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Divergent formulas (conservation laws) in the differential geometry of plane curves*

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Abstract. In this paper, it is discovered that in the differential geometry of arbitrary smooth plane curves there exists a solenoidal vector field $\boldsymbol{S}^*$, i.e., the field with a property $\operatorname{div} \boldsymbol{S}^* = 0$ (in a certain area D), having the following geometric meaning. Let $\{L_\tau\}$ be a set of arbitrary smooth non-intersecting plane curves L_τ with the Frene unit vectors $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(x, y)$ ($\boldsymbol{\tau}$ is the unit tangent vector, $\boldsymbol{\nu}$ is the unit normal of a curve L_τ), and $\{L_\nu\}$ be a set of orthogonal to them curves L_ν with the Frene unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\eta} = -\boldsymbol{\tau}$. The set $\{L_\tau\}$ fills by the continuous image some area D and satisfies some general conditions. The vector field $\boldsymbol{S}^*$ is expressed in terms of the Frene unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ of the curves L_τ and equals the sum of the curvature vector $d\boldsymbol{\tau}/ds = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = k\boldsymbol{\nu}$ of the curve $L_\tau \in \{L_\tau\}$ and the curvature vector $d\boldsymbol{\nu}/ds_\nu = (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} = k_\nu \boldsymbol{\eta} = -k_\nu \boldsymbol{\tau}$ of the curve $L_\nu \in \{L_\nu\}$. Here s and s_ν are natural parameters, i.e., the length of a curve being counted from its certain point along L_τ and L_ν, respectively, $k = k(x, y)$ and $k_\nu = k_\nu(x, y)$ are curvatures of the curves L_τ and L_ν, respectively. The symbol $(\boldsymbol{a} \cdot \nabla)\boldsymbol{a}$ denotes a derivative of the vector $\boldsymbol{a}$ in the direction $\boldsymbol{a}$. This property can be interpreted as existence in the differential geometry of plane curves of a conservation law (for the vector field $\boldsymbol{S}^*$ or for vector fields of unit vectors). A number of equivalent representations of the field $\boldsymbol{S}^*$ and equivalent forms of a conservation law $\operatorname{div} \boldsymbol{S}^* = 0$ is obtained.

1. Introduction

The given paper is an extension of papers [1–3].

First we will consider the basic geometric elements in terms of which the formulas obtained express: the Frene unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ (unit vectors of the tangent and the normal of the plane curve L_τ) and the curvature vector of a curve.

An important place in the classical differential geometry of curves is occupied by the Frene equations, which for a plane smooth regular curve L_τ, look like [4–6]

$$\frac{d\boldsymbol{\tau}}{ds} = k\boldsymbol{\nu}, \qquad \frac{d\boldsymbol{\nu}}{ds} = -k\boldsymbol{\tau}. \tag{1}$$

Here and below $\boldsymbol{\tau} = \boldsymbol{\tau}(s)$ is a unit tangential vector (the velocity vector) of the curve L_τ ($|\boldsymbol{\tau}| \equiv 1$), $\boldsymbol{\nu} = \boldsymbol{\nu}(s)$ is a unit vector of a normal ($|\boldsymbol{\nu}| \equiv 1$),

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$k = k(s)$ is a curvature of the curve L_τ, s is the length curve (a natural parameter), calculated from its certain point, $\dfrac{d\boldsymbol{\tau}}{ds} = \boldsymbol{\tau}_s = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau}$ and $\dfrac{d\boldsymbol{\nu}}{ds} = \boldsymbol{\nu}_s = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\nu}$ are derivatives of the vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ in the direction $\boldsymbol{\tau}$. The vector $\boldsymbol{\tau}_s = k\boldsymbol{\nu}$ is called the curvature vector of a curve L_τ [5, 6] or the vector of acceleration, a pair of unit vectors $(\boldsymbol{\tau}, \boldsymbol{\nu})$ is the Frene basis (frame). Equations (1) describe changing of the Frene unit vectors $(\boldsymbol{\tau}, \boldsymbol{\nu})$ along an arbitrary smooth regular fixed curve L_τ.

In this paper, we consider not the properties of a fixed curve L_τ, but properties of a set $\{L_\tau\}$ of the curves L_τ with the Frene basis $(\boldsymbol{\tau}, \boldsymbol{\nu})$, filling some area D with a continuous image in the plane with rectangular co-ordinates x, y. Concerning a set $\{L_\tau\}$ to be low everywhere, we will assume that the following conditions are satisfied:

(A) one and only one curve $L_\tau \in \{L_\tau\}$ passes at any point $(x, y) \in D$, so the curves L_τ do not intersect at any point $(x, y) \in D$ [2].

(B) at any point (x, y) of any curve $L_\tau \in \{L_\tau\}$, there exists a Frene basis $(\boldsymbol{\tau}, \boldsymbol{\nu})$, so that the Frene unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are one-valued vector functions of the variables x, y in the area D: $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(x, y)$. Thus, in D, two mutually orthogonal vector fields of the unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are defined. We consider the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ to be along the axes of co-ordinates x, y and unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ to form the right system of vectors.

(C) at any point $(x, y) \in D$, there exist quantities $\operatorname{div}\boldsymbol{\tau}$, $\operatorname{rot}\boldsymbol{\tau}$, $\operatorname{div}\boldsymbol{\nu}$, $\operatorname{rot}\boldsymbol{\nu}$, i.e., the vector fields $\boldsymbol{\tau}(x, y)$, $\boldsymbol{\nu}(x, y)$ being smooth enough.

