p^{-2k}/j\}$ and using the known expansions in a power series for the functions $\ln(1+x)$ and $(1+x)^{-1/2}$, we obtain the formula

$$\int\limits_0^{\varphi(p)}\ln n\,d\tau = \ln p\cdot\varphi(p) - \int\limits_{-n_0}^{p}\frac{\varphi(\tilde p)}{\tilde p+p}\,d\tilde p.$$

The other approaches for the inverse kinematic problem can be found in [13–16].

10. The group substance

The variables and the equations which we consider and obtain above, have the following group substance [1–7]. Let G be an infinite group of the point transformations of the space of five variables x, y, t, $u^1=\tau$, $u^2=n^2$ with the Lie algebra of infinitesimal operators X of its one-parameter subgroups of the form

$$X = \Phi(x,y)\frac{\partial}{\partial x} + \Psi(x,y)\frac{\partial}{\partial y} - 2\Phi_x(x,y)u^2\frac{\partial}{\partial u^2},$$

where Φ and Ψ are arbitrary conjugate harmonic functions. Then the expressions $J^1 - J^{11}$ defined in Sections 1, 3 are functionally independent differential invariants of the group G. The basis of differential invariants of the group G is formed by the invariants J^1, J^2. The operators A_1, A_2, A_3 defined by (4) are the operators of invariant differentiation of the group G. The operator of invariant differentiation A_1 gives differentiation with respect to the parameter of the point source, A_2 along the ray, A_3 along the normal to the ray. Equation (1) admits the group G. The system $J^7 = 1$, $J^5 = v(t, \tau, \tau_t)$ is the automorphic system AG and equation (6) and each of systems (7), (8) is the resolving system RE of the group bundle of equation (1) relative to G. The Gauss curvature $K(x,y)$ defined by (16) in equations (13)–(15) and in the statements of Sections 3–9, is a differential invariant of second order of the group G.

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