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Chapter 1

The constant geomagnetic field

Introduction

Recently, the theory and practice of geoelectromagnetism have been enriched with the new theoretical and experimental results that appeared in many publications. However, there are still unsolved problems in geoelectromagnetism that arose as far back as in the time of Gauss, who laid the groundwork for the physico-mathematical modeling of magnetic fields observed on the Earth [9].

The first and principal problem is that it was generally accepted to consider the supposition about potentiality of the Main Geomagnetic Field (MGF) and its variations in the Earth's atmosphere due to the absence in the air of detectable electric currents to be justified. It has appeared that in practice, when interpreting the MGF and fields of its variations, there arise non-potential fields in the air suggesting that their appearance should be investigated.

The second and no less principal problem concerns the source of the MGF. There are many hypotheses about the source of the MGF. However, most of them do not match the results obtained in the theory of electromagnetic field, on the one hand. And on the other hand, there have appeared experimental data casting some doubt on theoretical results substantiating the nature of the Earth's electromagnetic field [7, 20, 28].

The third problem deals with the interpolation of data about the electromagnetic fields observed on the Earth's surface by the world network of electromagnetic observatories as well as of the satellites data. Although with supercomputers the possibility to interpolate algorithms has sharply increased, the accuracy of the interpolation with the use of the Gauss–Schmidt decompositions does not still satisfy the researchers of the Earth's electromagnetic fields.

All these, at first glance, solvable problems inspired the author to be deeply involved in updated physico-mathematical grounds of theory of the Earth's electromagnetic field. This resulted in its revision from the new standpoints that deal not only with physical problems of description and interpretation of the Earth's electromagnetic field, but has also formulated a number of new mathematical tasks. In particular, there has arisen a problem of reconstruction of poloidal and toroidal vector fields on the sphere surface with the help of one scalar function and some other problems, whose solution is the subject of this study.

1.1. On reconstruction of the main geomagnetic field on the Earth's surface with allowance of its toroidal part

In the last few decades, a new direction in the Earth's EMF theory relating to a possible source of the MGF has been intensively developed. We mean the so-called dynamo-excitation of the MGF, where a toroidal magnetic field has a significant place. The interaction between the toroidal magnetic field that has no normal magnetic component on the Earth's surface and the poloidal field that does have such a normal magnetic component permits the generation of a magnetic field with rather an intensive toroidal magnetic field of 100 up to 500 Hz in the Earth's interior [6; 15, p. 166; 22]. The demand for a significant in intensity toroidal magnetic field inside the Earth has generated a myth of a possible existence of such a toroidal magnetic field in the Earth's interior. But such a field does not reach the Earth's surface and is not observed in the EMF measurents.

The fact is that the boundary conditions for the magnetic field indicate to its continuous transition through the Earth's surface because this surface is non-magnetic and no intensive electric currents are observed on it. Therefore the Earth's surface is not a screen for the MGF and EF as a whole.

In this connection there arises a basically new mathematical problem of a unique reconstruction of a solenoidal vector field in a sphere (a solenoidal magnetic field) with allowance for a vector field presence in the sphere but without normal component on its surface (a toroidal vector field). In our opinion, the solution to this problem should be sought for with a particular emphasis on the Helmholtz theorem "On finding a vector field from its rotor and divergence". In [12], the following definition is given: let V be a finite open spatial domain bounded by a regular surface S, whose positive normal is uniquely defined and continuous at each point of the surface.

The Helmholtz theorem. *If the divergence and rotor of the field $\boldsymbol{F}(\boldsymbol{r})$ are defined at each point $(\boldsymbol{r})$ of the domain V, then everywhere in V the function $\boldsymbol{F}(\boldsymbol{r})$ can be presented as a sum of the conservative field $\boldsymbol{F}_1(\boldsymbol{r})$ and the solenoidal field $\boldsymbol{F}_2(\boldsymbol{r})$:*

$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{F}_1(\boldsymbol{r}) + \boldsymbol{F}_2(\boldsymbol{r}), \tag{1.1}$$

where

$$\nabla \times \boldsymbol{F}_1(\boldsymbol{r}) = 0, \qquad \nabla \cdot \boldsymbol{F}_2(\boldsymbol{r}) = 0 \tag{1.2}$$

(*the Helmholz decomposition theorem*).

The function $\boldsymbol{F}(\boldsymbol{r})$ is uniquely defined under an additional condition of setting a normal component $\boldsymbol{F}(\boldsymbol{r}) \cdot \dfrac{d\boldsymbol{S}}{|d\boldsymbol{S}|}$ of the function $\boldsymbol{F}(\boldsymbol{r})$ at each point of the surface S (theorem of uniqueness).

The effective detection of the function $\boldsymbol{F}(\boldsymbol{r})$ from these data reduces to solving partial differential equations under certain boundary conditions.

When solving the problem of reconstructing the MGF on the Earth's surface, the initial conditions of the Helmholtz theorem are essentially reduced, because an intensive potential magnetic field inside the Earth, for which $\nabla \times \boldsymbol{H} = 0$, is not observed due to the presence of a non-potential toroidal part. Moreover, when reconstructing a solenoidal magnetic field, observed on the Earth's surface, it is necessary to take into account its toroidal part. In other words, in this case it is assumed that on the Earth's surface only a soledoidal magnetic field is observed, whose tangential to the Earth's surface components contain a toroidal part. The normal component of the field exists, is continuous and assigned at each point of the Earth's surface. In this case, it is possible to reformulate the Helmholtz theorem for this problem and to seek for a proof of the new theorem without solving the respective differential equations and without new boundary conditions in addition to proving the existence of the normal component $H_N(\boldsymbol{r})$ at each point of the Earth's surface.

Thus, according to the condition of solenoidality (without divergence) of the magnetic field

$$\nabla \cdot \boldsymbol{H} = 0, \tag{1.3}$$

the source of the magnetic field is usually a vector potential $\boldsymbol{A}$, given by the expression:

$$\boldsymbol{H} = \nabla \times \boldsymbol{A}. \tag{1.4}$$

Therefore it becomes possible to divide the vector potential into two parts using the following orthogonal decomposition:

$$\boldsymbol{A} = (Q\boldsymbol{r}) + \nabla \times (Q\boldsymbol{r}), \tag{1.5}$$

where $Q(r, \theta, \phi)$ is a scalar function of the class C^∞, and (r, θ, ϕ) are spherical coordinates with the center in the Earth's center. The orthogonality of decomposition (1.5) is evident:

$$\big(0, 0, (Qr)\big) \cdot \big(\nabla_\theta (Q\boldsymbol{r}), \nabla_\phi (Q\boldsymbol{r}), 0\big) \equiv 0. \tag{1.6}$$

However, in this case there arises a mathematical problem of reconstructing a solenoidal vector field $\boldsymbol{H}$ in a sphere using one scalar function. In this case it is required to prove that it is possible to uniquely recover in the sphere not only the vector field having an external normal component, but also to prove the fact that there is no such normal component (a toroidal vector field). Such a generalized theorem should be proved in order that toroidal and poloidal magnetic fields be introduced that are interconnected by the known relation [13]:

$$\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P, \tag{1.7}$$

where $\boldsymbol{H}_T$ is a toroidal magnetic field, $\boldsymbol{H}_P$ is a poloidal magnetic field. The validity of formula (1.7) follows from the definitions:

$$\begin{gathered} \boldsymbol{H}_T = \nabla \times (Q\boldsymbol{r}), \qquad \boldsymbol{H}_P = \nabla \times \nabla \times (Q\boldsymbol{r}), \\ \nabla \times \boldsymbol{H}_T = \nabla \times \nabla \times (Q\boldsymbol{r}) = \boldsymbol{H}_P. \end{gathered} \tag{1.8}$$

The mathematical proof of the problem of reconstructing the vector field $\boldsymbol{H}$ in the sphere using one scalar function is formulated as the following theorem.

Theorem 1. *The solenoidal vector field $\boldsymbol{H}$ in the spherical domain V (with the surface V and radius R in the sphere) is uniquely restored by the formula*

$$\boldsymbol{H} = \boldsymbol{H}_T + \boldsymbol{H}_P = \nabla \times (Q\boldsymbol{r}) + \nabla \times \nabla \times (Q\boldsymbol{r}), \tag{1.9}$$

if the normal component $H_N(\boldsymbol{r})$ on S is known, and the function $Q(r,\theta,\phi) \in C^\infty$, whose mean $\langle Q\rangle = 0$ on S, and $\boldsymbol{H}, \boldsymbol{H}_T, \boldsymbol{H}_P \neq 0$ and $\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P$ everywhere.

Here:

$$\langle Q\rangle = \int_0^{2\pi}\int_0^{\pi} Q \sin\theta \, d\theta \, d\phi = 0.$$

Really, if the vector field $\boldsymbol{H}$ corresponds to (1.9), then to prove the uniqueness of decomposition (1.9) it is needed to express the function Q via the original normal component of the vector field $\boldsymbol{H}_P$ or $\nabla \times \boldsymbol{H}_T$. For example,

$$\begin{aligned} (\boldsymbol{r} \cdot \boldsymbol{H}_P) &= \boldsymbol{r} \cdot \nabla \times \nabla \times (Q\boldsymbol{r}) = \boldsymbol{r} \cdot \left\{\nabla\nabla \cdot (Q\boldsymbol{r}) - \nabla^2(Q\boldsymbol{r})\right\} \\ &= \boldsymbol{r} \cdot \left\{\nabla[\boldsymbol{r} \cdot \nabla Q + 3Q] - 2\nabla Q - \boldsymbol{r}\nabla^2 Q\right\} \\ &= -r^2\nabla^2 Q + r \cdot \nabla(r \cdot \nabla Q) + r \cdot \nabla Q \\ &= -r^2\nabla^2 Q + \frac{\partial}{\partial r}\left(r^2\frac{\partial Q}{\partial r}\right) = -DQ, \end{aligned} \tag{1.10}$$

$$(\boldsymbol{r} \cdot \nabla \times \boldsymbol{H}_T) = \boldsymbol{r} \cdot \nabla \times \nabla \times (Q\boldsymbol{r}) = -DQ.$$

Here D is a direct operator (the Beltrami operator) to be defined from (1.10), that is,

$$D = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}, \tag{1.11}$$

and is a part of the Laplace operator without derivatives. From (1.10) it follows that

$$Q = -D^{-1}(\boldsymbol{r} \cdot \boldsymbol{H}_P) = -D^{-1}(\boldsymbol{r} \cdot \nabla \times \boldsymbol{H}_T), \tag{1.12}$$

where D^{-1} is an operator inverse to the operator D subject to determination.