To a given set of the curves $\{L_\tau\}$ in D there corresponds a set $\{L_\nu\}$ of the curves L_ν, orthogonal to the curves L_τ. The tangent unit vector of the curve L_ν coincides with the normal unit vector $\boldsymbol{\nu}$ of the curve L_τ, and the normal unit vector $\boldsymbol{\eta}$ of the curve L_ν coincides with a tangent unit vector $\boldsymbol{\tau}$ of the curve L_τ to within a sign. Sets of the curves $\{L_\tau\}$ and $\{L_\nu\}$ will be called *mutually orthogonal*. For a curve $L_\nu \in \{L_\nu\}$ the Frene equations look like

$$\frac{d\boldsymbol{\nu}}{ds_\nu} = k_\nu \boldsymbol{\eta}, \qquad \frac{d\boldsymbol{\eta}}{ds_\nu} = -k_\nu \boldsymbol{\nu},$$

where s_ν is a natural parameter (a variable length) of a curve L_ν, $\dfrac{d\boldsymbol{\nu}}{ds_\nu} = (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu}$ and $\dfrac{d\boldsymbol{\eta}}{ds_\nu} = (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\eta}$ are derivatives of the vectors $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ in the direction $\boldsymbol{\nu}$, k_ν and $k_\nu\boldsymbol{\eta}$ are the curvature and the curvature vector of a curve L_ν. The curves L_τ are vector lines of the vector field $\boldsymbol{\tau}$, and the curves L_ν are vector lines of the vector field of normals $\boldsymbol{\nu}$ of the curves L_τ.

In the given paper, it is revealed that for any set $\{L_\tau\}$ of smooth plane curves L_τ with the Frene unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ with the specified properties

(A)–(C) or for any two mutually orthogonal sets of smooth curves $\{L_\tau\}$ and $\{L_\nu\}$ with such properties, we have the divergent identity (in D):

$$\operatorname{div} \boldsymbol{S}^* = 0 \quad \Leftrightarrow \quad \boldsymbol{S}^* = \operatorname{rot} \boldsymbol{A}, \tag{2}$$

where

$$\boldsymbol{S}^* = \frac{d\boldsymbol{\tau}}{ds} + \frac{d\boldsymbol{\nu}}{ds_\nu} = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} + (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} = k\boldsymbol{\nu} + k_\nu \boldsymbol{\eta}, \tag{3}$$

$\boldsymbol{\eta} = -\boldsymbol{\tau}$, $\boldsymbol{A}$ is a vector field. Identity (2) means that for such a set of curves $\{L_\tau\}$ there always exists a vector field $\boldsymbol{S}^*(\boldsymbol{\tau}) = k\boldsymbol{\nu} + k_\nu \boldsymbol{\eta}$, representing the sum of the vector field of the curvature vector $k\boldsymbol{\nu}$ of a curve $L_\tau \in \{L_\tau\}$ and the vector field of the curvature vector $k_\nu \boldsymbol{\eta}$ of an orthogonal curve $L_\nu \in \{L_\nu\}$ which is solenoidal. This property can be interpreted as existence in the differential geometry of plane curves of a conservation law for the vector field $\boldsymbol{S}^*(\boldsymbol{\tau})$, having the differential form (2). (As for any smooth vector field $\boldsymbol{a}$, the identity $\operatorname{div} \boldsymbol{a} = 0$ is a differential conservation law with the integral form for the flux (in the plane case) $\int_S (\boldsymbol{a} \cdot \boldsymbol{n})\, dS = 0$, where S is an arbitrary piecewise smooth closed curve in the plane x, y, dS is an element of the length S, $\boldsymbol{n}$ is a unit normal to S).

Thus, the differential conservation law for an arbitrary set of smooth plane curves L_τ in the differential form (2), (3) in terms of the Frene unit vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ or curvature vectors is discovered. We notice that this conservation law can be expressed in terms of only one curvature vector $\boldsymbol{\tau}_s = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = k\boldsymbol{\nu}$ of curves L_τ as the normal unit vector $\boldsymbol{\nu}$ can be expressed by the known formula $\boldsymbol{\nu} = \boldsymbol{\tau}_s / |\boldsymbol{\tau}_s|$ ($|\boldsymbol{\tau}_s| = k$). Therefore the vector $\boldsymbol{S}^*$ in formulas (2), (3) can be expressed (see note 1 below) only in terms of one curvature vector $\boldsymbol{\tau}_s = k\boldsymbol{\nu}$ by the formula

$$\boldsymbol{S}^* = \boldsymbol{\tau}_s + \{(\boldsymbol{\tau}_s \cdot \nabla)\boldsymbol{\tau}_s - \boldsymbol{\tau}_s(\operatorname{grad} \ln |\boldsymbol{\tau}_s| \cdot \boldsymbol{\tau}_s)\}/|\boldsymbol{\tau}_s|^2. \tag{4}$$

In addition, for the field $\boldsymbol{S}^*$, the following expressions are obtained: only in terms of one unit vector $\boldsymbol{\tau}$:

$$\boldsymbol{S}^* = -\{\boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau}\};$$

only in terms of one unit vector $\boldsymbol{\nu}$:

$$\boldsymbol{S}^* = -\{\boldsymbol{\nu} \operatorname{div} \boldsymbol{\nu} + \boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\nu}\};$$

other expressions are obtained in terms of $\boldsymbol{\tau}$, $\boldsymbol{\nu}$.

Obviously, it is possible to consider as initial geometric object not a set of the plane curves $\{L_\tau\}$, and an arbitrary vector field $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$ of the unit vectors $\boldsymbol{\tau}$ in the plane x, y. Then formulas (2), (3) give a differential conservation law for an arbitrary vector field of the unit vectors $\boldsymbol{\tau}$ with the vector lines L_τ.

