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Numerical modeling of seismic waves for the radial-heterogeneous spherical Earth

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The paper presents an efficient algorithm based on the combination of the integral Laguerre transforms for the temporal derivatives with the Legendre transforms and finite difference method for the spatial variables. Several examples of synthetic seismograms computed for the SH waves propagating in the radial- heterogeneous spherical Earth are presented.

1. Introduction

Generally speaking, there are two main approaches to the numerical simulation of seismic waves propagating in the heterogeneous spherical Earth. The first approach is the numerical simulation of seismic fields in the time domain. Such methods are based on a finite difference scheme of the second order approximation for temporal derivatives and spectral or finite difference methods for spatial variables.The second approach employs the frequency domain simulation for the laterally heterogeneous media. The Fourier integral transforms are used for approximation of the temporal derivatives. The space frequency domain simulation does not have any stability problem, while the accuracy of the simulation based on the space-time domains is determined by a stability limit dependent on the greatest velocity in the model. Also, it is possible to employ a function of frequency as the damping coefficient, therefore the simulation of the damping effects appears more flexible. Discretization of the frequency domain equations leads to the large matrix equations and their solution for many temporal frequencies due to the enormous computational costs. A brief reviews of these methods are presented in papers [1, 2]. In this paper, we present an approach which allows us to reduce such enormous computational costs. This algorithm employs the integral Laguerre transform with respect to the time coordinate instead of the integral Fourier transform with the Legendre transforms and the finite difference method for the spatial variables [3]. The obtained algebraic equations have a matrix independent of number n – the degree of the Laguerre polynomials. Only the right-hand side of the system has the recurrent dependence on the parameter n. As this takes place, the matrix is only once transformed as compared to the frequency-domain forward modeling. For solving the obtained system we use a sweep method. In this paper, for briefness and simplicity, the algorithm is illustrated only for the SH waves propagation, although it has been developed for simulation of P and SV waves propagation as well.

2. Statement of the problem

Let us consider a spherical layer with the coordinates R, θ, φ ($r_0 \le R \le R_0$, $0 < \theta \le \pi$, $0 \le \varphi < 2\pi$). The displacement vector is the following:

$$\vec{U} = U_\varphi(r, \theta, t)\vec{e}_\varphi. \tag{1}$$

The SH wave propagation in the radial-heterogeneous spherical Earth may be described by the following equation

$$\frac{\mu}{r^2}\frac{\partial}{\partial\theta}\Big(\frac{\partial U_\varphi}{\partial\theta} - U_\varphi \cot\theta\Big) + \frac{\partial}{\partial r}\Big[\mu\Big(\frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r}\Big)\Big] + \\ 3\frac{\mu}{r}\Big(\frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r}\Big) + \frac{2\mu}{r^2}\Big(\frac{\partial U_\varphi}{\partial\theta} - U_\varphi\cot\theta\Big)\cot\theta = \rho\frac{\partial^2 U_\varphi}{\partial t^2}, \tag{2}$$

subject to the initial conditions

$$U_\varphi|_{t=0} = \frac{\partial U_\varphi}{\partial t}\Big|_{t=0} = 0, \tag{3}$$

and the boundary conditions

$$\mu\Big(\frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r}\Big)\Big|_{r=R_0} = F(\theta, t), \tag{4}$$

$$\mu\Big(\frac{\partial U_\varphi}{\partial r} - \frac{U_\varphi}{r}\Big)\Big|_{r=r_0} = 0, \tag{5}$$

where the function $F(\theta, t)$ is given by the impulse SH-torque:

$$F(\theta, t) = \frac{\delta(\theta)}{2\pi R_0^2 \sin\theta} f(t). \tag{6}$$

The Lame parameter $\mu(r)$ and the density $\rho(r)$ are piecewise continuous functions of the coordinate r.

3. Theory

The method is based on the Legendre transform with respect to the coordinate θ and the expansion of the coordinate t in series with respect to the basic functions

$$\phi_n^\alpha(ht) = \int_{ht}^{\infty}\left(\int_{x}^{\infty} e^{-y} L_n^\alpha(y)\,dy\right)dx$$
$$= e^{-ht}\Big(L_n^\alpha(ht) - 2L_{n-1}^\alpha(ht) + L_{n-2}^\alpha(ht)\Big), \tag{7}$$

where $n = 0,\dots,N$ and $L_n^\alpha(y)$ are the Laguerre polynomials. The solution to equation (2) is given by

$$U_\varphi(r,\theta,t) = r\sqrt{h}(ht)^\alpha e^{-\frac{ht}{2}} \sum_{l=0}^{L}\sum_{n=0}^{N} S_l(r,n) L_n^\alpha(ht) P_l(\cos\theta), \tag{8}$$

where $P_l(\cos(\theta))$ are the Legendre polynomials, and the functions $S_l(r,n)$ are equal to:

$$S_l(r,n) = \sqrt{h}\frac{n!}{(n+\alpha)!}\int_0^\infty e^{-\frac{ht}{2}} U_\varphi(r,l,t) L_n^\alpha(ht)\,dt. \tag{9}$$