Taking formulas (1.10) into account, it is possible to show that if in the formula for $\boldsymbol{A}(\boldsymbol{r})$ two arbitrary scalar functions of the form $\boldsymbol{A}(\boldsymbol{r}) = (P\boldsymbol{r}) + \nabla \times (Q\boldsymbol{r})$ are used, then because of the fact that the vector field $\boldsymbol{H}_T$ does not contain a normal component to the surface of the sphere S and cannot be uniquely defined according to the above-mentioned Helmholtz theorem, it is required to make use of the condition of Theorem 1 ($\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P$). In this case the direct operators $-DQ = (\boldsymbol{r} \cdot \boldsymbol{H}_P)$ and $-DP = (\boldsymbol{r} \cdot \nabla \times \boldsymbol{H}_T)$ will lead to the following coinciding inverse operators: $-D^{-1}(\boldsymbol{r} \cdot \boldsymbol{H}_P)$ and $-D^{-1}(\boldsymbol{r} \cdot \nabla \times \boldsymbol{H}_T) = -D^{-1}(\boldsymbol{r} \cdot \boldsymbol{H}_P)$. This means that $P = Q$ and in the expression for $\boldsymbol{A}(\boldsymbol{r})$ it is sufficient without loss of generality to use one arbitrary scalar function, for example, the function Q.

Thus, in order to find Q, it is necessary to define the direct D and the inverse D^{-1} operators. The inverse operator D^{-1} is defined as follows. Let $\psi(r, \theta, \phi)$ and $f(r, \theta, \phi)$ be arbitrary scalar functions related by

$$D\psi(r, \theta, \phi) = f(r, \theta, \phi). \tag{1.13}$$

In this case the functions $\psi(r, \theta, \phi)$ and $f(r, \theta, \phi)$ belong to C^∞ with mean on the surface S equal to zero: $\langle\psi\rangle = 0$, $\langle f\rangle = 0$.

Based on [12, p. 675], let us denote

$$S_n(\theta, \phi) = \sum_{m=0}^{n} A_n^m P_n^m(\cos\theta) e^{im\phi}, \tag{1.14}$$

where $P_n^m(\cos\theta)$ are spherical functions, A_n^m are complex constants, $S_n(\theta, \phi) \in C^\infty$.

Now let us represent auxiliary functions ψ and f by their standard decompositions in spherical functions [12]:

$$\begin{gathered} \psi = \sum_{n=1}^{\infty} \psi_n(r) S_n(\theta, \phi) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \psi_n(r) A_n^m P_n^m(\cos\theta) e^{im\phi}, \\ f = \sum_{n=1}^{\infty} f_n(r) \bar{S}_n^m(\theta, \phi). \end{gathered} \tag{1.15}$$

Summation over n begins with unit, a free term is absent due to the imposed condition of zero means for the functions Q, ψ, f on the sphere S. In this case the functions Q, ψ, f for $r \le R$ are proportional to r^n, and for $r \ge R$, respectively, to $\frac{1}{r^{n+1}}$.

We apply the direct operator D to the function $S_n(\theta,\phi)$ using (1.10):

$$DS_n(\theta,\phi) = -n(n+1)S_n(\theta,\phi). \tag{1.16}$$

Really, with allowance for (1.10) and conditions for the functions $\psi_n(r)$ and $f_n(r)$ (1.15), we can write down

$$-r^2\Big(\frac{\partial^2}{\partial r^2}r^n S_n(\theta,\phi) + \frac{2}{r}\frac{\partial}{\partial r}r^n S_n(\theta,\phi)\Big) = Dr^n S_n(\theta,\phi),$$

hence $-n(n+1)S_n(\theta,\phi) = DS_n(\theta,\phi)$,

$$-r^2\Big(\frac{\partial^2}{\partial r^2}\frac{1}{r^{n+1}}S_n(\theta,\phi) + \frac{2}{r}\frac{\partial}{\partial r}\frac{1}{r^{n+1}}S_n(\theta,\phi)\Big) = D\frac{1}{r^{n+1}}S_n(\theta,\phi),$$
$$-n(n+1)S_n(\theta,\phi) = DS_n(\theta,\phi).$$

Such a representation seems to be most convenient, as, according to (1.10), the functions are differentiated only with respect to the angular coordinates in the operator D.

Now apply the operator D to (1.12) and using the above-obtained decompositions of the functions ψ and f we obtain

$$D\psi = -\sum_{n=1}^{\infty}\psi_n(r)n(n+1)S_n(\theta,\phi) = \sum_{n=1}^{\infty}f_n(r)\bar{S}_n(\theta,\phi). \tag{1.17}$$

The functions $S_n(\theta,\phi)$ and $\bar{S}_n(\theta,\phi)$ differ only in complex coefficients.

Let us equate the common terms in (1.17) keeping in mind the absolute and uniform convergence of the expansions series of the functions ψ, f to spherical functions [12, 18]. Then divide the right-hand and the left-hand sides by the factor $n(n+1)$. As a result we obtain

$$\psi_n(r)S_n(\theta,\phi) = -f_n(r)\frac{\bar{S}_n(\theta,\phi)}{n(n+1)}. \tag{1.18}$$

Summing up all the harmonics in (1.17), we derive

$$\psi = -D^{-1}f = -\sum_{n=1}^{\infty}f_n(r)\frac{S_n(\theta,\phi)}{n(n+1)}. \tag{1.19}$$

Formula (1.19) determines the inverse operator D^{-1}. The inverse operator in (1.19), in our opinion, is more appropriate for the problem in question than the integral inverse Beltrami operator containing the Green function [18]. Applying it to formulas (1.12), we have:

$$Q = -\sum_{n=1}^{\infty} rH_{Prn}(r)\frac{\tilde{S}_n(\theta,\phi)}{n(n+1)} = -\sum_{n=1}^{\infty} r(\nabla\times\boldsymbol{H}_T)_{rn}(r)\frac{S'_n(\theta,\phi)}{n(n+1)}. \tag{1.20}$$

If $r = R$ is taken into consideration, then we have $H_{Prn}(R) = H_{Nn}(R)$,

$$\begin{gathered} H_N(r) = \sum_{n=1}^{\infty} H_{Nn}(R)S''_n(\theta,\phi), \\ \nabla\times\boldsymbol{H}_T = \nabla\times\nabla\times(Q\boldsymbol{r}) = \boldsymbol{H}_P, \end{gathered} \tag{1.21}$$

hence

$$Q = -\sum_{n=1}^{\infty} RH_{Nn}(R)\frac{\tilde{S}_n(\theta,\phi)}{n(n+1)}, \tag{1.22}$$

where R is a radius of the sphere.

Thus, formula (1.22) defines the function Q on the surface of the sphere. At any point inside the sphere and at any point outside it, this function depends on the coordinate r by the known way (1.15). So, Theorem 1 extends the impact of the Helmholtz theorem by including into the unique definition on the sphere surface not only the poloidal magnetic field but also the toroidal magnetic field related to the poloidal field by (1.7). Moreover, for Theorem 1, as well as for the Gauss theorem [9], it is required to define one scalar function on the surface of the sphere. In the Gauss theorem, the potentiality of the magnetic field in the air $\nabla\times\boldsymbol{H} = 0$, $\boldsymbol{H} = -\nabla V$ is used, therefore the scalar function $V \in C^{\infty}$ reconstructs on the sphere surface only a poloidal magnetic field that is potential in the air.The toroidal magnetic field is not potential everywhere where it is observed according to definition (1.7). Nevertheless, with Theorem 1 it appears possible to reconstruct, also, the toroidal magnetic field on the sphere surface which is the generalization of the Gauss theorem although in this case a scalar function remains alone. This circumstance has important consequences for interpreting electromagnetic fields observed on the Earth [6].

1.2. Boundary conditions for the main geomagnetic field and conditions of existence of its toroidal and poloidal parts

Based on Theorem 1 and physical data for the properties of the Earth's surface which is not a screen for the MGF due to the absence on its surface of intensive surface currents and magnetic masses with a significant magnetic permeability, the boundary conditions for the MGF on the Earth's surface will take the form:

$$(\boldsymbol{H}_P^1 - \boldsymbol{H}_P^2)|_{r=R} = 0, \qquad (\boldsymbol{H}_T^1 - \boldsymbol{H}_T^2)|_{r=R} = 0. \tag{1.23}$$

Here indices 1 and 2 mean the Earth and the air, respectively.

Boundary conditions (1.23) allow the toroidal field (if it is present) to freely penetrate into the Earth's atmosphere and be measured there by magnitometers both over the world network of magnetic observatories and at separate points of the Earth's surface for regional investigations.

Theorem 2. *A toroidal magnetic field presents in the location where a normal component of the poloidal magnetic field H_{PN} exists.*

Really,

$$\oint_L (\boldsymbol{H}_T \cdot \boldsymbol{dl}) = \int_W (\nabla \times \boldsymbol{H}_T \cdot \boldsymbol{ds}) = \int_S H_{PN} ds \Big|_{H_{PN} \neq 0} \neq 0. \tag{1.24}$$

Theorem 2 essentially extends the domain of existence of the toroidal magnetic field as compared to the results obtained in [13, 15], where the proof of the absence of the toroidal magnetic field in the Earth's atmosphere is based on the well-known formula resulting from the standard Maxwell equations:

$$\oint_L (\boldsymbol{H} \cdot \boldsymbol{dl}) = \int_W (\nabla \times \boldsymbol{H} \cdot \boldsymbol{ds}) = \int_S j_n ds|_{j_n=0} = 0. \tag{1.25}$$

In the Earth's atmosphere, the air conductivity is quite insignificant, therefore the density of the current can be considered to be close to zero ($10^{-12} \div 10^{-14}\,\mathrm{A/m^2}$), and the magnetic field in (1.25) — to be potential. However, Theorem 1 and result (1.24) indicate to the fact that a non-potential toroidal magnetic field can exist and be measured along with a poloidal potential field in the Earth's atmosphere. In this case, the refining theorems are formulated as follows.

