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Genetic and heuristic algorithms of synthesis of optimal multidimensional circulant networks*

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A solution to combinatorial optimization problem of constructing optimal circulant networks with the minimum diameter under given degree and number of graph nodes is considered. The circulant networks and their different applications are the object of intense investigations. Under degree of a graph $\delta > 4$ and large number of nodes it is necessary to develop new effective methods of the optimal circulant designs synthesis. The application to decision of given problem a heuristic algorithm and a genetic algorithm based on simulation of natural evolution process is considered. The comparison of the algorithms and their different versions of realization is obtained. The results of computer experiments and catalogues of optimal circulants with $N > 1000$ and $\delta > 4$ are presented.

Introduction

In this work we consider fundamental optimization problem of the design of efficient interconnection networks for parallel computer systems – the construction of optimal networks having the minimum diameter for a given number of nodes (order) N and the degree δ of a regular graph. The choice of optimal graphs as interconnection networks allows to optimization such parameters of functioning as transmission delays, reliability and connectivity, speed of routing algorithms and etc. under given equipment expenditures. The circulant graphs [1–12] are characterized by high scalability, survival and modularity and realized as interconnection networks in multimodule supercomputer systems (MPP, Intel Paragon, Cray T3D etc.). For degree of a graph $\delta = 4$ the problem of synthesis of optimal graphs has an analytical solution [3–5]. But under $\delta > 4$ and any N it is known as NP-hard and an analytical approach to its solution meets some difficulties so for its decision it is necessary to develop new effective algorithms. We consider a heuristic algorithm using the reduced search and a genetic algorithm based on simulation of evolutionary process of development of living organisms. The genetic algorithms were successfully used [13, 18] for decision the number of problems in combinatorial optimization, artificial neural network learning,

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graph theory and the Cayley graphs degree/diameter problem [14]. With development of genetic algorithms in conformity to the synthesis of optimal circulants considerable advance can be obtained in decision of the problem for the degrees $\delta > 4$ and large orders ($N > 1000$).

1. Optimal circulants

Definition 1. A *circulant network* is the graph $G(N; s_1, s_2, \ldots, s_n)$ with N nodes, labeled as $0, 1, 2, \ldots, N-1$, having $i \pm s_1$, $i \pm s_2, \ldots, i \pm s_n$ (mod N) nodes adjacent to each node i.

The numbers $S = (s_i)$ $(0 < s_1 < s_2 < \ldots < s_n < (N+1)/2)$ are the generator set of the finite Abelian automorphism group associated to a graph. The degree of a node in an undirected graph G is $\delta = 2n$, where n is the dimension of a graph. The graph $G(N; s_1, s_2, \ldots, s_n)$, when $s_1 = 1$ is known as loop network [2, 6–9, 12, 15]. The optimization problem is in a search for a graph with the minimum diameter and minimum mean distance among all circulants ($C(N, n)$) having N nodes and the dimension n. The diameter of G is defined by $d = \max_{ij} d_{ij}$, where d_{ij} is the length of shortest path from a node i to a node j. The average distance of G is $\bar{d} = \frac{1}{N(N-1)} \sum_{ij} d_{ij}$. Let $d(N) = \min_S \{d(G(N; S))\}$. Let for any graph $G \in C(N, n)$, $K_{n,m}$ denote the number of nodes to be attained from the node 0 by using at most m steps along generators, and $K^*_{n,m}$ be the upper bound for $K_{n,m}$. Denote $L_{n,m} = K_{n,m} - K_{n,m-1}$, $L^*_{n,m}$ be the upper bound for $L_{n,m}$. The values of $K^*_{n,m}$, $L^*_{n,m}$ for any n, m were determined in [6, 16].

Note that a term "optimal" is used in the literature in different senses. For example, in [2, 7–9] a graph is optimal if $d(G) = d(N)$ and tight optimal if $d(G) = \mathrm{ulb}(N)$ (exact lower bound of $d(N)$). We will use the following terminology.

Definition 2. A graph $G \in C(N, n)$ is *extremely optimal*, if $L_{n,m} = L^*_{n,m}$ for any $0 \leq m \leq d^* - 1$ and $L_{n,d^*} = N - K^*_{n,d^*-1}$, where a diameter $d^* = \mathrm{ulb}(N)$ is given from the correlation $K^*_{n,d^*-1} < N \leq K^*_{n,d^*}$.