Identity (2) is purely geometric in its meaning. However, it can be translated "to a physical language" in the case when the set $\{L_\tau\}$ is a set of vector lines of some smooth vector field $\boldsymbol{v} = \boldsymbol{v}(x, y) = |\boldsymbol{v}|\boldsymbol{\tau}$ with the modulus $|\boldsymbol{v}|$ and a direction $\boldsymbol{\tau}$ ($|\boldsymbol{\tau}| \equiv 1$) and with a property $|\boldsymbol{v}| \neq 0$. In [12], it is established that identity (2) in terms of such a field $\boldsymbol{v}$ is equivalent to the identity

$$\operatorname{div}\Big\{\frac{\boldsymbol{v}\operatorname{div}\boldsymbol{v} + \boldsymbol{v}\times\operatorname{rot}\boldsymbol{v}}{|\boldsymbol{v}|^2} - \frac{1}{2}\operatorname{grad}\ln|\boldsymbol{v}|^2\Big\} = 0, \tag{5}$$

representing a differential conservation law for the above-mentioned arbitrary field $\boldsymbol{v}(x, y)$. Formula (5) is found in [7] by means of the group analysis for a potential plane field $\boldsymbol{v} = \operatorname{grad} u(x, y)$, and in [2] — for an arbitrary smooth plane field $\boldsymbol{v}(x, y)$.

If the vector field $\boldsymbol{v}$ is a model of a physical field and satisfies a differential equation (E) of the mathematical physics, then we obtain an additional (as compared to (5)) differential relation (E) for the field $\boldsymbol{v}$. Using it, with the help of (5) it is possible to obtain conservation laws for the physical field $\boldsymbol{v}$. In this connection, in [2], new conservation laws for a plane movement of an ideal liquid (solutions of the Euler's hydrodynamic equations [8]) are found, and in [3], conservation laws for a time field τ (the eikonal equation solutions) in the kinematic seismics (geometric optics) are discovered. In paper [12], the proof of these conservation laws for the time field τ is given and by means of formula (2) their geometric interpretation from the point of view of the differential geometry in terms of curvature vectors of vector lines (rays) of the corresponding physical field $\operatorname{grad}\tau$ and orthogonal to them curves (fronts) is given.

Let $(\boldsymbol{a}\cdot\boldsymbol{b})$ and $\boldsymbol{a}\times\boldsymbol{b}$ denote scalar and a vector products of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, ∇ be the Hamiltonian operator ("a nabla"), $(\boldsymbol{v}\cdot\nabla)\boldsymbol{a}$ be a derivative of the vector $\boldsymbol{a}$ in the vector $\boldsymbol{v}$ direction, $\Delta u = u_{xx} + u_{yy}$.

2. Conservation law for an arbitrary field of unit vectors

Assume D to be a domain in the plane x, y; $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ is the right-hand side system of unit vectors along the axes x, y, z of the rectangular coordinates; $\{L_\tau\}$ is a set of the curves L_τ, defined in D, with the Frene unit vectors $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(x, y)$ with a natural parameter s and with properties (A)-(C). The role of the binormal $\boldsymbol{\beta}$ of the curve L_τ is played by the unit vector $\boldsymbol{k}$. The vectors $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and the curvature k of the curve L_τ can be represented as [5, p. 104]

$$\boldsymbol{\tau} = \tau_1(x, y)\boldsymbol{i} + \tau_2(x, y)\boldsymbol{j} = \boldsymbol{\tau}(\alpha) = \cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j}, \tag{6}$$

$$\boldsymbol{\nu} = -\sin\alpha\,\boldsymbol{i} + \cos\alpha\,\boldsymbol{j}, \tag{7}$$

$$k = \frac{d\alpha}{ds} = (\operatorname{grad}\alpha\cdot\boldsymbol{\tau}) = k(x, y). \tag{8}$$

Here $\alpha = \alpha(x, y)$ is the angle the vector $\boldsymbol{\tau}$ makes with the axis Ox, so that $\cos\alpha = \tau_1$, $\sin\alpha = \tau_2$, i.e., $\alpha(x, y)$ is the polar angle of a point ($\xi = \tau_1$, $\eta = \tau_2$) in the plane ξ, η:

$$\alpha \stackrel{\text{def}}{=} \arctan\frac{\tau_2}{\tau_1} + (2k + \delta)\pi, \qquad k \in \mathbb{Z}, \tag{9}$$

$\delta = 0$ and $\delta = 1$ in quadrants I, IV and II, III of the plane ξ, η, respectively; owing to the definition of the operation rot [9] we have $\operatorname{rot}\{\alpha(x, y)\boldsymbol{k}\} = \alpha_y\boldsymbol{i} - \alpha_x\boldsymbol{j}$. The angle the vector $\boldsymbol{\nu}$ makes with the axis Ox is $\alpha_\nu = \alpha + \pi/2$. In [2, Theorem 3] we have obtained

Theorem 1 (Conservation law for an arbitrary vector field of unit vectors). *For any vector field of the unit vectors $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$ ($|\boldsymbol{\tau}| \equiv 1$) with components,*

$$\tau_j(x, y) \in C^2(D) \qquad (j = 1, 2) \tag{10}$$

we have the identity

$$\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0, \tag{11}$$

where

$$\boldsymbol{S}(\boldsymbol{\tau}) \stackrel{\text{def}}{=} \boldsymbol{\tau} \operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau} \times \operatorname{rot}\boldsymbol{\tau} = \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} - k\boldsymbol{\nu} = -\{(\boldsymbol{\tau}\cdot\nabla)\boldsymbol{\tau} + (\boldsymbol{\tau}\times\nabla)\times\boldsymbol{\tau}\}, \tag{12}$$

moreover

$$\boldsymbol{S}(\boldsymbol{\tau}) = \operatorname{rot}\{\alpha(x, y)\boldsymbol{k}\}, \tag{13}$$

$$\operatorname{rot} \boldsymbol{S}(\boldsymbol{\tau}) = -(\Delta\alpha)\boldsymbol{k}. \tag{14}$$