After applying transform (9) the subject to the boundary conditions (5), we arrive at the following equation:

$$\frac{d}{dr}\left(r\mu(r)\frac{dS_l(r,n)}{dr}\right) + 3\mu(r)\frac{dS_l(r,n)}{dr} - \left(\mu_l + \frac{h^2 r\rho(r)}{4}\right)S_l(r,n) = \Phi_l(r,n), \tag{10}$$

$$\Phi_l(r,n) = \frac{n}{n+\alpha}\left[2\Phi_l(r,n-1) - \frac{n-1}{n+\alpha-1}\Phi_l(r,n-2) + h^2 r\rho(r) S_l(r,n-1)\right],$$
$$r\mu(r)\frac{dS_l(r,n)}{dr}\bigg|_{r=R_0} = F_l(n), \qquad r\mu(r)\frac{dS_l(r,n)}{dr}\bigg|_{r=r_0} = 0,$$
$$\mu_l = \frac{\mu(r)}{r}(l^2-2), \qquad F_l(n) = \frac{2l+1}{4\pi R_0^2} f_n,$$

where f_n are coefficients of the expansion of the function $f(t)$ in the Laguerre polynomials series. The coefficients of equation (10) are independent of the number n, and the right-hand side $\Phi_l(r,n)$ has the recurrent dependence on the parameter n. For solving equations (10) we use a standard conservative difference scheme of the second order accuracy

$$\mu(r_{k+\frac12})r_{k+\frac12}\frac{S_l(r_{k+1},n) - S_l(r_k,n)}{\Delta^2} - \mu(r_{k-\frac12})r_{k-\frac12}\frac{S_l(r_k,n) - S_l(r_{k-1},n)}{\Delta^2} +$$
$$3\mu(r_k)\frac{S_l(r_{k+1},n) - S_l(r_{k-1},n)}{2\Delta} - Q_{r_k} S_l(r_k,n) = \Phi_l(r_k,n), \tag{11}$$

$$
\begin{aligned}
&r_0\mu(r_{\frac{1}{2}})\frac{S_l(r_1,n)-S_l(r_0,n)}{\Delta}-\frac{\Delta}{2}Q_{r_0}S_l(r_0,n)=\frac{\Delta}{2}\Phi_l(r_0,n),\\
&R_0\mu(r_{K-\frac{1}{2}})\frac{S_l(r_K,n)-S_l(r_{K-1},n)}{\Delta}+\frac{\Delta}{2}Q_{r_K}S_l(R_0,n)=D_{R_0},
\end{aligned}
\tag{12}
$$

where $r_k = r_0 + k\Delta$, $D_{R_0} = (1+\frac{2\Delta}{R_0})F_l(n) - \frac{\Delta}{2}\Phi_l(R_0,n)$,

$$
Q_{r_k} = \left(\frac{\mu(r_k)}{r_k}(l^2-2) + \frac{h^2 r_k \rho(r_k)(r_k)}{4}\right). \tag{13}
$$

The scalar sweep method is used for solving system (11), (12). The forward coefficients of the sweep method are independent of n. A stability criterion restricts the parameter h:

$$
h^2 > 8\frac{\mu(r_k)}{{r_k}^2\rho(r_k)}.
$$

By choosing the parameter α we satisfy the initial conditions.

4. Some examples of computation

The accuracy of the numerical solutions in question is involved in the following testing. We use an exact solution for the homogeneous sphere ($0 \le R \le R_0$). After applying the Legendre transform and replacing $U = r^{-\frac{1}{2}}\tilde{U}$ in (3), (5) for the constant μ, ρ we make use of the finite Hankel transform of order $\nu = l + \frac{1}{2}$:

$$
\chi(k_i,\nu_l,t) = \int_0^{R_0} rK_{\nu_l}(k_i r)\tilde{U}(r,\nu_l,t)\,dr, \tag{14}
$$

where

$$
K_{\nu_l}(k_i r) = \frac{\sqrt{2}}{R_0}\left[\frac{1}{4R_0^2} + \left(1 - \frac{\nu_l^2}{k_i^2 R_0^2}\right)\right]^{-1/2}\frac{J_{\nu_l}(k_i r)}{J_{\nu_l}(k_i R_0)}, \tag{15}
$$

and k_i are the positive roots of the equation

$$
J'_{\nu_l}(k_i R_0) - \frac{1}{2R_0 k_i}J_{\nu_l}(k_i R_0) = 0. \tag{16}
$$

The inverse transform is defined as

$$
\tilde{U}(r,\nu_l,t) = \sum_{i=1}^{\infty} K_{\nu_l}(k_i r)\chi(k_i,\nu_l,t), \tag{17}
$$