An extremely optimal graph achieves the lower bounds of a diameter and an average distance, has a maximum of reliability and connectivity among all graphs from $C(N, n)$ [16], requires a minimum number of rounds for realization of broadcast and gossiping [17].

Definition 3. A graph G is *optimal* if $d(G) = \mathrm{ulb}(N)$.

Definition 4. A graph G is *suboptimal* if $d(G) = \mathrm{ulb}(N) + 1$.

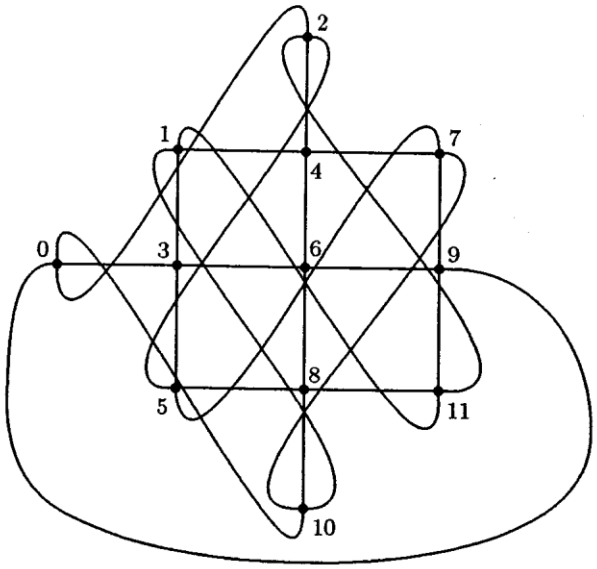

Figure 1. The circulant graph $G(12; 2, 3)$ shown as a lattice

The diameters of optimal circulants are computed from the expression for $K^*_{n,m}$. In the case when $n = 2$ [3–5] the exact lower bound for $d(N)$ should be $\mathrm{ulb}(N) = \lceil(\sqrt{2N-1}-1)/2\rceil$, where $\lceil x\rceil$ denotes the smallest integer number which is not less then x. It is shown in [3–5] that, for any $N > 4$, the optimal graph $G(N; s_1, s_2)$ exists for the values of $s_1 = \mathrm{ulb}(N)$ and $s_2 = \mathrm{ulb}(N)+1$. The example of two-dimensional circulant with a given description is shown in Figure 1.

The necessary and sufficient conditions of existence of extremely optimal double-loop networks [15] and of optimal ones [7, 9, 10] are pointed, and dense infinite families of values of N for them are determined.

In [12], for undirected loop networks of degree 6, the intervals of existence of optimal graphs with $N \leq 1237$ and their description are obtained.

2. Heuristic algorithms of the synthesis

The heuristic algorithms of getting the optimal (suboptimal) circulant graphs (loop networks) with any order and degree which use different methods of determining an optimality of a graph on its description are considered. The following heuristics are used under the synthesis of graphs to reduce an exhaustion of generators $S = (s_1, \ldots, s_n)$:

- the values of s are removed from a set of possible generators for which $N/\,\mathrm{GCD}(N, s) < 2(d-1)$ at a synthesis of any circulants (GCD is the greatest common devisor of given integers);

- the values of generators are taken with $2(d-1) \leq s$ for an exhaustion and test at a synthesis of loop networks.

The basic part of computations is required for finding a diameter of a graph. Two different algorithms of finding a diameter and some their technical modifications were performed in the complex of heuristic and genetic algorithms programs.

2.1. The first algorithm. Let N nodes of a tested graph G with numbers $0, 1, \ldots, N-1$ be cycling in a loop. An entry $h[i]$ of the array h contains a distance from 0-th node to i-th one. At the beginning of computations, $h[0] = 0$ and $h[i] = -1$ for all others i. Then the loop follows in which the paths are formed from 0 node to all nodes, the lengths of these paths and the number of nodes which may be attained from the node 0 at given loop step are computed. The loop is executed until there are the nodes in which a path can be built. In addition, it is defined is the graph G connected or not. In the last case a diameter of G gets the infinity value. As soon as the loop step exceeds the value d^* for optimal (or $d^* + 1$ for suboptimal) circulants search, the loop is finished.

