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On existing optimal three-dimensional circulant networks*

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The optimal circulant graphs have minimum diameter for the given order N and degree v and, respectively, optimal features with respect to communication delays, reliability and connectivity under implementation as interconnection networks in multimodule supercomputer systems. The questions of existence of optimal (nearly optimal) circulants with the degree $v = 6$ are investigated. An algorithm for enumeration of graphs, the experimental results obtained for optimal and nearly optimal loop circulant graphs of degree 6 and some conclusions about the probability of their existence are presented.

1. Introduction

Circulant graphs and their various applications are intensively investigated in computer science, graph theory and discrete mathematics [1–5]. They are realized as interconnection networks in a number of parallel supercomputer systems and used in multimodule memory designs. Multidimensional circulants are a subclass of the class of $R_s(N, v, g)$ graphs [6, 7] – parametrically prescribed, regular, and those described by semigroups, where g is the girth and s is the number of equivalence classes of a graph. The class of $R_s(N, v, g)$ graphs includes the majority of known graphs presented earlier as structures of computer systems. Circulant graphs with the dimension $n = v/2$, if the degree v is even, coincide with the class of $R_s(N, v, g)$ graphs provided $g = 4$ and $s = 1$. Note, when $g = 4$ the class of $R_s(N, v, g)$ graphs also includes such known network topology as hypercubes.

It is shown in [7] that the optimal circulants diameter is $\approx 0.32 \log_2 N$ whereas the hypercubes diameter is $\log_2 N$, provided they have the same vertex and edge complexity. Optimal circulants have also a smaller average interprocessor distance than hypercubes for the same number of processors under implementation as interconnection networks in parallel computer systems. For $v = 4$, the problem of optimal circulants existence has been solved [8, 9, 2].

In this paper, we present an investigation of the problem of existing the optimal and nearly optimal circulant graphs of degree 6.

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2. Definitions and problem formulation

Along with high scalability, survival and modularity one of the advantages of circulant graphs is the opportunity to be represented by means of a compact parametric description.

Definition 1. A circulant is an undirected graph $G(N; s_1, s_2, \ldots, s_{v/2})$ with N nodes, labeled as $0, 1, 2, \ldots, N-1$, having $i \pm s_1$, $i \pm s_2$, $\ldots$, $i \pm s_{v/2}$ (mod N) nodes, adjacent to each node i.

The numbers $S = (s_i)$ $(0 < s_1 < \ldots < s_{v/2} < N/2)$ are the generators of a finite Abelian group of automorphisms connected to the graph. Circulant graphs $G(N; 1, s_2, \ldots, s_{v/2})$, with an identity generator, are known as loop networks [1, 10, 4]. They have a hamiltonian cycle. Note, a circulant has a hamiltonian cycle if one of its generators is relatively prime to N.

Let n_k determine the number of nodes in the level k of a circulant G with the degree v, n_k^* being the upper bound for n_k. Let $u_k = \sum_{i=0}^{k} n_i$, denote the number of nodes in G which are reachable by at most k steps from the node 0, and u_k^* being the upper bound for u_k. Recurrence relations and exact formulas for an evaluation of n_k^* and u_k^* have been obtained [12, 11, 2]:

$$n_0^* = 1, \quad n_k^* = \sum_{i=0}^{v/2-1} C_{v/2}^{i} C_{k-1}^{v/2-i-1} 2^{v/2-i} \text{ under } k \geq 1;$$

$$u_k^* = \sum_{i=0}^{v/2} C_{v/2}^{i} C_{k}^{v/2-i} 2^{v/2-i}.$$

One of the main problems concerning circulant graphs and their use in computer science is a basic construction optimization problem, the conversation to the well-known $(Degree/Diameter)$-graph problem: *Given the vertex number N and the degree v find a circulant with smallest diameter d.*

The diameter of G is $d = d(G) = \max_{ij} d_{ij}$, where d_{ij} is the length of a shortest path from a node i to a node j. The average distance of G is $\bar{d} = \frac{1}{N(N-1)} \sum_{ij} d_{ij}$. Let d^* denote the exact lower bound for $d(N) = \min_S \{d(G(N; S))\}$. The diameter d^* is defined by the correlation $u_{d^*-1}^* < N \leq u_{d^*}^*$.

Definition 2. A graph G is *optimal* if $d(G) = d^*$, a graph G is *suboptimal* if $d(G) = d^* + 1$.