Theorem 3. *The vortices of a toroidal magnetic field generate a poloidal magnetic field in any medium.*

Actually, according to (1.8)

$$\nabla \times \boldsymbol{H}_T = \nabla \times \nabla \times (Q\boldsymbol{r}) = \boldsymbol{H}_P. \tag{1.26}$$

Theorem 4. *The vortices of a poloidal magnetic field generate a toroidal magnetic field only in a conducting medium.*

Really, according to (1.8)

$$\nabla \times \boldsymbol{H}_P = \nabla \times \nabla \times \nabla \times (Q\boldsymbol{r}) = -\nabla \times (\Delta Q\boldsymbol{r}) = \chi \boldsymbol{H}_T. \tag{1.27}$$

Here it is considered that

$$\Delta Q = -\chi Q = -\frac{\gamma}{\eta} Q, \qquad t = 0. \tag{1.28}$$

In this case γ (m/s) is the diffusion rate in a medium, $\eta = 1/(\sigma\mu)$ is magnetic viscosity with the magnetic permeability μ (gn/m) and the conductivity $(\mathrm{Om} \cdot \mathrm{m})^{-1}$ of a medium σ.

Theorems 3 and 4 uniquely answer the question about the kind of nature in the non-potential part found in [7, 28]. This is a toroidal magnetic field. According to Theorems 3 and 4, the poloidal field vortices generate a toroidal field in a conducting medium, while the vortices of a toroidal magnetic field, which according to (1.8) is not potential everywhere, generate a poloidal magnetic field but not the electric current. That is why a non-potential toroidal field can exist in the atmosphere, while there may be no electric current in the Earth's atmosphere. This fact is substantiated by Theorems 1 and 2.

1.3. On the main geomagnetic field generation

According to the well-known Kauling "anti-dynamo theorem" [11], the cylindrical symmetry of the Earth's magnetic field observed in the first approximation in the MGF, inhibits its dynamo-generation due to occurrence of the reflexive (reflecting) symmetry. The results obtained in [10, 11, 22] concerning the impossibility of dynamo-excitation of magnetic fields in a one-disk generator $\boldsymbol{H}_T$ (Bullard) as well as in turbulent planar fluxes (Zeldovich) and in the fields and fluxes with cylindrical symmetry (Kauling), were called "anti-dynamo theorems" [14]. In order to overcome the effect of the recurrent symmetry, Braginsky [8] has introduced small supplements into the induction equation with the Larmor dynamo term [24]. Then with a strong toroidal field in the Earth's interior there could occur a poloidal magnetic field by way of "drawing off" one from another.

For revising these well-known results related to the MGF generation, it is necessary to present two important theorems, essentially refining the possibilities of generation of $\boldsymbol{H}_P$ and $\boldsymbol{H}_T$ in the Earth's interior for $\boldsymbol{H}$, $\boldsymbol{H}_T$, $\boldsymbol{H}_P \neq 0$.

Theorem 5. *A non-vortex vector field* $\nabla \times \boldsymbol{H} = 0$ *provided that* $\nabla \cdot \boldsymbol{H} = \rho$ *when* $\rho = \mathrm{const}$ *does not admit the generation of the vector field* $\boldsymbol{H}_T = \nabla \times (Q\boldsymbol{r})$.

Really, the helicity $\boldsymbol{H} \cdot \nabla \times \boldsymbol{H}$ of the non-vortex vector field $\nabla \times \boldsymbol{H} = 0$ equals zero, while by definition, the helicity of the vector field $\boldsymbol{H}_T$ is not equal to zero: $\boldsymbol{H}_T \cdot \nabla \times \boldsymbol{H}_T = \boldsymbol{H}_T \cdot \boldsymbol{H}_P \neq 0$. The absence of coincidence of helicities due to the reflexive symmetry effect excludes the mutual generation of the vector fields $\boldsymbol{H}_T$ and $\boldsymbol{H}_P$ [13].

Theorem 6. *In the solenoidal vector field* $\nabla \cdot \boldsymbol{H} = 0$, $\nabla \times \boldsymbol{H} = \boldsymbol{P}$ *(*$\boldsymbol{P}$ *is a vector field) the vector field* $\boldsymbol{H}_T = \nabla \times (Q\boldsymbol{r})$ *can be generated.*

The proof is apparent from the fact that the helicities of the vector fields $\boldsymbol{H}$ and $\boldsymbol{H}_T$ are not equal to zero: $\boldsymbol{H}\cdot\nabla\times\boldsymbol{H}\neq 0$ due to $\nabla\times\boldsymbol{H}=\boldsymbol{P}$, while $\boldsymbol{H}_T\cdot\nabla\times\boldsymbol{H}_T\neq 0$ due to $\nabla\times\boldsymbol{H}_T=\boldsymbol{H}_P$. The presence of helicities in both vector fields is favorable to their mutual generation at the expense of the relations:

$$\nabla\times\boldsymbol{H}_T=\boldsymbol{H}_P,$$
$$\nabla\times\boldsymbol{H}_P=\nabla\times\nabla\times\nabla\times(Q\boldsymbol{r})=-\nabla\times(\Delta Q\boldsymbol{r})=\chi\nabla\times(Q\boldsymbol{r})=\chi\boldsymbol{H}_T,$$

where $\Delta Q=-\chi Q$, $\chi=\text{const}$ [11, 13].

The physical interpretation of Theorem 5 is in that with magnetic masses as sources of the magnetic field, a toroidal magnetic field cannot occur. On the other hand, if along with a poloidal field a toroidal magnetic field is measured on the Earth's surface, then the source of such a magnetic field is electric current that according to Theorem 6 possess the possibility of a simultaneous generation of $\boldsymbol{H}_P$ and $\boldsymbol{H}_T$. This fact will be considered in greater detail in the next section.

1.3.1. On a toroidal electric current in the Earth's interior

The next theoretical and practical problem that arises under the assumption that a source of the MGF is toroidal electric currents in the Earth's interior is the elucidation of whether the Earth's toroidal currents are able to cause tangential components of a toroidal magnetic field and whether in this case toroidal fluxes and magnetic fields are stable. The answer to the first one of these principal questions is given by the following theorem.

Theorem 7. *The source of a toroidal magnetic field on the Earth's surface is toroidal components of the electric current, flowing in the spherical layers or on the spherical surfaces inside and outside of the Earth.*

The proof of this theorem goes back to the two circumstances. The first is in applying the operator of the total current $\boldsymbol{j}^{\Pi}=(\nabla\nabla\cdot-\nabla\times\nabla\times)\boldsymbol{A}$ in a source, and the second — in mapping this operator onto the axis of the spherical coordinate system (r,θ,ϕ) fixed in the Earth's center. Such a mapping brings about two toroidal components of the electric current of the form

$$\begin{aligned}-j_\theta^{\Pi}={}&\frac{\partial^2 A_\theta}{\partial r^2}+\frac{2\partial A_\theta}{r\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 A_\theta}{\partial\phi^2}+\frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial\theta^2}-\frac{\cos\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\theta}-\\&\frac{A_\theta}{r^2\sin^2\theta}-2\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}+\frac{2\partial A_r}{r^2\partial\theta},\end{aligned}\tag{1.29}$$
$$-j_\phi^{\Pi}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial A_\phi}{\partial\phi}+\frac{1}{r^2\sin\theta}\frac{\partial^2 A_\phi}{\partial\phi^2}+\frac{1}{r}\frac{\partial^2 rA_\phi}{\partial r^2}+\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}-$$

$$\frac{\cos\theta}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\phi-\frac{1}{r}\frac{\partial^2 A_\phi}{\partial\theta\partial\phi}+\frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi}.$$

If we turn to formula (1.5), it appears possible to express spherical components of the vector potential via the scalar function Q in the following manner:

$$A_\theta=\frac{1}{\sin\theta}\frac{\partial Q}{\partial\phi},\qquad A_\phi=-\frac{\partial Q}{\partial\theta},\qquad A_r=rQ. \tag{1.30}$$

By definition (1.8) the toroidal field components are of the form

$$H_{T\theta}=\frac{1}{\sin\theta}\frac{\partial Q}{\partial\phi},\qquad H_{T\phi}=-\frac{\partial Q}{\partial\theta},\qquad H_{Tr}\equiv 0. \tag{1.31}$$

An analysis of formula (1.29) shows that projections of the equation for the total current on the axis of the spherical coordinate system fixed in the Earth's center have the following terms:

$$\frac{2\partial A_r}{r^2\sin\theta\partial\phi}=\frac{2}{r\sin\theta}\frac{\partial Q}{\partial\phi}=\frac{2}{r}H_{T\theta},\quad \frac{2\partial A_r}{r^2\partial\theta}=\frac{2}{r}\frac{\partial Q}{\partial\theta}=-\frac{2}{r}H_{T\phi}. \tag{1.32}$$

With allowance for formulas (1.31) and equation (1.32) they are just the doubled components of the toroidal magnetic field referred to a current radius, which gives them dimensionality of the current density.

Thus, the toroidal electric current with its spherical components always generates a toroidal magnetic field that occurs on the Earth's surface according to boundary conditions (1.23) and is measured on the world network of magnetic observatories by magnitometers, immediately fixing the observed intensity of the MGF. In [6], the presence both of a toroidal and a poloidal magnetic fields in the MGF is proved for the period of 1965. Hence it follows that the MGF is excited by the toroidal electric current. The poloidal and toroidal magnetic fields in the MGF are generated by this current, are present in the atmosphere and are contained in its measured values including the data received by the world network of magnetic observatories and other magnetometric measurements except for magnetometric prospecting, in which magnetic masses are a source of a magnetic field.

Formulas (1.32) also contain the proof of dimension of the toroidal magnetic field $\boldsymbol{H}_T$, which according to (1.32) is given in A/m, Gauss, or (?). The left-hand side of (1.32) automatically arises when projecting the Laplace operator onto the axis of the spherical coordinate system. That is why it is contained in the operator $(\nabla\nabla\cdot -\nabla\times\nabla\times)$ independent of definition of the toroidal magnetic field from formulas (1.8). This is just the proof of the dimension $\boldsymbol{H}_T$ in A/m, as multiplication of the magnetic field by $\frac{2}{r}$ aligns dimensionalities of the right- and left-hand sides of (1.32) in its dimension in A/m^2 and are among the density terms of the total electric current.