2.2. The second algorithm is based on the constructive method of synthesis of circulants [3, 8, 15], using the geometrical visualization and generating the constructions of optimal circulants. It is a generalization of the algorithm proposed in [8] for two-dimensional circulants. Consider, for example, three-dimensional circulants. A circulant $G(N; s_1, s_2, s_3)$ may be constructed as an octahedron-similar frame of lattice unit squares in Z^3 in the following way. Label each lattice point (i, j, k) by the number $m = s_1 i + s_2 j + s_3 k \pmod N$, m is a graph node number. As a result, every label m belongs to $\{0, 1, \ldots, N-1\}$ and is repeated infinitely many times in Z^3, resulting in a tessellation of octahedron-similar constructions of Z^3. All N node labels of optimal graphs with $d = d^*$ must lay inside an octahedron with semidiagonals equal to d^* (for suboptimal graphs, d is incremented by one). So, we scan all cubes in octahedron exchanging coordinates of lattice points in the following way: $i \in [-d, d]$, and, correspondingly, $j \in [-(d-|i|), d-|i|]$, and $k \in [-(d-|i|-|j|), d-|i|-|j|]$, and label given numbers of nodes. If the number of labelled (distinct) nodes equals to N, then the optimal description is found. For $n > 3$, The generalization of the algorithm is evident for $n > 3$.

2.3. The results of comparison and execution. As it was shown by a computer realization, the first algorithm to obtain a diameter demands more memory capacity than the second one but is better on execution time on the average in 2–2.7 times. The results of work of heuristic and genetic

algorithms presented in the tables below were given under the realization of the first version of obtaining a diameter.

By means of heuristic algorithm, for every tested N, the generator set was found out for optimal description of a graph G and the cases when it coincides with extremely optimal one were identified. Simultaneously, suboptimal description of the graph G has been defined.

3. Genetic algorithms of the synthesis

The genetic algorithms are based on a simulation of the survival of the fittest in the population of entities each of which presents the point in a space of solutions of the optimization graph problem. The entities are presented by strings of genes (generator sets). The function $\mathcal{F}$ named fitness function evaluates the degree of an approximation of a graph diameter to its exact lower bound.

The purpose of the genetic algorithm is to search a global minimum of the fitness function, when the initial structure of the population is given, through applying generic operators (selection, crossover, and mutation) to it.

3.1. The data structure. We use two populations while searching the optimum, an old and a new one. The old population is produced on previous iteration (for the first iteration it is filled with randomly chosen generator sets) and is used for filling a new population. Under the synthesis of the optimal descriptions for N, changing in some range, the generator sets that were the best ones given for the value N are used as the first population for the value $N+1$.

The parameters of genetic algorithm are: N is a graph order, n is a graph dimension, M is a number of graphs in the population, i_M is the number of iterations, p_m is the probability of a mutation, p_c is the probability of applying the crossover to pair of entities.

Each population is the set of generator sets for a given N. Each generator set is represented with the vector of the length $n+1$. The genes contain the integers from 1 to $[N/2]$. The $[n+1]$-term describes the diameter of a graph G. The minimization $\mathcal{F}$ is done by the diameter. Iterations are finished if the best value of fitness function has not changed during certain time or after a given number of steps.

3.2. The mutation. The mutation is applied to randomly chosen generators in all sets of generators of the current population (their number is equal to the parameter of mutation) and produces new ones. The content of each chosen generator is replaced with randomly chosen number from 1 to $[N/2]$. The mutations of two types are applied:

- a replacement of a generator with a random number,
- a replacement of a generator with a random number from some neighborhood of the replaced generator.

3.3. The crossover. We apply the crossover to two generator sets and get two new ones. We exchange the contents of corresponding generators with some probability. Some arbitrary pairs are chosen from the population consisting of M graphs. In every pair, the graph generators are partitioned into two parts in randomly chosen place and exchange the parts (Figure 2).