Proof. Let us present it, as in [2] the statement is given without proof. Using formulas (6), (7) for $\boldsymbol{\tau}$, $\boldsymbol{\nu}$, we obtain:

$$\begin{aligned}
\operatorname{div}\boldsymbol{\tau} &= (\cos\alpha)_x + (\sin\alpha)_y = -\sin\alpha\ \alpha_x + \cos\alpha\ \alpha_y = (\boldsymbol{\nu}\cdot\operatorname{grad}\alpha);\\
\boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau} &= -\boldsymbol{\tau}_s = -(\cos\alpha\ \boldsymbol{i} + \sin\alpha\ \boldsymbol{j})_s = -(-\sin\alpha\ \alpha_s\boldsymbol{i} + \cos\alpha\ \alpha_s\boldsymbol{j})\\
&= -\alpha_s(-\sin\alpha\ \boldsymbol{i} + \cos\alpha\ \boldsymbol{j}) = -(\operatorname{grad}\alpha\cdot\boldsymbol{\tau})\boldsymbol{\nu} \quad \Rightarrow\\
\boldsymbol{S}(\boldsymbol{\tau}) &= \boldsymbol{\tau}\operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{\nu}\cdot\operatorname{grad}\alpha) - \boldsymbol{\nu}(\operatorname{grad}\alpha\cdot\boldsymbol{\tau})\\
&= \operatorname{grad}\alpha\times(\boldsymbol{\tau}\times\boldsymbol{\nu}) = \operatorname{grad}\alpha\times\boldsymbol{k} = \operatorname{rot}(\alpha\boldsymbol{k}).
\end{aligned}$$

We used the known formula [9, §17]: $\operatorname{grad}\{|\boldsymbol{a}|^2/2\} = (\boldsymbol{a}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times\operatorname{rot}\boldsymbol{a}$. From it, with $\boldsymbol{a} = \boldsymbol{\tau}$ and $|\boldsymbol{\tau}| \equiv 1$ follows $\boldsymbol{\tau}\times\operatorname{rot}\boldsymbol{\tau} = -\boldsymbol{\tau}_s = -k\boldsymbol{\nu}$. Also, we have applied formula [9, §7] for the double vector product $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$. Formula (13) has been obtained. From it follow identities (11) and (14). □

Corollary 1. *For the Frene unit vectors* $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ *of the set* $\{L_\tau\}$ *of curves with properties* (A)–(C) *and* (10), *we have identities* $\operatorname{div} \boldsymbol{S}(\boldsymbol{\tau}) = 0$, $\operatorname{div} \boldsymbol{S}(\boldsymbol{\nu}) = 0$, *where* $\boldsymbol{S}(\boldsymbol{\nu}) \overset{\text{def}}{=} \boldsymbol{\nu} \operatorname{div} \boldsymbol{\nu} + \boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\nu}$.

Lemma 1. *Let a set of plane curves* $\{L_\tau\}$ *with the Frene unit vectors* $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ *have in the area* D *the properties* (A)–(C), $\tau_j(x, y) \in C^1(D)$ $(j = 1, 2)$, $\{L_\nu\}$ *is a set of the curves* L_ν, *orthogonal to the curves* L_τ, *with the Frene unit vectors* $\boldsymbol{\nu}$, $\boldsymbol{\eta}$. *Then in* D, *the following identities take place*:

$$\operatorname{div} \boldsymbol{\tau} = (\boldsymbol{\nu} \cdot \operatorname{grad} \alpha), \qquad \operatorname{rot} \boldsymbol{\tau} = (\boldsymbol{\tau} \cdot \operatorname{grad} \alpha)\boldsymbol{k}, \tag{15}$$

$$\operatorname{div} \boldsymbol{\nu} = -(\boldsymbol{\tau} \cdot \operatorname{grad} \alpha), \qquad \operatorname{rot} \boldsymbol{\nu} = (\boldsymbol{\nu} \cdot \operatorname{grad} \alpha)\boldsymbol{k}. \tag{16}$$

For the unit normal vector $\boldsymbol{\eta}$ *of a curve* $L_\nu \in \{L_\nu\}$, *its curvature* k_ν *and the curvature* k *of a curve* $L_\tau \in \{L_\tau\}$, *we have formulas*

$$\boldsymbol{\eta} = -\boldsymbol{\tau}, \qquad k_\nu = \operatorname{div} \boldsymbol{\tau} = (\boldsymbol{\nu} \cdot \operatorname{grad} \alpha), \tag{17}$$

$$k = -\operatorname{div} \boldsymbol{\nu} = (\boldsymbol{\tau} \cdot \operatorname{grad} \alpha) \quad \Rightarrow \quad \operatorname{rot} \boldsymbol{\tau} = k\boldsymbol{k}, \quad \operatorname{rot} \boldsymbol{\nu} = k_\nu \boldsymbol{k}, \tag{18}$$

$$\boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} = -k_\nu \boldsymbol{\eta}, \qquad \boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau} = -\boldsymbol{\tau}_s, \tag{19}$$

explaining the geometric meaning of the quantities entering expression (12).