The stability of the toroidal current and its total magnetic field is solved by the following theorem.

Theorem 8. *Toroidal electric currents (fluxes) are stable as related to external and internal actions on their magnetic field.*

The proof of the theorem goes back to studying a pair of equations: the Navier–Stokes and the induction equations, in which it appeared possible to take into account the new properties of a medium, for example, magnetic and kinematic viscosities, density as well as forces whose influence on velocity components is essential, such as the Lorentz force, the Coriolis force, the Archimedes force, the pressure force, etc. If one tries to describe complicated processes in the Earth's liquid core, which result in the MGF generation, it becomes possible to write down a more general system of equations for a totality of forces causing the fluxes of a substance inside the liquid core to move or affecting this movement:

$$\begin{cases} \rho\Big[\dfrac{\partial}{\partial t}\boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} + 2[\boldsymbol{\omega}\times\boldsymbol{U}]\Big] = \nu\rho\Delta\boldsymbol{U} + \nabla P' + [\boldsymbol{j}\times\boldsymbol{B}] + \boldsymbol{f}, \\ \dfrac{\partial}{\partial t}\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{U} - (\boldsymbol{U}\cdot\nabla)\boldsymbol{B} - \eta\Delta\boldsymbol{B}. \end{cases} \tag{1.33}$$

Equations (1.33) are written for 1 m^2 of a medium. Here:

ρ — density of a medium,

μ — magnetic permeability,

ν, η — kinematic and magnetic viscosities, respectively,

$P' = P - \rho g h$ — pressure, where $\rho g h$ is a hydrostatic pressure,

$\boldsymbol{U}$ — a vector of the velocity flux,

$\boldsymbol{B}$ — a vector of magnetic induction,

$\boldsymbol{j}$ — a vector of the current density,

$\boldsymbol{\omega}$ — a vector of the angular rotation velocity,

$\boldsymbol{f}$ — a vector of other potential forces,

$[\boldsymbol{j}\times\boldsymbol{B}] = \frac{1}{\mu}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} - \nabla\Big(\dfrac{B^2}{2\mu}\Big)$ — the Lorentz force,

$\nu\rho\Delta\boldsymbol{U}$ — viscosity force,

$\rho\dfrac{\partial\boldsymbol{U}}{\partial t}$ — centrifugal force,

$\rho(U\cdot\nabla)\boldsymbol{U}$ — a force of viscous impulse transfer,

$2\rho[\boldsymbol{\omega}\times\boldsymbol{U}]$ — the Coriolis force.

System (1.33) represents a balanced system of forces, affecting fluxes and fields in the Earth's liquid core. The first is the Navier–Stokes equation. The second one is the equation of induction for incompressible fluid in the magnetic field. The joint solution to system (1.33) for unknown functions of the velocity $\boldsymbol{U}$ and the induction $\boldsymbol{B}$ with a priori known all other parameters of the system could allow answering the principal question of the generation theory: what fluxed support the current strength of the Earth's MGF. The fluxes for the equation of induction that are calculated or a priori assigned in different publications essentially vary, thus not allowing the creation of their complete picture. The theory of kinematic dynamo-excitation of the MGF contains, in addition, a number of contradicting estimations of the parameters from system (1.33), whose refinement presents certain difficulties. As was noted above, the main unsolved issues when constructing flows and fields in the dynamo-theory still remain: unknown values of strength of the toroidal magnetic field in the vicinity of a source, unknown values of viscosity of a medium in a source, an unstudied possibility of a turbulent current in the vicinity of the source, its size.

Answers to some of these questions can be found in the generation theory if, of course, system (1.33) would be solved. Constructing the general solution to system (1.33) or deriving from it partial or similar or asymptotic solutions could be helpful in the creation of the verifiable theory of the MGF generation. The presence in system (1.33) of completely unknown terms such as forces $\boldsymbol{f}$, that is, all the other a priori unknown forces not contained in system (1.33), can be compensated by their estimations when studying the closeness of the generation process. It is also possible to make use of the fact that the strength of the MGF is currently weakly varying from year to year.

In order to investigate the solvability of system (1.33) for the vector of the magnetic induction $\boldsymbol{B}$ from specified velocities of the flow $\boldsymbol{U}$ and vice versa, let us employ equivalent estimations of differential operators by introducing the notations:

$$\begin{aligned}
&\Delta\boldsymbol{B} \sim \frac{\boldsymbol{B}}{L^2}, \qquad (\boldsymbol{B}\cdot\nabla)\boldsymbol{U} \sim \Big(\boldsymbol{B}\cdot\frac{1}{L}\Big)\boldsymbol{U},\\
&\frac{\partial}{\partial t}\boldsymbol{B} \sim 2\pi\frac{\boldsymbol{B}}{T}, \qquad (\boldsymbol{U}\cdot\nabla)\boldsymbol{B} \sim \Big(\boldsymbol{U}\cdot\frac{1}{L}\Big)\boldsymbol{B},\\
&\boldsymbol{F} = \rho\Big[\frac{2\pi}{T}\boldsymbol{U} + \Big(\boldsymbol{U}\frac{1}{L}\Big)\boldsymbol{U} + 2[\boldsymbol{\omega}\times\boldsymbol{U}]\Big] - \frac{\nu\rho}{L^2}\boldsymbol{U} - \frac{\boldsymbol{e}P'}{L} - \boldsymbol{f},
\end{aligned} \tag{1.34}$$

where L is a characteristic size of the domain with a source, T is a period of inversion (changes in polarity) of the MGF, $\boldsymbol{e}$ is a unit vector along the pressure gradient.

Then system (1.33) can be rewritten in an equivalent form

$$\begin{cases} \dfrac{1}{\mu}\Big(\boldsymbol{B}\dfrac{1}{L}\Big)\boldsymbol{B} - \dfrac{\boldsymbol{n}}{2L\mu}\boldsymbol{B}^2 = \boldsymbol{F}, \\ \Big(\dfrac{2\pi}{T} - \dfrac{\eta}{L^2}\Big)\boldsymbol{B} = \Big(\boldsymbol{B}\dfrac{1}{L}\Big)\boldsymbol{U} - \Big(\boldsymbol{U}\dfrac{1}{L}\Big)\boldsymbol{B}, \end{cases} \tag{1.35}$$

where $\boldsymbol{n}$ is a unit vector along the magnetic field.

If we rewrite system (1.35) in the component-by-component manner in a spherical coordinate system, we will obtain the sought for system of equivalent equations that are linear with respect to the square of the induction vector B^2:

$$\begin{gathered} \Big[\frac{2\pi L^2-\eta T}{\mu L^2 T U_r}\Big(1+\frac{LT(U_\theta+U_\phi)}{2\pi L^2-\eta T}\Big)+\frac{1}{2L\mu}\Big]B_r^2-\frac{1}{2\mu L}B_\theta^2-\frac{1}{2\mu L}B_\phi^2=F_r, \\ -\frac{1}{2\mu L}B_r^2+\Big[\frac{2\pi L^2-\eta T}{\mu L^2 T U_\theta}\Big(1+\frac{LT(U_r+U_\phi)}{2\pi L^2-\eta T}\Big)+\frac{1}{2\mu L}\Big]B_\theta^2-\frac{1}{2\mu L}B_\phi^2=F_\theta, \\ -\frac{1}{2\mu L}B_r^2-\frac{1}{2\mu L}B_\theta^2+\Big[\frac{2\pi L^2-\eta T}{\mu L^2 T U_\phi}\Big(1+\frac{LT(U_r+U_\theta)}{2\pi L^2-\eta T}\Big)+\frac{1}{2\mu L}\Big]B_\phi^2=F_\phi. \end{gathered} \tag{1.36}$$

System (1.36) can be resolved algebraically with respect to the squares of the magnetic induction components B_r^2, B_θ^2, B_ϕ^2.

If we introduce the notations

$$\begin{aligned} \tilde{U}_r &= \frac{4\pi L^2 - 2\eta T + L(TU_r + 2U_\theta + 2U_\phi)}{2\mu L^2 T U_r}, \\ \tilde{U}_\theta &= \frac{4\pi L^2 - 2\eta T + L(2U_r + 2U_\phi + TU_\theta)}{2\mu L^2 T U_\theta}, \\ \tilde{U}_\phi &= \frac{4\pi L^2 - 2\eta T + L(TU_\phi + 2U_r + 2U_\theta)}{2\mu L^2 T U_\phi}, \end{aligned} \tag{1.37}$$

then the determinant of system (1.36) can be written down as follows:

$$\Delta = \begin{vmatrix} \tilde{U}_r & -\dfrac{1}{2\mu L} & -\dfrac{1}{2\mu L} \\ -\dfrac{1}{2\mu L} & \tilde{U}_\theta & -\dfrac{1}{2\mu L} \\ -\dfrac{1}{2\mu L} & -\dfrac{1}{2\mu L} & \tilde{U}_\phi \end{vmatrix} = \tilde{U}_r\tilde{U}_\theta\tilde{U}_\phi - \frac{\tilde{U}_r+\tilde{U}_\theta+\tilde{U}_\phi}{4\mu^2L^2} - \frac{1}{4\mu^3L^3}. \tag{1.38}$$

Determinant (1.38) is not equal to zero even if all the components of the function $\tilde{\boldsymbol{U}}$ equal zero. This means that a solution to system (1.35) exists, is unique and can be obtained for the squares of magnetic induction with known right-hand sides including the velocities U and some other unknown values.

Thus, with flows of a conducting fluid in the Earth's core, there always occurs a magnetic field whose values are assigned just by these flows as well as by some other parameters of a medium and forces in the liquid core.

In this case, the internal symmetry of the induction equations with respect to coordinate systems is of no importance. Therefore when solving the complete system of equations (system (1.33)), i.e., the Navier–Stokes and the induction equations simultaneously, the symmetry of flows does not affect the generation of the field components. This fact makes possible to formulate the generation theory not basing on the self-generation of a poloidal field by a toroidal field. The latter, as in the case of a poloidal field, arises in the liquid core at the expense of properties of a medium, flows in it and, basically, due to spherical features of a source of the magnetic field.