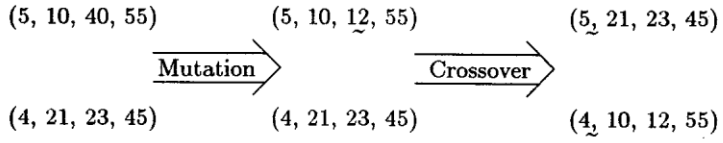

Figure 2. Example of a mutation and a crossover

3.4. The selection provides the principle of the survival of the fittest. It is applied to the old population as a whole. The selection consists in sorting the population on the significance of fitness function (diameter of a graph) and copying of several best generator sets to the new population which then is filled up by using of crossovers and mutations. The diameters of all graphs in population are computed. The graphs with smaller diameters are remembered separately to don't lose them under a mutation or a crossover. The graphs with largest diameters are displaced by the graphs of smaller diameters.

4. Experimental results

Some results of work of heuristic and genetic algorithms in the form of optimal descriptions for selected graphs are presented in Tables 1 and 2. The bound values of N corresponding to diameter alteration are shown. The following values are used for the genetic algorithm: $p_m = 0.18$, $p_c = 0.5$, $M = 200$, $i_M = 3000$.

For heuristic (HA) and genetic (GA) algorithms, the measurements t_{av} of the average execution time (getting of the first suboptimal description) in ranges $N_1 \leq N \leq N_2$ are shown in Table 3 (in seconds for Pentium-166 MMX). These times were defined in different ranges of N including a beginning, a middle and an end of ranges for a given diameter. The choice of these intervals is explained by the fact that a probability of existence of optimal (suboptimal) descriptions for circulants are decreased under increase of N for d given. The minimum probability is observed at the bounds of transitions from a diameter to another one. The execution time of the

Table 1. The fragment of catalogue of optimal (suboptimal) circulants with $n = 3$

N	d (d^*)	s_1	s_2	s_3	N	$d = d^*$	s_1	s_2	s_3
1561	11 (10)	43	645	650	1562	11	130	301	350
2047	12 (11)	19	575	974	2048	12	237	464	541
2625	13 (12)	1096	1244	1283	2626	13	65	239	562
3303	14 (13)	46	238	651	3304	14	229	1047	1182
4089	15 (14)	133	1309	1617	4090	15	347	1130	1240

Table 2. The selected suboptimal circulants of different degrees

N	n	d (d^*)	s_1	s_2	s_3	s_4	s_5	s_6	s_7
11521	3	22 (20)	823	5135	5137				
13287	3	23 (21)	1138	1586	6042				
15225	3	24 (22)	105	337	1247				
1024	5	6 (5)	49	64	367	462	476		
4096	6	7 (6)	321	753	836	1380	1893	1990	
16384	7	8 (6)	490	1277	1645	2512	3832	4317	5448

Table 3. Comparison of heuristic and genetic algorithms for synthesis of suboptimal circulants

n	$N_1 - N_2$	d (d^*)	$t_{av}(HA)$	$t_{av}(GA)$
3	1562 – 1571	12 (11)	0.6	0.5
	1800 – 1809	12 (11)	1.3	2.7
	2038 – 2047	12 (11)	225.1	54.2
3	2048 – 2057	13 (12)	1.2	0.8
	2332 – 2341	13 (12)	3.2	9.0
	2616 – 2625	13 (12)	476.3	198.3
4	682 – 691	7 (6)	0.9	0.3
	980 – 989	7 (6)	9.2	0.5
	1280 – 1289	7 (6)	92.1	14.6
4	1290 – 1299	8 (7)	5.9	0.6
	1760 – 1769	8 (7)	58.0	1.1

algorithms essentially increases in comparison with the beginning of the range.

It was shown that graphs of the degree $\delta = 6$–10 and the diameter $d \leq 10$ have suboptimal descriptions.

The explicit catalogues of optimal and suboptimal three-dimensional circulants and graphs of larger degrees are represented at Web-page: `http://rav.sscc.ru/~emilia/`.

5. Conclusion

The results of research on the development of genetic and heuristic algorithms for optimization graph problem are presented. They show that the proposed algorithms are able to perform effectively a search for an optimal (suboptimal) solution to the given problem for circulant graphs of large dimensions and numbers of nodes. The genetic algorithms are good for search of perspective areas in spaces of solutions but are not so good for exact search in these areas. Under increasing n dimension the appropriateness of using genetic algorithm is increased. On the other hand, heuristics are good for search of tight solutions but have larger run time than genetic algorithms.

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