Proof. Formulas (15), (16) follow from (6), (7). As the angle, the vector $\boldsymbol{\nu}$ makes with the axis Ox, is $\alpha_\nu = \alpha + \pi/2$, the curvature k_ν of the curve L_ν with the tangential unit vector $\boldsymbol{\nu}$ is

$$k_\nu \overset{\text{def}}{=} \frac{d\alpha_\nu}{ds_\nu} = (\boldsymbol{\nu} \cdot \operatorname{grad} \alpha_\nu) = (\boldsymbol{\nu} \cdot \operatorname{grad} \alpha) = \operatorname{div} \boldsymbol{\tau}.$$

For the vector $\dfrac{d\boldsymbol{\nu}}{ds_\nu}$ from (6), (7), we come to

$$\frac{d\boldsymbol{\nu}}{ds_\nu} = -\boldsymbol{\nu} \times \operatorname{rot} \boldsymbol{\nu} = -(\operatorname{grad} \alpha \cdot \boldsymbol{\nu})\boldsymbol{\tau} = -k_\nu \boldsymbol{\tau}.$$

Comparing this equality to the Frene equation for the curve L_ν of the form

$$\frac{d\boldsymbol{\nu}}{ds_\nu} = k_\nu \boldsymbol{\eta},$$

for the unit normal vector $\boldsymbol{\eta}$ of the curve L_ν we obtain $\boldsymbol{\eta} = -\boldsymbol{\tau}$. Moreover, the curvature k of the curve L_τ with the tangential unit vector $\boldsymbol{\tau}$ owing to (8) is

$$k = \frac{d\alpha}{ds} = (\boldsymbol{\tau} \cdot \operatorname{grad} \alpha) = -\operatorname{div} \boldsymbol{\nu}. \qquad \square$$

Corollary 2. *Under conditions of Lemma 1 for the vector fields of the Frene unit vectors $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$ and $\boldsymbol{\nu} = \boldsymbol{\nu}(x, y)$ and for the vector field $\boldsymbol{S}(\boldsymbol{\tau})$ of the form of* (12)*, in the area D the following identities hold*:

$$\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{S}(\boldsymbol{\nu}) = \boldsymbol{\nu}\,\mathrm{div}\,\boldsymbol{\nu} + \boldsymbol{\nu} \times \mathrm{rot}\,\boldsymbol{\nu}, \tag{20}$$

$$\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{S}_{\mathrm{div}} \overset{\mathrm{def}}{=} \boldsymbol{\tau}\,\mathrm{div}\,\boldsymbol{\tau} + \boldsymbol{\nu}\,\mathrm{div}\,\boldsymbol{\nu}, \tag{21}$$

$$\boldsymbol{S}(\boldsymbol{\tau}) = \boldsymbol{S}_{\mathrm{rot}} \overset{\mathrm{def}}{=} \boldsymbol{\tau} \times \mathrm{rot}\,\boldsymbol{\tau} + \boldsymbol{\nu} \times \mathrm{rot}\,\boldsymbol{\nu} \quad \Rightarrow \tag{22}$$

$$\boldsymbol{S}_{\mathrm{div}} = \boldsymbol{S}_{\mathrm{rot}},$$

$$\boldsymbol{S}(\boldsymbol{\tau}) = -\boldsymbol{S}^* \quad \Rightarrow \tag{23}$$

$$\mathrm{div}\,\boldsymbol{S}(\boldsymbol{\tau}) = -\,\mathrm{div}\,\boldsymbol{S}^*, \tag{24}$$

where the vector field $\boldsymbol{S}^*$ is defined by formula (3)*. Formulas* (12), (20)–(22) *give various representations for the field $\boldsymbol{S}(\boldsymbol{\tau})$. Hence, the vector field $\boldsymbol{S}^*$ is expressed in terms of the unit vectors $\boldsymbol{\tau}$ and (or) $\boldsymbol{\nu}$ from any of the formulas*:

$$\boldsymbol{S}^* = -(\boldsymbol{\tau}\,\mathrm{div}\,\boldsymbol{\tau} + \boldsymbol{\tau} \times \mathrm{rot}\,\boldsymbol{\tau}) = -\boldsymbol{\tau}\,\mathrm{div}\,\boldsymbol{\tau} + k\boldsymbol{\nu}, \tag{25}$$

$$\boldsymbol{S}^* = -(\boldsymbol{\tau}\,\mathrm{div}\,\boldsymbol{\tau} + \boldsymbol{\nu}\,\mathrm{div}\,\boldsymbol{\nu}), \tag{26}$$

$$\boldsymbol{S}^* = -(\boldsymbol{\tau} \times \mathrm{rot}\,\boldsymbol{\tau} + \boldsymbol{\nu} \times \mathrm{rot}\,\boldsymbol{\nu}), \tag{27}$$

$$\boldsymbol{S}^* = -(\boldsymbol{\nu}\,\mathrm{div}\,\boldsymbol{\nu} + \boldsymbol{\nu} \times \mathrm{rot}\,\boldsymbol{\nu}), \tag{28}$$

$$\boldsymbol{S}^* = -\,\mathrm{rot}\{\alpha(x, y)\boldsymbol{k}\} = -(\alpha_y\,\boldsymbol{i} - \alpha_x\,\boldsymbol{j}). \tag{29}$$

Formula (25) *gives representations of the field $\boldsymbol{S}^*$ only in terms of one Frene tangential unit vector $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$, and formula* (28) *— only in terms of one normal unit vector $\boldsymbol{\nu}$. Formula* (29) *expresses the field $\boldsymbol{S}^*$ in terms of the inclination angle α of the vector $\boldsymbol{\tau}$ to the axis Ox.*