Spherical features of a source underlies the occurrence of a toroidal field both in the liquid core and outside of it, for example, in the Earth's atmosphere. Spherical features of a source facilitate the initiation of a toroidal part of the MGF from components of the toroidal current density, which are tangential to the spherical surface of the liquid core. Radial flows are not needed for the generation of a toroidal part of a magnetic field. Thus, it is impossible to prove its solvability in terms of the dynamo kinematic theory. It is necessary to change the induction equation by introducing small supplements to violate the inner symmetry of the equation. In this case, however, there will be an ambiguity of the results due to violation of the induction law. To overcome this difficulty, it is necessary to add to the induction equation an essentially more complicated Navier–Stokes equation for forces affecting these flows and, in essence, initiating these flows due to the Earth's rotation. The system of the Navier–Stokes and the induction equations is solvable with respect to components of the magnetic field B^2. Existence and uniqueness of this solution that result from non-equality to zero of the basic determinator of the equivalent system of linear algebraic equations to which the Navier–Stokes and the induction equations are simultaneously reduced.

Moreover, let us prove the solvability of the inverse problem, which is in that it is possible to define velocity components of the flows $\boldsymbol{U}$ or to assess them from the assigned components of the observed magnetic field $\boldsymbol{H}$ with known or assessed other parameters.

In order to prove the solvability of system (1.33) for $\boldsymbol{U}$ with known components of the vector $\boldsymbol{B}$ we proceed in a similar manner. Let us make use of estimations (1.34) and the notations

$$\begin{gathered}\boldsymbol{f}^b = \Big(\frac{\eta}{L^2} - \frac{2\pi}{T}\Big)\boldsymbol{B},\\ \boldsymbol{F}^b = \frac{1}{\mu L}(B_r + B_\theta + B_\phi)\boldsymbol{B} - \frac{\boldsymbol{n}}{2\mu L}B^2 + \frac{\boldsymbol{e}P'}{L} + \boldsymbol{f}.\end{gathered} \tag{1.39}$$

With allowance for (1.39), system (1.33) can be reduced to the following similar form in the spherical coordinate system:

$$\begin{aligned}
&\frac{2\pi\rho}{T}U_r + (U_r + U_\theta + U_\phi)\frac{\rho}{L}U_r - \omega_\phi U_\theta 2\rho - \frac{\nu\rho}{L^2}U_r = F_r^b,\\
&(B_r + B_\theta + B_\phi)\frac{1}{L}U_r - (U_r + U_\theta + U_\phi)\frac{1}{L}B_r = f_r^b,\\
&\frac{2\pi\rho}{T}U_\theta + (U_r + U_\theta + U_\phi)\frac{\rho}{L}U_\theta + \omega_\phi U_r 2\rho - \frac{\nu\rho}{L^2}U_\theta = F_\theta^b,\\
&(B_r + B_\theta + B_\phi)\frac{1}{L}U_\theta - (U_r + U_\theta + U_\phi)\frac{1}{L}B_\theta = f_\theta^b,\\
&\frac{2\pi\rho}{T}U_\phi + (U_r + U_\theta + U_\phi)\frac{\rho}{L}U_\phi - \frac{\nu\rho}{L^2}U_\phi = F_\phi^b,\\
&(B_r + B_\theta + B_\phi)\frac{1}{L}U_\phi - (U_r + U_\theta + U_\phi)\frac{1}{L}B_\phi = f_\phi^b.
\end{aligned}\tag{1.40}$$

In (1.40), it is convenient to solve the second relations as related to the sums of velocity components:

$$\begin{aligned}
(U_r + U_\theta + U_\phi) &= \frac{B_r + B_\theta + B_\phi}{B_r} - \frac{Lf_r^b}{B_r},\\
(U_r + U_\theta + U_\phi) &= \frac{B_r + B_\theta + B_\phi}{B_\theta} - \frac{Lf_\theta^b}{B_\theta},\\
(U_r + U_\theta + U_\phi) &= \frac{B_r + B_\theta + B_\phi}{B_\phi} - \frac{Lf_\phi^b}{B_\phi}.
\end{aligned}\tag{1.41}$$

Substituting (1.41) into the corresponding equations of (1.40), we obtain a system of equations for the velocity components:

$$\begin{aligned}
&\frac{\rho(B_r + B_\theta + B_\phi)}{LB_r}U_r^2 + \Big(\frac{2\pi\rho}{T} - \frac{\nu\rho}{L^2} - \frac{\rho f_r^b}{B_r}\Big)U_r - 2\rho\omega_\phi U_\theta = F_r^b,\\
&\frac{\rho(B_r + B_\theta + B_\phi)}{LB_\theta}U_\theta^2 + \Big(\frac{2\pi\rho}{T} - \frac{\nu\rho}{L^2} - \frac{\rho f_\theta^b}{B_\theta}\Big)U_\theta - 2\rho\omega_\phi U_r = F_\theta^b,\\
&\frac{\rho(B_r + B_\theta + B_\phi)}{LB_\phi}U_\phi^2 + \Big(\frac{2\pi\rho}{T} - \frac{\nu\rho}{L^2} - \frac{\rho f_\phi^b}{B_\phi}\Big)U_\phi = F_\phi^b,
\end{aligned}\tag{1.42}$$

System of equations (1.42) is solvable as related to the velocity components U_r, U_θ, U_ϕ, which appear in it. The third equation (1.42) is quadratic for components of the velocity U_ϕ, therefore the solution can be immediately written down:

$$\begin{aligned}
U_\phi^{(1,2)} = {}&\frac{LB_\phi}{2\rho(B_r + B_\theta + B_\phi)}\Bigg\{\Bigg\{-\Big(\frac{2\pi\rho}{T} - \frac{\nu\rho}{L^2} - \frac{\rho f_\phi^b}{B_\phi}\Big)\Bigg\} \pm\\
&\sqrt{\Bigg[\Big(\frac{2\pi\rho}{T} - \frac{\nu\rho}{L^2} - \frac{\rho f_\phi^b}{B_\phi}\Big)^2 + \frac{4\rho(B_r + B_\theta + B_\phi)}{LB_\phi}F_\phi^b\Bigg]}\Bigg\}.
\end{aligned}\tag{1.43}$$

According to equation (1.43) there may be values of the velocity U_ϕ for the positive or zero value of the radicand formula. Only in the case when the discriminant in the radicant formula is less than zero, there may not be real velocity values. Each one of the components U_r, U_θ has two solutions. It is trivial to obtain them by equating successively to zero the needed parts of components entering the first two equations (1.42):

$$\begin{aligned} U_r^{(1)} &= \frac{F_\theta^b}{2\rho\omega_\phi}, \quad U_r^{(2)} = -\frac{LB_r}{\rho(B_r + B_\theta + B_\phi)}\Big(\frac{2\pi\rho}{T} - \frac{\nu\rho}{L^2} - \frac{\rho f_r^b}{B_r}\Big), \\ U_\theta^{(1)} &= \frac{F_r^b}{2\rho\omega_\phi}, \quad U_\theta^{(2)} = -\frac{LB_\theta}{\rho(B_r + B_\theta + B_\phi)}\Big(\frac{2\pi\rho}{T} - \frac{\nu\rho}{L^2} - \frac{\rho f_\theta^b}{B_\theta}\Big). \end{aligned} \tag{1.44}$$

The solvability of system (1.42) with respect to the velocity components U_r, U_θ, U_ϕ allows us to affirm that it is possible to find velocity components of currents in the liquid core from the current strength of the MGF having preliminarily estimated the physical parameters of the liquid core and certain forces entering the right-hand sides of formulas (1.43) and (1.44). Moreover, an exact solution to system (1.33) (if, of course, it is possible to obtain it) will allow the direct simulation of the configuration of flows resulting in the currently observed MGF. Solutions to (1.43) and (1.44) make possible to numerically assess velocities of a steady-state liquid flow in the core with two formulas at once, and then to choose the most suitable solution corresponding to the current status of the Earth's liquid core and its properties. In this case, essential "roughening" of differential operators with the aid of estimations (1.34) allows one to obtain only the assessments of velocity components but not their accurate values. Nevertheless, in such a sophisticated situation when a direct experiment that could shed light upon some physical parameters of the Earth's liquid core and processes in it is impossible, these velocity assessments are of importance.

To simplify the estimations of velocity components, let us take into consideration the fact that among all the forces affecting flows, the pressure forces have a dominant role. On this basis, it appears possible to essentially simplify the assessments in the following way:

$$\begin{gathered} U_\phi^{(1,2)} \approx \pm\Big[\Big(\frac{P}{\rho} - gh\Big)\frac{H_\phi}{|\boldsymbol{H}|}\Big]^{1/2} \mp 2L\omega_\phi, \\ U_r^{(1)} \approx -\frac{P'}{2L\rho\omega_\phi} + 2L\omega_\phi, \quad U_r^{(2)} \approx \Big(\frac{\nu+\eta}{L} - \frac{4\pi L}{T}\Big)\frac{H_r}{|\boldsymbol{H}|}, \\ U_\theta^{(1)} \approx +\frac{P'}{2L\rho\omega_\phi} - 2L\omega_\phi, \quad U_\theta^{(2)} \approx \Big(\frac{\nu+\eta}{L} - \frac{4\pi L}{T}\Big)\frac{H_\theta}{|\boldsymbol{H}|}. \end{gathered} \tag{1.45}$$

Knowing values of the observed field $\boldsymbol{H}$ on the Earth's surface, having estimated from planetary considerations the pressure, density and some other parameters in the upper zone F of the liquid core, using (1.45) we are able to estimate velocity components of steady-state flows in this part of the liquid core that result in the generation of the observed MGF. To this end it is sufficient to select from six solutions to (1.45) the three most adequate for the modern understanding of the physics of processes in the core. The major is the first equation from (1.45) because it detects the stability of the flow with respect to the action (both external and internal) onto a magnetic field or on the velocity of the flow. Their inversely proportional interaction under strong external or an internal influence on the magnetic field makes the velocity of the flow change in the inverse order, then it returns to the original value at the account of the constant speed of the Earth's rotation. A similar situation takes place if the flow velocity is under action. In this case a magnetic field aligns the flow velocity.