Optimal circulant networks have the minimum communication delays, the maximum reliability and connectivity [12, 2] and the minimum number of steps for a realization of communication algorithms [13], but do not exist for some values N and $v > 4$ [1, 3, 10, 14]. For $v = 4$, an analytical solution to the problem of existence and synthesis of optimal circulants has been found in [8, 9, 2]. In [10, 14, 3, 15] some conditions have been found for an existence of optimal loop circulant graphs with $v = 4$, and dense infinite families of values of N, which are optimal, have been defined by analytical formulas. The intervals for the existence and the descriptions of optimal three-dimensional loop circulant graphs were obtained by computer search in [4] for $N \leq 1000$ and there was shown that there exist suboptimal graphs for all $N \leq 1000$. But, in general, the known search algorithms do not give solutions for large N and $v > 4$. A variant of the problem of existence: *is it true that* $d(N) \leq d(N') + 1$ *for any* $N < N'$*?* set up for the two-dimensional case [16] and for any dimension of loop network [4]. A survey of foreign works for the problems of construction and existence of optimal circulants has been made in [1].

In this paper, we take the first step to investigate the problem of existence in multidimensional case. In Section 3, there present an algorithm for enumeration of loop circulant graphs of degree 6, the experimental results obtained for optimal and nearly optimal loop circulant graphs of degree 6 and some conclusions about the probability of their existence.

3. A study of existing optimal and nearly optimal circulant graphs

Let Ω_N be a set of all possible circulant graphs G with descriptions of the form $\{N; 1, s_2, s_3\}$ for a given N. As shown in [3], it is sufficient to sort out all $1 < s_2 < s_3 < N/2$. The algorithm for the enumeration of all graphs from Ω_N and determination of the number of graphs with given diameter consists of the following steps.

Algorithm 1:

Step 1: $s_1 := 1$; $s_2 := 2$, $s_3 := 3$;
Step 2: to take the current description $\{N; 1, s_2, s_3\}$ of $G \in \Omega_N$
(the generators s_2, s_3 are sorted by the lexicographic order);
Step 3: to determine the value of diameter of a graph G;
Step 4: to calculate the number of graphs with obtained diameter;
Step 5: to take the next description of $G \in \Omega_N$ in lexicographic order;
Step 6: if $s_2 \neq N/2 - 2$ & $s_3 \neq N/2 - 1$, then go to Step 2;
Step 7: stop.

An algorithm for determining the diameter (Step 3) is described in [17]. The values of diameters for the graphs $G \in \Omega_N$ are varied from d^* to some value generated by the worst graph, that is the graph with the description $\{N; 1, 2, 3\}$.