Remark 1. The vector field $\boldsymbol{S}^*$ of the form of (3) can also be expressed in terms of one curvature vector $\boldsymbol{\tau}_s = k\boldsymbol{\nu}$ of the curves L_τ from formula (4). Really, owing to the general vector analysis formula [9, §17] of the form of $(\boldsymbol{v} \cdot \nabla)\varphi\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{v} \cdot \mathrm{grad}\,\varphi) + \varphi(\boldsymbol{v} \cdot \nabla)\boldsymbol{a}$ with $\boldsymbol{v} = \boldsymbol{\tau}_s/k$, $\boldsymbol{a} = \boldsymbol{\tau}_s$, $\varphi = 1/k$ we have

$$\begin{aligned}\frac{d\boldsymbol{\nu}}{ds_\nu} &= (\boldsymbol{\nu} \cdot \nabla)\boldsymbol{\nu} = \Big(\frac{\boldsymbol{\tau}_s}{k} \cdot \nabla\Big)\frac{\boldsymbol{\tau}_s}{k} = \boldsymbol{\tau}_s\Big(\frac{\boldsymbol{\tau}_s}{k} \cdot \mathrm{grad}\,\frac{1}{k}\Big) + \frac{1}{k}\Big(\frac{\boldsymbol{\tau}_s}{k} \cdot \nabla\Big)\boldsymbol{\tau}_s \\ &= \frac{1}{k^2}\big\{(\boldsymbol{\tau}_s \cdot \nabla)\boldsymbol{\tau}_s - \boldsymbol{\tau}_s(\mathrm{grad}\ln k \cdot \boldsymbol{\tau}_s)\big\}.\end{aligned}$$

With allowance for $k = |\boldsymbol{\tau}_s|$ and (3), obtain (4). Substituting formula (4) into (2) gives one more form of conservation law (2) in terms of one vector of curvature $\boldsymbol{\tau}_s = k\boldsymbol{\nu}$.

3. Conservation law for an arbitrary set of smooth curves and its equivalent forms

Theorem 1 and Corollary 2 follows

Theorem 2 (A conservation law for a set of curves). *Let conditions of Lemma 1 and condition* (10) *be satisfied. Then in the area D identity* (2) *is valid for the vector field $\boldsymbol{S}^*$ of the form* (3)*, i.e., the sum $\boldsymbol{S}^*$ of the curvature vectors $k\boldsymbol{\nu}$ and $k_\nu \boldsymbol{\eta} = -k_\nu \boldsymbol{\tau}$ of the two plane curves L_τ and L_ν from mutually orthogonal sets of the curves $\{L_\tau\}$ and $\{L_\nu\}$ is a solenoidal vector field in D, moreover, $\boldsymbol{S}^* = -\operatorname{rot}\{\alpha(x,y)\boldsymbol{k}\}$, and in* (2) *we have $\boldsymbol{A} = -\alpha(x,y)\boldsymbol{k}$. The given statement is equivalent to the following.*

Under conditions of the theorem for any set $\{L_\tau\}$ of plane smooth curves L_τ with the Frene unit vectors $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(x,y)$, the sum $\boldsymbol{S}^*$ of the curvature vectors $k\boldsymbol{\nu}$ and $k_\nu\boldsymbol{\eta}$ of the vector lines L_τ of a vector field of tangential unit vectors $\boldsymbol{\tau}(x,y)$ and of the vector lines L_ν of a vector field of the normals $\boldsymbol{\nu}(x,y)$ satisfies in D identity $\operatorname{div}\boldsymbol{S}^* = 0$*, i.e., $\boldsymbol{S}^*$ is a solenoidal vector field in D, thus* $\boldsymbol{S}^* = -\operatorname{rot}(\alpha\boldsymbol{k})$, $\operatorname{rot}\boldsymbol{S}^* = (\Delta\alpha)\boldsymbol{k}$.

For the vector field $\boldsymbol{S}^*$ we have anyone of representations (3), (4), *and* (25)–(29).

The following statement explains the geometric meaning of vector lines of the field $\boldsymbol{S}^*$.

Corollary 3. *Under conditions of Lemma 1, we have* $(\boldsymbol{S}^* \cdot \operatorname{grad}\alpha) = 0$*, i.e., vector lines of the vector field $\boldsymbol{S}^*$ of the form of* (3) *coincide with the level lines of the scalar field of the inclination angles $\alpha(x,y)$ of the unit vectors $\boldsymbol{\tau}(x,y)$ to the axis Ox.*

The proof results from the identity $(\operatorname{rot}(\alpha\boldsymbol{k}) \cdot \operatorname{grad}\alpha) = 0$.

From the Frene equations (1) and formulas (11)–(13) follows

Theorem 3. *Let conditions of Lemma 1, condition* (10) *be satisfied and $|k| = |\boldsymbol{\tau}_s| \neq 0$ in D. Then for any set of curves $\{L_\tau\}$ with the Frene unit vectors $\boldsymbol{\tau} = \boldsymbol{\tau}(x,y)$, $\boldsymbol{\nu} = \boldsymbol{\nu}(x,y)$ and the curvature $k = k(x,y)$ in D, we have the identity*

$$\begin{aligned} \boldsymbol{Q}^* \overset{\text{def}}{=} \frac{\boldsymbol{\nu}\operatorname{div}\boldsymbol{\tau}_s - \boldsymbol{\tau}\operatorname{div}\boldsymbol{\nu}_s}{k} &= \frac{\boldsymbol{Q}_0}{k^2} = \operatorname{grad}\ln|k| + \operatorname{rot}(\alpha\boldsymbol{k}) \\ &= \operatorname{grad}\ln|k| - \boldsymbol{S}^* \quad \Rightarrow \end{aligned} \tag{30}$$