The effect of inverse proportionality between the flow velocity and magnetic intensity excludes an infinite increase of the strength of current in a source (inductive "acceleration" of current). Due to increasing the magnetic intensity with "acceleration" of current, the velocity of the flow falls thus preventing an infinite increase in the MGF strength. The effect in question was not taken into account in the induction hypothesis of the MGF excitation in [19, 23], that is why this, in essence, true hypothesis has been the subject of much controversy.

In the discussed interaction between the flow velocity and the magnetic intensity, the stability of the system "a flow — a magnetic field" is realized, thus proving Theorem 8.

If we substitute into the first formula from (1.45) the values of known parameters for the liquid core and for the zone F of the core equal to [6]:

$$
\begin{aligned}
&P = 2.445 \cdot 10^{12}\ \frac{\mathrm{g}}{\mathrm{cm}\cdot\mathrm{sec}^2}, \quad g = 226\ \frac{\mathrm{cm}}{\mathrm{sec}^2}, \quad \rho = 11.4\ \frac{\mathrm{g}}{\mathrm{cm}^3},\\
&h = 4.9 \cdot 10^{8}\ \mathrm{cm}, \quad L = 2.9 \cdot 10^{8}\ \mathrm{cm}, \quad \nu = 10^{3}\ \frac{\mathrm{m}^2}{\mathrm{sec}},\\
&\omega_\phi = 7.3 \cdot 10^{-5}\ \frac{1}{\mathrm{sec}}, \quad T = 1.8 \cdot 10^{15}\ \mathrm{sec}, \quad \eta = 2.6\ \frac{\mathrm{m}^2}{\mathrm{sec}},\\
&\frac{H_\phi}{|\boldsymbol{H}|} = 0.0216, \quad \frac{H_r}{|\boldsymbol{H}|} = 0.8, \quad \frac{H_\theta}{|\boldsymbol{H}|} = 0.5,
\end{aligned}
$$

then the assessment for components of the absolute linear velocity of the flow takes the value

$$
U_\phi \approx 50 \frac{\mathrm{m}}{\mathrm{sec}}, \qquad U_r, U_\phi \approx 10^{-4} \frac{\mathrm{m}}{\mathrm{sec}}. \tag{1.46}
$$

The above values clearly emphasize the movement of the flow tracing the rotation. All other components in (1.45) are "suppressed" by the rotation. The angular rotation velocity of the flow is calculated by the known formula and looks like

$$\omega_{U_\phi} = \frac{U_\phi}{R_F} = 3.4 \cdot 10^{-5}\ \frac{1}{\text{sec}}. \tag{1.47}$$

This angular velocity of the flow (1.47) is somewhat less than that of the Earth's rotation: $\omega_\phi = 7.3 \cdot 10^{-5}\frac{1}{\text{sec}}$. According to the relativity principle, this important circumstance allows the electric field to exist in the flow. An observer, inflexibly related with the uniformly rotating Earth, does not "see" the electric field of a flow if its rotation coincides with that of the Earth. Either an advance or a delay of a flow is needed. In our example, this is just a delay that provides the existence of the electric field in the flow. And this is a direct confirmation of the presence of the toroidal electric current in the zone F of the Earth's liquid core, whose electrodynamic characteristics will be calculated in a sequel.

1.3.2. On a distance from the Earth's surface to the toroidal current

The next step in analyzing the MGF generation is detecting the location of the toroidal electric current in the Earth.

Based on the results of the spherical analysis of the MGF [6], it is possible to calculate a distance to a source of the MGF assuming this source to be a toroidal electric current.

The first assumption to be made is in that the magnetic moment of the internal poloidal magnetic field coincides with that of the ring electric current locating somewhere in the Earth's core whose field with its lines of force is an exact replica of the lines of force of the dipole part of the internal MGF. In terms of mathematics this looks like

$$|\boldsymbol{M}| = \mu_1^1 4\pi 10^{-3} R_0^3 = I\pi r_k^2\ \text{A/m}^2, \tag{1.48}$$

where μ_1^1 is a coefficient of the dipole term of expansion of the Earth's internal poloidal field [6], I is the strength of current in the contour with current, r_k is a radius of the contour, R_0 is the Earth's radius, $4\pi 10^{-3}$ is a translating factor from the dimension in the Gauss units to the dimension in A/m.

The second assumption is that the intensities of fields on the axis of the contour coinciding with the magnetic axis connecting the Earth's North and South Poles are supposed to be equal. In the first approximation this can be expressed as

$$H_{Pr}^i(0, R_0) 4\pi 10^{-3} = \frac{2\pi I r_k^2}{(R_0^2 + r_k^2)^{3/2}}\ \text{A/m}. \tag{1.49}$$

Excluding the force of current from formulas (1.48) and (1.49) results in

$$2\mu_1^1 = H_{Pr}^i(0, R_0)\Big(1 + \frac{r_k^2}{R_0^2}\Big)^{3/2}. \tag{1.50}$$

Expanding in series the formula in brackets in the right-hand side of (1.50) and restricting ourselves to the first two terms of the expansion, we obtain

$$2\mu_1^1 = H_{Pr}^i(0, R_0)\Big(1 + \frac{3r_k^2}{2R_0^2}\Big). \tag{1.51}$$

According to the calculations made in [6], the values of the internal poloidal magnetic field on the Pole and the coefficient of its dipole part become known:

$$H_{Pr}^i(0, R_0) = 0.59473 \text{ Gs}, \qquad \mu_1^1 = 0.32006 \text{ Gs}. \tag{1.52}$$

In this case $1 + \frac{3r_k^2}{R_0^2} = 1.07632$. Hence,

$$r_k = R_0 \cdot 0.22557 = 1437 \text{ km}. \tag{1.53}$$

Consequently, the depth to the source is equal to

$$h = 4934 \text{ km}. \tag{1.54}$$

The calculated radius of the source with electric current accurately to 4.6 % coincides with the radius of the zone F of the liquid core, which, according to the data available, is 1,371 km. The latter value was repeatedly reported in various geological and geophysical publications [15, 21]. The layer thickness F, which is equal, approximately, to 100 km is also known from scientific literature.

In our calculations the layer thickness with electric current will be defined below with allowance for the Reynolds kinematic number that determines the boundary between the laminar particles flow in the zone F, the liquid core and its turbulence. According to [15], the Reynolds kinematic number in the zone F cannot exceed 100–150.

1.3.3. On calculation of the MGF intensity in the zone *F* of the Earth's liquid core

Based on the calculated value of the radius of the source of current exciting the MGF equal to 1,437 km, it is possible to calculate the value of the magnetic moment of the toroidal magnetic field on this radius and on that of 1,371 km by the formula

$$\boldsymbol{H}_{TF} \cdot R_F^3 = \boldsymbol{M}_{TF}, \tag{1.55}$$

where $\boldsymbol{H}_{TF}$ is the toroidal magnetic intensity in the zone F, $\boldsymbol{M}_{TF}$ is a magnetic moment, R_F is a radius of the zone F.

On the Earth's surface, a magnetic moment of the toroidal field is calculated as

$$\boldsymbol{H}_T \cdot R_0^3 = \boldsymbol{M}_T. \tag{1.56}$$

Since the MGF magnetic moment is constant, then comparing (1.55) and (1.56) it is possible to define the relation between magnetic fields in the zone F with respect to the magnetic field on the surface:

$$\begin{gathered} |\boldsymbol{H}_{TF}| = |\boldsymbol{H}_T|\Big(\frac{R_0}{R_F}\Big)^3 = |\boldsymbol{H}_T|\Big(\frac{6371}{1437}\Big)^3 \approx |\boldsymbol{H}_T| \cdot 90|_{r=1437}, \\ |\boldsymbol{H}_{TF}| = |\boldsymbol{H}_T|\Big(\frac{6371}{1371}\Big)^3 = |\boldsymbol{H}_T| \cdot 100|_{r=1371}. \end{gathered} \tag{1.57}$$

Thus, a toroidal field in the zone F of the liquid core with its radius is 100 times greater than the intensity values on the Earth's surface and 90 times greater at a depth of the calculated distance to the source. The cubic degree of the dependence of the relation of the Earth's radii and the source enables us to calculate the poloidal field intensity in the vicinity of the source:

$$\begin{aligned} \big|\boldsymbol{H}_P\big||_{1371} = 0.6 \cdot 100 = 60 \text{ Gs}, \\ \big|\boldsymbol{H}_P\big||_{1437} = 0.6 \cdot 90 = 54 \text{ Gs}, \end{aligned} \tag{1.58}$$

where the MGF intensity on the Earth's surface is approximately equal to $|\boldsymbol{H}_P|$=0.6 Gs. The absolute value of the toroidal field intensity in a source, 1,437 km in radius, is equal to

$$|\boldsymbol{H}_T| = |\boldsymbol{H}_P| \cdot 0.06 = 3.2 \text{ Gs}.$$

It was obtained from the result in [6] where the relation between the fields on the Earth's surface was calculated from their observed maximum values taken in 1965: $|\boldsymbol{H}_T|/|\boldsymbol{H}_P| = 3345/54886 = 0.06$. On a radius of 1,371 km we obtain the following value

$$|\boldsymbol{H}_T| = |\boldsymbol{H}_P| \cdot 0.06 = 3.6 \text{ Gs}. \tag{1.59}$$

It is believed that the intensity of the Earth's toroidal magnetic field in the zone F of the liquid core does not exceed 4 Gs. Further these values of $\boldsymbol{H}_P$ and $\boldsymbol{H}_T$ at a depth will be used in calculation of the electric current parameters in the MGF source.

1.3.4. On excitation of electric current in the liquid core

An investigation of the character and intensity of the toroidal magnetic field in the liquid core, in the zone F to be exact, where in our opinion a source of the MGF locates, has clearly shown that this field is generated by spherical features of the source, i.e., the toroidal electric current. Therefore, the question arises about the way of occurrence of the toroidal electric current in the zone F of the liquid core. Clearly, this is a flow of charged particles, most likely, as judged from the direction of the MGF lines of force, electrons, carrying off by the Earth's rotation from East to West. By the right-hand rule the MGF has usual North and South Poles.

Clearly, there should be a certain external magnetic field to provide "initialization", for example, of an inductive electric current at the time of its occurrence [19, 23].