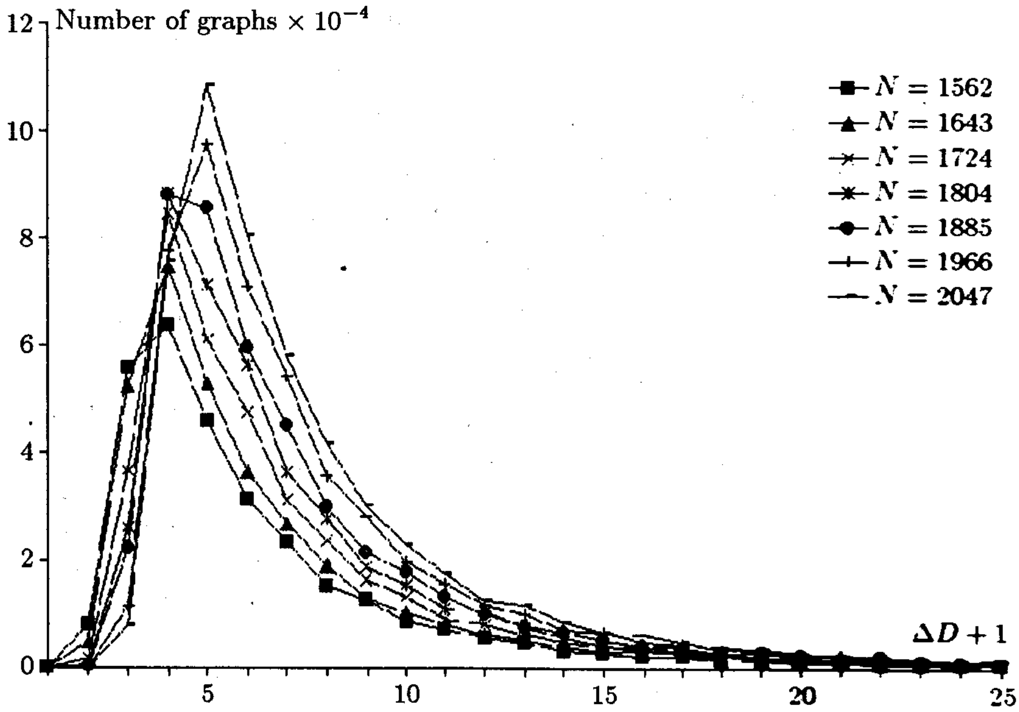

Figure 1

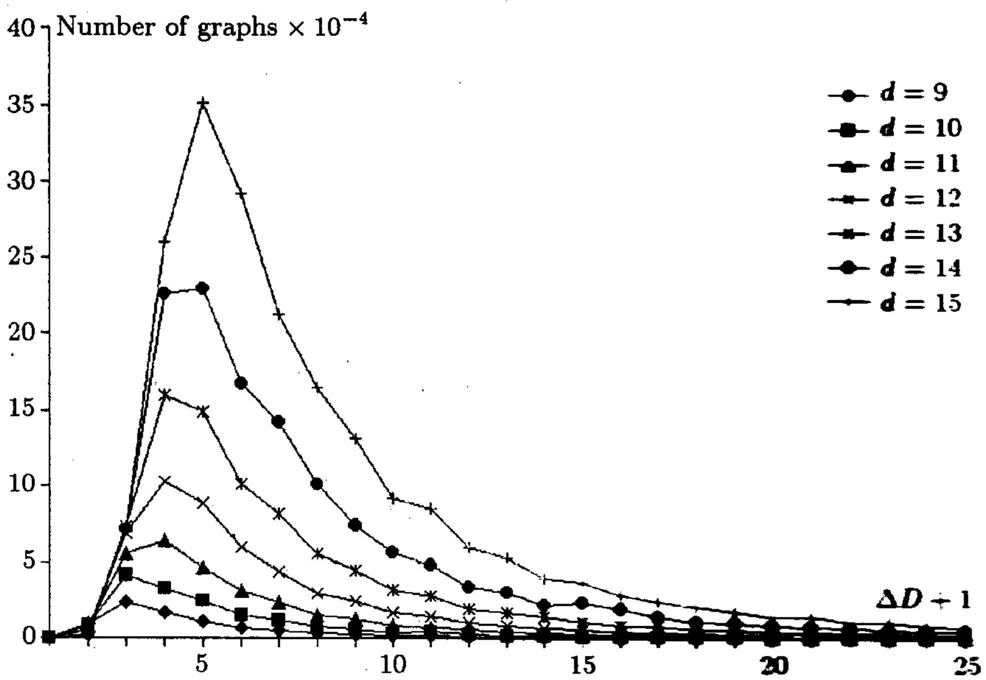

Figure 2

The data obtained by Algorithm 1 allow us to determine the distribution function of the graphs $G \in \Omega_N$ on the diameters or, more precisely, on the degree of deviation of graph diameter from its exact lower bound (the parameter ΔD in Figures 1–3). Note, in Figures 1–3 there present fragments of the distribution function for $0 \leq \Delta D \leq 24$, it can be neglected by the other values.

In Figure 1, the distribution function is presented for different values of N, $1562 \leq N \leq 2047$ which is varied under fixed diameter (the example corresponds to the value $d = 11$). In Figure 2, a diameter is a variable parameter, $9 \leq d \leq 15$, and the value of N is the constant of the form $N = 4/3d^3 - 2d^2 + 8/3d$ which is equal to the minimum value of N for a given d.

The maximum value of the number of graphs (see Figure 1) is achieved under $\Delta D = 3$ ($N = 1562, 1643, 1724, 1804, 1885$) and $\Delta D = 4$ ($N = 1966, 2047$). In Figure 2, the maximum value of the number of graphs is achieved under $\Delta D = 2$ ($d = 9, 10$), $\Delta D = 3$ ($d = 11, 12, 13$), and $\Delta D = 4$ ($d = 14, 15$). The data given for the middle and maximum values of N for a given d show that the character of the distribution function is the same.

The probability P of existing loop circulant graph with given diameter was also obtained (see Figure 3). We see that the probability of existing P corresponding to maximum number of graphs from Ω_N with the given diameter is reduced under increase of diameter.

We analyze the existence of optimal and suboptimal graphs because we are interested in optimizing the diameter among all possible choices of generators for the given N.