$$\begin{aligned} \operatorname{div}\boldsymbol{\tau}_s &= ([\operatorname{grad}\ln|k| + \operatorname{rot}(\alpha\boldsymbol{k})] \cdot \boldsymbol{\tau}_s), \\ \operatorname{rot}\boldsymbol{\tau}_s &= [\operatorname{grad}\ln|k| + \operatorname{rot}(\alpha\boldsymbol{k})] \times \boldsymbol{\tau}_s, \end{aligned} \tag{31}$$

where $\boldsymbol{Q}_0 \overset{\text{def}}{=} \boldsymbol{\tau}_s \operatorname{div} \boldsymbol{\tau}_s + \boldsymbol{\tau}_s \times \operatorname{rot} \boldsymbol{\tau}_s = \boldsymbol{\nu}_s \operatorname{div} \boldsymbol{\nu}_s + \boldsymbol{\nu}_s \times \operatorname{rot} \boldsymbol{\nu}_s$*. Formulas* (30) *give an explicit expansion of the vector field* $\boldsymbol{Q}^*$ *in the potential* $(\operatorname{grad} \ln |k|)$ *and the rotational* (*solenoidal*) $(\operatorname{rot}(\alpha \boldsymbol{k}) = -\boldsymbol{S}^*)$ *components, and formulas* (31)*, i.e., expressions for the source and vortex intensities of the vector field of the curvature vector* $\boldsymbol{\tau}_s = k\boldsymbol{\nu}$*. Also, the formulas resulting from* (31) *are valid that are obtained by replacing* $\boldsymbol{\tau}_s$ *for* $\boldsymbol{\nu}_s$ *everywhere.*

Proof. Applying to each of the Frene equations (1) the operation div, multiplying the first of the obtained scalar equalities by $\boldsymbol{\nu}$, the second – by $\boldsymbol{\tau}$, subtracting and dividing by k, we arrive at:

$$\begin{aligned}\frac{\boldsymbol{\nu} \operatorname{div} \tau_s - \boldsymbol{\tau} \operatorname{div} \boldsymbol{\nu}_s}{k} &= \boldsymbol{\nu} \operatorname{div} \boldsymbol{\nu} + \boldsymbol{\tau} \operatorname{div} \boldsymbol{\tau} + \frac{\operatorname{grad} k}{k} \\ &= \boldsymbol{S}(\tau) + \operatorname{grad} \ln |k| = -\boldsymbol{S}^* + \operatorname{grad} \ln |k|.\end{aligned}$$

From this point on, owing to (13) or (29) follows formula (30). Formulas (31) are obtained as a result of scalar and vector multiplication of equality (30) by the vector $\boldsymbol{\tau}_s = k\boldsymbol{\nu}$. □

Corollary 4. *Let conditions of Theorem 3 be satisfied and* $\tau_j(x, y) \in C^3(D)$ $(j = 1, 2)$*. Then identities* (2) *and* (11) *can be represented in the form of an equivalent identity* (*i is an imaginary unit*)

$$\operatorname{div}\{\boldsymbol{Q}^* - \operatorname{grad} \ln |k|\} = 0 \quad \Leftrightarrow \quad \operatorname{div} \boldsymbol{Q}^* = \Delta \ln |k|.$$

In addition,

$$\operatorname{rot} \boldsymbol{Q}^* = -(\Delta\alpha)\boldsymbol{k} \quad \Rightarrow \quad \Delta \operatorname{Ln}\{|k| \, \mathrm{e}^{\pm i\alpha}\} = \operatorname{div} \boldsymbol{Q}^* \mp i(\operatorname{rot} \boldsymbol{Q}^* \cdot \boldsymbol{k}).$$

Remark 2. In all the above formulas the vector $\boldsymbol{\nu}$ can be expressed in terms of the vector $\boldsymbol{\tau}$ under the formula $\boldsymbol{\nu} = \boldsymbol{\tau}_s/|\boldsymbol{\tau}_s| = \boldsymbol{\tau}_s/k$ or $\boldsymbol{\nu} = -(\boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau})/|\boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau}|$ owing to $\boldsymbol{\tau}_s = (\boldsymbol{\tau} \cdot \nabla)\boldsymbol{\tau} = -(\boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau})$ and $k = |\boldsymbol{\tau}_s| = |\boldsymbol{\tau} \times \operatorname{rot} \boldsymbol{\tau}|$.

4. On the way of deriving conservation laws (2) and (11)

The presented proofs of conservation law (11) for a vector field of unit vectors and conservation law (2) for a set of plane curves are fairly simple. However, as is known, to find (to formulate) a theorem and to prove it in mathematics, generally speaking, are different tasks. The author doesn't know yet, how it would be possible to think with the keep of only vector analysis and differential geometry about the existence of a conservation law of the form of (11) for an arbitrary field $\boldsymbol{\tau}$ of unit vectors (or the Frene unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$) or of the conservation law of the form of (2) for the field $\boldsymbol{S}^*$ (the sum of curvature vectors of curves from the two mutually orthogonal sets of curves) which is equivalent. Actually, the author has come to formulas (2) and (11) through the group analysis of the following sequence of operations (group terms are understood in the sense of [11]).