In order to calculate the value of an original external field, it is needed to know the Reynolds numbers both kinematic and magnetic [6]. The Reynolds kinematic number for the liquid core is selected as 100–150 units based on the fact that these values define the interface between laminarity and turbulence of the flow in the liquid core [15]. Such Reynolds numbers do not forbid from the occurrence of turbulence, possibly, in local temperature anomalies against a general laminar flow of charged particles. Generally speaking, such an assumption is a compromise between a laminar flow and local turbulence that can occur according to anomalies in the MGF observed over the ages. Nevertheless, the Reynolds kinematic number allows the assessment of the width of the charged particles flow based on the relation:

$$R_e = \frac{U_\phi \cdot l}{\nu}, \qquad l = 150 \cdot 10^3/50 = 3 \cdot 10^3 \text{ m}, \tag{1.60}$$

where ν is kinematic viscosity in the liquid core equal to $10^3\,\text{m}^2/\text{sec}$.

Such a low intensity of the flow, equal to three km is due to a relatively insignificant kinematic viscosity that essentially narrows the flow in spite of rather a large Reynolds kinematic number which, in turn, brings about "widening" the flow intensity. The interaction of the contradictory tendencies stabilizes the flow resulting in its steady-state existence confirmed by the MGF stability.

The known intensity (width) of a charged particles flow allows the assessment of the Reynolds magnetic number R_m from the formula

$$R_m = l \cdot \sigma_F \cdot \mu_0 \cdot U_\phi = 3 \cdot 10^3 \cdot 5 \cdot 10^5 \cdot 4\pi \cdot 10^{-7} = 9.42 \cdot 10^4, \tag{1.61}$$

where $\sigma_F = 5 \cdot 10^5\ (\text{Om} \cdot \text{m})^{-1}$ is specific conductivity in the zone F of the core [15].

Such a large Reynolds magnetic number allows us to neglect ohmic losses. In this case the inductive excitation will be defined only by a reactive component. Then the original "initialization" magnetic field can be calculated based on the Braginsky formula [8]:

$$\boldsymbol{H} = \boldsymbol{H}_0 \cdot R_m, \quad |\boldsymbol{H}_0| = \frac{|\boldsymbol{H}|}{R_m} = \frac{60 \cdot 10^{-5}}{9.42} \approx 64 \text{ nT}. \tag{1.62}$$

Such a small in intensity initialization field for induction could exist at the dawn of occurrence of the MGF, on the one hand, and it is just this field that currently supports the MGF in connection with the influence of the constant component of the solar magnetic field through the "solar wind", on the other hand. Till the Sun exists, the MGF will exist as well. The stability of the latter as related to external and internal effects is given by formula (1.45). The stability is attained at the expense of the inverse proportion between the velocity of the flow and the magnetic field intensity. Demagnetization results in increasing the velocity of the flow, while strengthening of the magnetic field results in decreasing the velocity of the flow which then is equavilized by the Earth's stable rotation, the latter being a pledge of the MGF stability as a whole.

The Earth's poles are focused by a weak toroidal magnetic field by the formula

$$[\boldsymbol{H}_T \times \boldsymbol{H}_P] = H_{Pr} H_{T\phi} e_\theta - H_{Pr} H_{T\theta} e_\phi + (H_{T\theta} H_{P\phi} - H_{T\phi} H_{P\theta}) e_r. \tag{1.63}$$

A maximum value of the poloidal magnetic field on the poles is amplified by components of the toroidal magnetic field "screwing" onto the lines of force of the poloidal field by formula (1.8). This leads to the focusing effect on the MGF poles. In the present model of the source of the MGF, a change of polarity in the MGF, often being a subject of discussions among geologists, can take place only at the expense of changing the direction of the Earth's rotation to the opposite one, on the one hand, or liquidation for whatever reason of the Poles focusing on the other hand. Such events for the Earth as a planet are highly improbable.

1.3.5. On electrodynamic parameters of the MGF source

In order to evaluate, based on the data obtained, electrodynamic parameters of the source it is necessary to previously evaluate geometric parameters of the flow, that is, its cross-section square and the volume it occupies. Based on the fact that the source generates the toroidal magnetic field, measured on the whole Earth's surface, the source will be considered to be distributed in the spherical layer three kilometers thick from Pole to Pole (1.60). Then the square (in square meters) of a cross-section of the semi-layer will be equal to

$$S_F = \frac{\pi}{2}(R_2^2 - R_1^2) = 13.5 \cdot 10^9 \text{m}^2, \tag{1.64}$$

where $R_2 = 1.437 \cdot 10^6$, $R_1 = 1.434 \cdot 10^6$ m.

The volume of the layer is

$$V_F = \frac{4}{3}\pi(R_2^3 - R_1^3) = 7.8 \cdot 10^{16}\text{m}^3. \tag{1.65}$$

Based on the equality of magnetic moments expressed through the magnetic field and the current force by the formula

$$|\boldsymbol{H}_F| \cdot R_F^3 = S_F \cdot I,$$

it is possible to calculate the current force in the source:

$$I = \frac{|\boldsymbol{H}_F| \cdot R_F^3}{S_F} = \frac{60 \cdot 4\pi \cdot 10^{-3}(1.437)^3 \cdot 10^{18}}{13.5 \cdot 10^9} = 1.66 \cdot 10^8\text{A}. \tag{1.66}$$

Thus, the current force in the source is about one trillion A. The density of the current in the source can be assessed as follows:

$$|\boldsymbol{j}| = \frac{I}{S_F} = \frac{1.66 \cdot 10^8}{13.5 \cdot 10^9} = 1.23 \cdot 10^{-2}\text{A/m}^2. \tag{1.67}$$

The number of particles in a cubic meter in the source is assessed through the current density and the velocity of the particles flow:

$$ne = \frac{j^\phi}{U_\phi}, \tag{1.68}$$

where the amount of the charge Q in a cubic meter equals

$$ne = \frac{1.23 \cdot 10^{-2}}{50} = 2.46 \cdot 10^{-4} Q/\text{m}^3.$$

If we consider the electric current in the source to be generated by the electron flux, then the amount of particles (electrons) in a cubic meter will be equal to

$$n = \frac{2.46 \cdot 10^{-4}}{1.6 \cdot 10^{-19}} = 15 \cdot 10^{14}, \tag{1.69}$$

where $e = 1.6 \cdot 10^{-19}$ is the electron charge. The electric field in a layer is evaluated as follows. The specific conductivity of the liquid core is known from literature [15]. It is about $\sigma_F = 5 \cdot 10^5$ S/m, then the electric field can be assessed by the formula

$$|\boldsymbol{E}_F| = \frac{|\boldsymbol{j}_F|}{\sigma_F} = \frac{1.23 \cdot 10^{-2}}{5 \cdot 10^5} = 2.46 \cdot 10^{-8}\text{V/m}. \tag{1.70}$$

Such an electric field is provided by the delay of the electrons flux with respect to the angular velocity of the Earth's rotation. The amount of charge in a layer is assessed from the amount of charge in a cubic meter multiplied by the volume of this layer (1.65), that is

$$neV_F = 2.46 \cdot 10^{-4} \cdot 7.8 \cdot 10^{16} = 1.92 \cdot 10^{13} \text{ Q}. \tag{1.71}$$

A charge in a layer rotates with angular velocity (1.47), therefore the current force it generates in one of directions will be equal to half the charge multiplied by the angular velocity of the flux rotation:

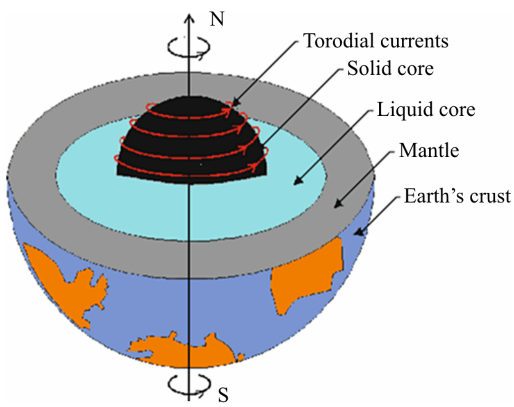
$$I = \frac{1}{2} neV_F \omega_{U_\phi} = 0.5 \cdot 1.92 \cdot 10^13 \cdot 3.4 \cdot 10^{-5} \approx 3.26 \cdot 10^8 \text{ A}, \tag{1.72}$$

or equal to $I \approx 6 \cdot 10^8$ A through the full charge in the layer. These values of the current force in order of the value coincide with the current force calculated from the equality of moments calculated from the magnetic field (1.66).

Thus, electrodynamic parameters of the MGF source, representing a toroidal electric conductivity current, provided by a flux of free electrons in the zone F of the liquid core, generate values of the poloidal and toroidal magnetic fields observed on the Earth. The electric current is supported by the Earth's stable rotation and by insignificant inductive initialization due to the solar magnetic field and interplanetary magnetic field. Until the above-mentioned rather weak magnetic fields exist on the Earth, there will exist the Earth's stable MGF. With its stability, the MGF has provided a long-term evolution of the Earth's biosphere (Figure 1.1).

Figure 1.1. The source of the Earth's main geomagnetic field

Parameters of the source:
The distance to the source . . . 4934 km
Internal radius of torus . . . . . . 1437 km
Diametrical size of torus . . . . .3 km
Current force . . . . $1.7 \cdot 10^8$ A
Current density . . . . . $1.23 \cdot 10^{-2}$ A/m^2
Intensity of electric field
in the source . . . . $2.5 \cdot 10^{-8}$ V/m
Intensity of poloidal magnetic field
in the source . . . . 60 Gs
Intensity of toroidal magnetic field
in the source . . . . .3.6 Gs



1.4. On separating the main geomagnetic field into the external and internal parts

When interpreting data of the world network of geomagnetic stations, it is proposed to separate the magnetic fields of inner sources, located in the Earth's interior, from those of possible exterior to the Earth sources.

This problem was solved in [27]. The Schmidt formulas were generally applied when separating the Earth's potential magnetic fields. In combination with the result earlier obtained by Gauss, the result leading to the separation of magnetic fields into external and internal is sometimes called the Gauss–Schmidt theorem. Now it is time to proceed to the generalization of this theorem to the toroidal vector (magnetic) fields.