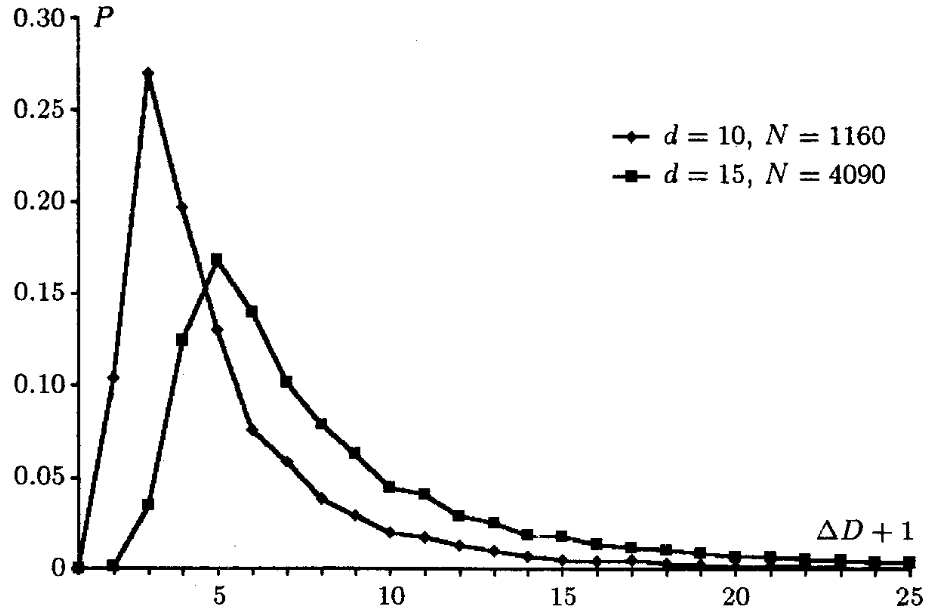

Figure 3

In Figure 4, the diagram of the total number of optimal graphs is presented as a function of N, $10 \leq N \leq 10000$. The character of its variation is represented completely for all N for the given d up to $N < 1000$. Under $N > 1000$ the values of parameter being investigated are shown only in the points of N relevant to the beginning and end of the range of N for a given d. On the whole the considered parameter is decreased with the graph diameter increase.

The sum total of suboptimal graphs (the upper line) and optimal graphs (the lower line) as a function of N is shown in Figure 5. As we can see the character of variation for present parameter remained the same for suboptimal graphs as for optimal graphs. But a size of the range of gradual reduction of this parameter is larger. It is seen that at the beginning of the range of N values for a given d the sum total of optimal graphs is approximately the same as the sum total of suboptimal graphs at the end of preceding one.

Thus, the sums total of optimal and suboptimal graphs for a given N are rapidly reduced for loop circulant networks with $v = 6$ and is tending to zero under $d > 21$. The given data in the range of $10 < N < 10000$ indicate in behalf of the hypothesis stated in [4] for loop circulant networks with degree 6: if $N < N'$ then $d(N) \leq d(N') + 1$.

The optimal and nearly optimal graphs (with a diameter exceeding the exact lower bound by two units at most) are synthesized [17] by means random search and genetic algorithms. Parametric descriptions of nearly optimal circulant graphs for some values of N and $v \geq 6$, obtained by genetic algorithm, are represented in [7]. For instance, the nearly optimal graphs descriptions have been synthesized for the following circulant graphs with large values of N and v: $N = 1048576$, $v = 20$ and $d = 10$; $N = 2097152$, $v = 20$ and $d = 11$.

For $v = 6$ the descriptions were obtained fragmentarily up to $N = 55000$. The values of execution time have been defined for these algorithms in different intervals of changing N including the beginning, middle and end of the range for a given d. It is seen by the analysis of time values needed for the random search and genetic algorithms that the probability of existence of optimal (suboptimal) circulants are decreased under increase of N for a given d. The minimum probability is observed at the bounds of transitions from d to $d + 1$. This fact is good coordinated with the diagrams for circulants with $v = 6$ presented in this section. The efficiency of random search and genetic algorithms of synthesis depends on the probability of existing the desired graphs for a given degree and order. Further investigations with respect to existence of circulants with given diameter will make the problem of synthesis of optimal (nearly optimal) circulants more clear.

Finally, we list some open questions for further investigations:

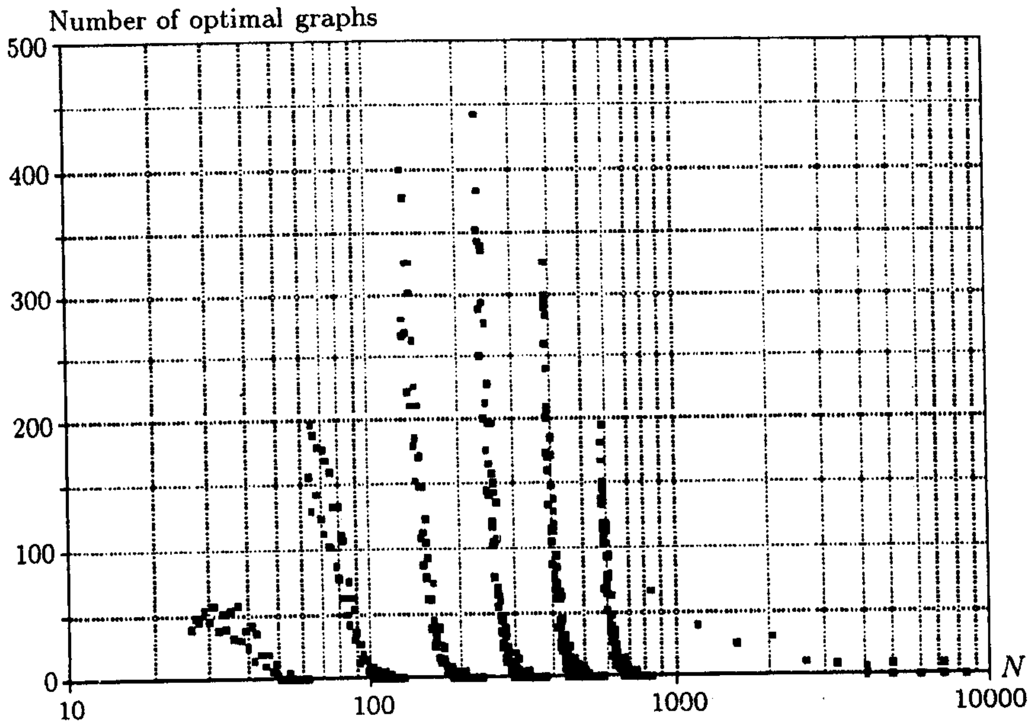

Figure 4

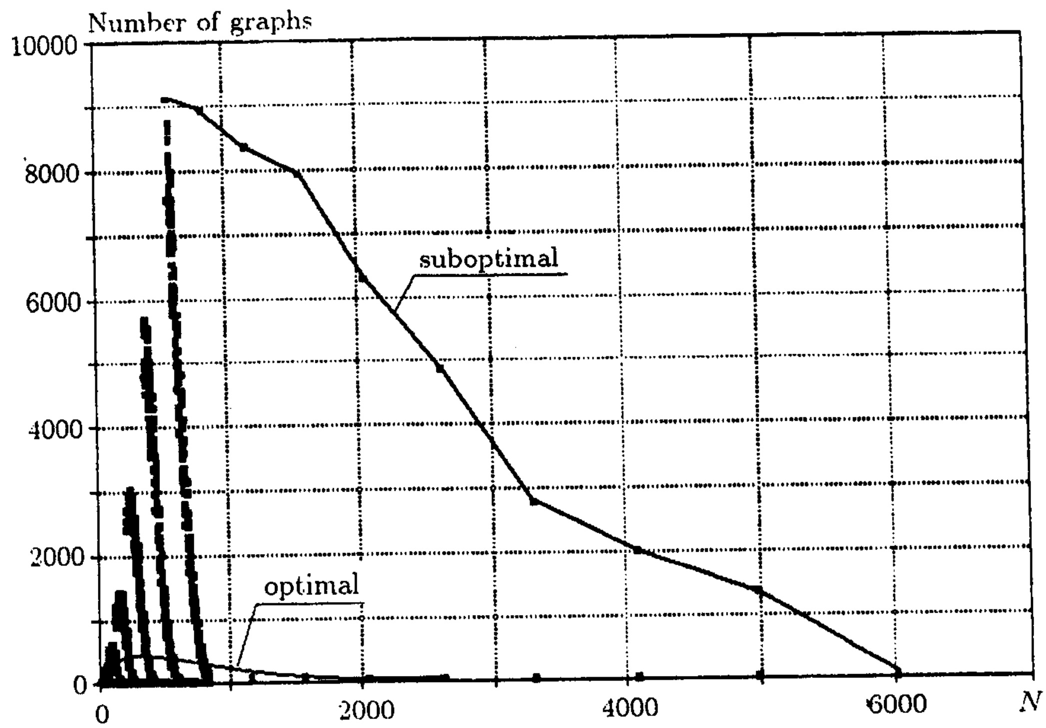

Figure 5

1. Study the generalization of problem of existence of optimal (nearly optimal) loop circulants for $v > 6$. The use of parallel version of Algorithm 1 (for $v > 6$) assumes for its solution.
2. Study the existence of optimal (nearly optimal) circulants of arbitrary form $G(N; s_1, s_2, \ldots, s_{v/2})$. For example, for the two-dimensional case there are much more such optimal circulants, when $s_1 > 1$, than optimal loop circulants [3, 10]. Is it true for $v \geq 6$?

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