First, a certain differential relation or connection (this is equality (5) in [7]) between differential invariants of the Lie group G was obtained when seeking for a basis of differential invariants of a group G in [10]. The group G has been found and investigated [10] as the equivalence group, assumed by the wave equation $u_{xx} + u_{yy} = n^2(x, y)u_{tt}$, by the eikonal equation $u_x^2 + u_y^2 = n^2(x, y)$ and by other differential equations of mathematical physics in the space of the five variables t, x, y, $u^1 = u$, $u^2 = n^2$; the solution $u^1 = u^1(t, x, y)$ and a variable parameter of the equation $u^2 = n^2(x, y)$ being thus considered to be equivalent variables. It was possible to present this differential relation [7] in a compacter form as

$$K(x, y) = J^{11} \stackrel{\text{def}}{=} \frac{1}{2}\,\frac{\Delta \ln J^7}{n^2} - A_2 H - J^7 H^2; \tag{32}$$

we also have the equality

$$J^{16} \stackrel{\text{def}}{=} \frac{\Delta\alpha}{n^2} = -A_3 H, \tag{33}$$

where

$$J^7 \stackrel{\text{def}}{=} \frac{u_x^2 + u_y^2}{n^2}, \quad H \stackrel{\text{def}}{=} \frac{J^4}{J^7}, \quad J^4 \stackrel{\text{def}}{=} \frac{\Delta u}{n^2}, \quad J^{11} \stackrel{\text{def}}{=} -\frac{1}{2}\,\frac{\Delta \ln n^2}{n^2} = K(x, y),$$

$$A_2 \stackrel{\text{def}}{=} \frac{1}{n^2}\Big\{u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y}\Big\}, \qquad A_3 \stackrel{\text{def}}{=} \frac{1}{n^2}\Big\{u_y \frac{\partial}{\partial x} - u_x \frac{\partial}{\partial y}\Big\},$$

for any functions $u(x, y) \in C^3(D)$, $n(x, y) \in C^2(D)$.

As is known in the differential geometry [4, P. 83; 13, P. 113], the quantity $J^{11} = K(x, y) = -(\Delta \ln n^2)/(2n^2)$ is the Gaussian curvature of the surface in the 3D Euclidean space with a linear element (metric) $dl^2 = n^2(x, y)(dx^2 + dy^2)$.

The expressions J^j and A_i have the following group content, formulated in the form of theorems in [10].

Theorem 4. *Let G be an infinite group of point transformations of the space of the five variables t, x, y, $u^1 = u$, $u^2 = n^2$, such that the Lie algebra of infinitesimal operators X of its one-parameter subgroups has the form $X = \Phi(x, y)\,\partial/\partial x + \Psi(x, y)\,\partial/\partial y - 2\Phi_x(x, y)u^2\,\partial/\partial u^2$, where Φ, Ψ are any conjugate harmonic functions. The expressions J^4, J^7, $J^{11} = K(x, y)$ are functionally independent differential invariants of the group G, contained in its universal second order differential invariant J_2. The invariant J_2*

consists of 15 scalar differential invariants J^j, whose explicit forms are given in [10]. *The basis of differential invariants of the group G is formed by the invariants $J^1 = t$ and $J^2 = u$. The operators $A_1 = \partial/\partial t$, A_2, A_3 are invariant differentiation operators of the group G.*

Corollary 5. *The group meaning of equality* (32) *is that this equality expresses the differential invariant $J^{11} = K(x, y)$ (the Gaussian curvature), which is determined by only one function $u^2 = n^2$, in terms of other differential invariants of the group G, which are determined by the two functions $u^1 = u$, $u^2 = n^2$. The expressions $(\Delta \ln J^7)/n^2$ and $\Delta\alpha/n^2$ in* (32), (33) *are also differential invariants of the group G (of third order).*

Thus, equalities (32) and (33) represent relations (connections) between differential invariants of the Lie group G, defined in Theorem 4.

Further it appeared possible to establish (in [7]) that formula (32) can be written down in the form of the vector differential identity

$$\frac{1}{n^2}\,\mathrm{div}\left\{\frac{1}{2}\,\mathrm{grad}\ln\frac{|\,\mathrm{grad}\,u|^2}{n^2} - \frac{\Delta u}{|\,\mathrm{grad}\,u|^2}\,\mathrm{grad}\,u\right\} = K(x, y), \tag{34}$$

whence formula (5) follows for the case of a potential vector field $\boldsymbol{v} = \mathrm{grad}\,u$, looking like the differential identity

$$\mathrm{div}\left\{\frac{\Delta u}{u_x^2 + u_y^2}\,\mathrm{grad}\,u - \frac{1}{2}\,\mathrm{grad}\ln\left(u_x^2 + u_y^2\right)\right\} = 0, \tag{35}$$

which holds for any smooth scalar function $u = u(x, y)$ with the property $|\,\mathrm{grad}\,u| \neq 0 \Leftrightarrow u_x^2 + u_y^2 \neq 0$. The analysis of identity (35) has allowed finding formulas $\mathrm{div}\,\boldsymbol{S}(\boldsymbol{\tau}) = 0$ and $\boldsymbol{S}(\boldsymbol{\tau}) = \mathrm{rot}\,(\alpha\boldsymbol{k})$ [1, 2]. Then these formulas were generalized [2] to the case of an arbitrary vector field $\boldsymbol{v}$.

Therefore, conservation law (5) can be considered to be a generalization of corollary (35) from divergent form (34) of formula (32), expressing the Gaussian curvature $K(x, y) = J^{11}$ in terms of other invariants J^j of the group G, and conservation laws (2), (11) — the forms of this generalization.

After identity (11) has been already revealed, it it is possible to prove it in a simpler way, not using the group analysis, for example, by means of formulas of the vector analysis and the differential geometry, as is shown in this paper.

Thus, the search for relations between differential invariants of the Lie group, for example, the group of equivalence of some differential equations, can be a source of the new differential identities and formulas of vector analysis and differential geometry.

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