where

c

$$\chi(k_i,\nu_l,t) = K_{\nu_l}(k_iR_0)Q_l\left(\sin k_it\int_0^t f(\tau)\cos k_i\tau\,d\tau - \cos k_it\int_0^t f(\tau)\sin k_i\tau\,d\tau\right),$$

$$Q_l = \frac{2l+1}{2k_i\sqrt{R_0}}. \tag{18}$$

The solution obtained with the proposed algorithm is compared with the solution by formula (17). The time dependence of the source pulse used is given by

$$f(t) = \exp\left[-\left(\frac{2\pi\omega(t-t_0)}{\gamma}\right)^2\right]\sin(2\pi\omega(t-t_0)), \tag{19}$$

where ω is the predominant frequency, γ is the damping factor ($\gamma = 4$), and t_0 is chosen such that $f(t) \cong 0$ or, equivalently, t_0 is the half-width of the pulse. The source is located at $r = R_0$, $\theta = 0$. The same number of the Legendre polynomials are used for both solutions. A comparative error for our algorithm is about one percent. Figure 1 shows the synthetic seismograms for both solutions: the upper graph is the solution obtained by formula (17) and the lower graph is the solution computed using the proposed algorithm, $t_0 = 1.75$ s.

The Earth's model used is the PREM [4]. Figure 2 shows synthetic seismograms of the SH-waves for the PREM and for the above presented source pulse. The synthetic seismograms are computed up to the epicentral distance of 70°, the predominant period $T_0 = 30$ s. The time is measured in terms of the predominant period.

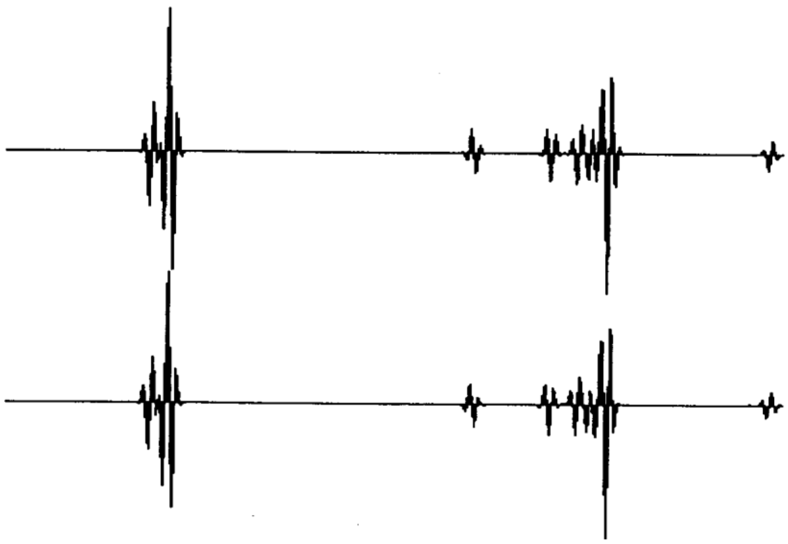

Figure 1. Synthetic seismograms for the homogeneous sphere, $\theta = \frac{\pi}{2}$

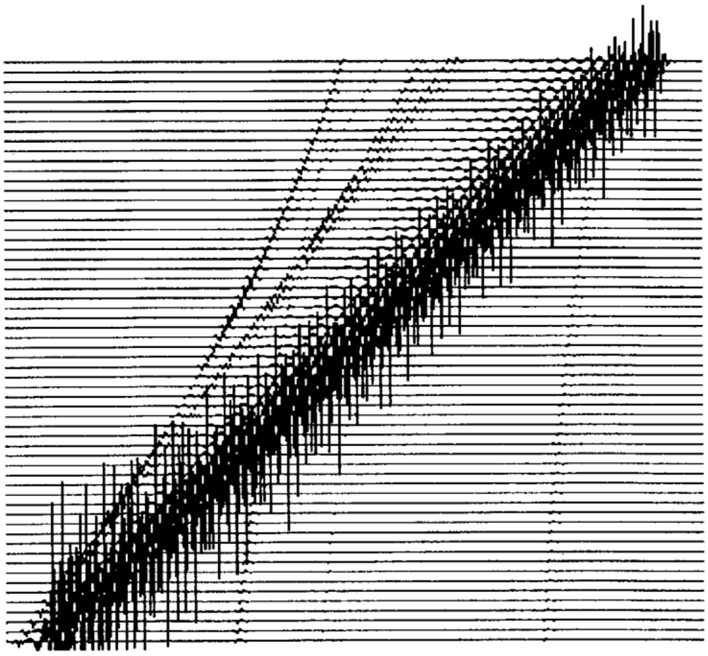

Figure 2. Synthetic seismograms for the PREM.
The maximum epicentral distance is 70°

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