Theorem 1 generalizes the Helmholtz theorem of reconstructing solenoidal vector fields in a sphere from their external normal component on the surface S of the sphere V in addition to the vector fields in a sphere having no such normal component to the surface S (a toroidal vector field) provided that $\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P$. The latter is identically fulfilled in the case of presenting the vector fields under study by one scalar function. Really, when $\boldsymbol{H}_T = \nabla\times(Q\boldsymbol{r})$, $\boldsymbol{H}_P = \nabla\times\nabla\times(Q\boldsymbol{r})$, obtain $\nabla\times\boldsymbol{H}_T = \nabla\times\nabla\times(Q\boldsymbol{r}) = \boldsymbol{H}_P$. In this case, the scalar function Q is uniquely defined by the normal component $H_N(\boldsymbol{r})$, specified at each point of the sphere surface S by formula (1.22). The generalized Helmholtz theorem, i.e., Theorem 1, allows us to formulate a theorem of the separation of solenoidal vector fields into the fields from external and internal sources, containing poloidal and toroidal components generalizing, in addition, the Gauss–Schmidt theorem.

Theorem 9. *The problem of the separation of poloidal and toroidal vector fields from the sources located outside the sphere V from the same fields but from the sources, located inside the sphere, is solvable in one way if the external normal component $H_N(\boldsymbol{r})$ and one of the two tangential components $H_t(\boldsymbol{r})$ of the total vector field $\boldsymbol{H}$ on the surface S of the sphere V are known.*

Really, let, according to formula (1.22), on S, the total scalar function $Q = Q^e + Q^i$, consisting from the external Q^e and the internal Q^i parts be known. Then with allowance for (1.15) and (1.22) on the surface S, we can write down

$$Q^e + Q^i = -\sum_{n=1}^{\infty} R\frac{S_n(\theta,\phi)}{n(n+1)}\Big[A_n\frac{1}{R^{n+1}} + B_nR^n\Big], \tag{1.73}$$

where A_n and B_n are complex constants of the external A_n and the internal B_n parts of the function Q, respectively, R is the radius of the sphere. The toroidal and poloidal components of the magnetic field on the sphere surface S are calculated with definitions (1.8):

$$\begin{aligned}\boldsymbol{H}_T &= \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(Q^e+Q^i)e_\theta - \frac{\partial}{\partial\theta}(Q^e+Q^i)e_\phi,\\ \boldsymbol{H}_P &= \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial\theta}(Q^e+Q^i)e_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial r}r\frac{\partial}{\partial\phi}(Q^e+Q^i)e_\phi - \\ &\quad \frac{1}{r\sin\theta}\Big(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}(Q^e+Q^i) - \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}(Q^e+Q^i)\Big)e_r.\end{aligned} \tag{1.74}$$

Keeping in mind the dependence of the function Q on a radius r for external and internal sources (1.15) and omitting elementary formulas, let us write down components of the poloidal field as follows:

$$\begin{aligned} H_{P\theta} &= -\frac{n}{r^{n+2}}\frac{\partial}{\partial\theta}Q^e + (n+1)r^{n-1}\frac{\partial}{\partial\theta}Q^i, \\ H_{P\phi} &= -\frac{n}{r^{n+2}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Q^e + (n+1)r^{n-1}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Q^i, \\ H_{Pr} &= \frac{1}{r\sin\theta}\Big[\cos\theta\frac{\partial(Q^e+Q^i)}{\partial\theta} + \sin\theta\frac{\partial^2(Q^e+Q^i)}{\partial\theta^2} - \frac{1}{\sin\theta}\frac{\partial^2(Q^e+Q^i)}{\partial\phi^2}\Big]. \end{aligned} \tag{1.75}$$

Taking into account (1.74), (1.75), and (1.22), write down tangential components of the total magnetic field and its normal component in the following form:

$$\begin{aligned} H_{t\theta} &= -\sum_{n=1}^{\infty} X_n(\theta,\phi)\frac{1}{n(n+1)}\Big[-A_n\frac{n}{R^{n+1}} + B_n(n+1)R^n\Big], \\ H_{t\phi} &= -\sum_{n=1}^{\infty} Y_n(\theta,\phi)\frac{1}{n(n+1)}\Big[-A_n\frac{n}{R^{n+1}} + B_n(n+1)R^n\Big], \\ H_N &= \sum_{n=1}^{\infty} Z_n(\theta,\phi)\frac{1}{n(n+1)}\Big[A_n\frac{1}{R^{n+1}} + B_nR^n\Big]. \end{aligned} \tag{1.76}$$

Here $X_n(\theta,\phi), Y_n(\theta,\phi), Z_n(\theta,\phi) \in C^\infty$ are complex angular functions that are derived from the known function $S_n(\theta,\phi)$, whose analytical form will be defined below. Formulas (1.76) indicate to the fact that any pair composed of one tangential and one normal components makes it possible to uniquely separate the coefficients of external and internal vector fields as related to the surface S since the determinant of separating equations differs from zero:

$$\begin{vmatrix} \dfrac{1}{R^{n+1}} & R^n \\ -\dfrac{n}{R^{n+1}} & (n+1)R^n \end{vmatrix} = \frac{(2n+1)}{R}. \tag{1.77}$$

The separate calculation of Q^e and Q^i allows us to reconstruct on the sphere surface S the poloidal and toroidal vector fields both from the sources located inside the sphere and from external sources. The numerical implementation of the separation algorithm with allowance for toroidal fields according to the evidence on the MGF dated back to 1965 is presented in monograph [6].

Formulas (1.76) and (1.77) make possible to be sure that Theorem 9 is valid. In addition, the Gauss–Shmidt result of the unique separation of potential magnetic fields is confirmed for the case of their presentation by one scalar function, which as it happened, also concerns solenoidal magnetic fields containing both the toroidal and the poloidal parts.

1.5. On algorithms of observations interpolation of the main geomagnetic field on the Earth's surface

1.5.1. The Gauss–Schmidt interpolation decompositions of the MGF

It turned out so that usually for the interpolation of the MGF observed on the Earth's surface at separate points, the Gauss–Schmidt algorithm was conventionally applied. In the modern interpretation, the Gauss method is based on the two assumptions. The first concerns the physical properties of the MGF in the Earth's atmosphere. In the Gauss method, the MGF in the atmosphere is considered to be potential: $\nabla \times \boldsymbol{H} = 0$. The second assumption resulting from the first one concerns sources of a magnetic field. These sources are scalar, because at that time they were considered to be magnetic masses. The magnetic field potential of magnetic masses $\boldsymbol{H} = -\nabla V$ satisfied the Laplace equation and could be determined by solving the following equation:

$$\nabla \times \boldsymbol{H} = 0, \quad \nabla \cdot \boldsymbol{H} = \rho'_m \quad \Rightarrow \quad \Delta V = -\rho'_m, \quad V = -\int\limits_W \frac{\rho'_m}{R_0}\, dw', \tag{1.78}$$

where ρ'_m is the magnetic masses density in the sphere (the Earth), $R_0(r, \theta, \phi, r', \theta', \phi')$ is a distance between any point in the sphere and any point outside it, i.e., in the atmosphere, r, θ, ϕ, r', θ', ϕ' are spherical coordinates with the center in the Earth's center, the dotted coordinates referring to the Earth's volume, W being the volume of the Earth.

In the modern mathematics, the Gauss interpolation series on the sphere surface are obtained by expanding the function $1/R_0$ in the spherical functions $P_n^m(\cos\theta)$ and trigonometric functions $\cos m\phi$, $\sin m\phi$ and integrating over the dotted coordinates. Let sources locate inside the sphere, and a magnetic field outside it, then

$$V^e = \sum_{n=1}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} (A_n^m \cos m\phi + B_n^m \sin m\phi) P_n^m(\cos\theta), \tag{1.79}$$

where

$$A_n^m = -\int\limits_W \rho'_m r'^{\,n} \cos m\phi' P_n^m(\cos\theta')\, dw',$$

$$B_n^m = -\int\limits_W \rho'_m r'^{\,n} \sin m\phi' P_n^m(\cos\theta')\, dw'.$$

In (1.79), it is convenient to introduce the notation with a radius of the sphere equal to R:

$$A_n^m = R^{-(n+2)} g_n^m, \qquad B_n^m = R^{-(n+2)} h_n^m. \tag{1.80}$$

Then potential (1.79) can be rewritten as follows:

$$V^e = R \sum_{n=1}^{\infty} \Big(\frac{R}{r}\Big)^{n+1} \sum_{m=0}^{n} (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\cos\theta), \tag{1.81}$$

and on the sphere surface we obtain

$$V^e = R \sum_{n=1}^{\infty} \sum_{m=0}^{n} (g_n^m \cos m\phi + h_n^m \sin m\phi) P_n^m(\cos\theta). \tag{1.82}$$

In this case, according to (1.78), components of the magnetic field intensity on the sphere surface will be of the form

$$\begin{aligned}
H_\phi^e &= -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(-g_n^m \sin m\phi + h_n^m \cos m\phi)\frac{mP_n^m(\cos\theta)}{\sin\theta},\\
H_\theta^e &= -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(g_n^m \cos m\phi + h_n^m \sin m\phi)\frac{\partial P_n^m(\cos\theta)}{\partial\theta},\\
H_r^e &= -\frac{\partial V}{\partial r} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(n+1)g_n^m \cos m\phi + (n+1)h_n^m \sin m\phi\big] P_n^m(\cos\theta).
\end{aligned} \tag{1.83}$$

When sources are outside the sphere and a magnetic field is concentrated inside the sphere, formed by an external source, components of the field are determined from the following potential:

$$V^i = \sum_{n=1}^{\infty} \frac{r^n}{R^{n-1}} \sum_{m=0}^{n} (j_n^m \cos m\phi + k_n^m \sin m\phi) P_n^m(\cos\theta). \tag{1.84}$$

In this case

$$j_n^m = C_n^m R^{n-1}, \qquad k_n^m = D_n^m R^{n-1}.$$

Components of the field from the external source inside the sphere:

$$\begin{aligned}
H_\phi^i &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(-j_n^m \sin m\phi + k_n^m \cos m\phi)\frac{mP_n^m(\cos\theta)}{\sin\theta},\\
H_\theta^i &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(j_n^m \cos m\phi + k_n^m \sin m\phi)\frac{\partial P_n^m(\cos\theta)}{\partial\theta},\\
H_r^i &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(j_n^m \cos m\phi + k_n^m \sin m\phi) n P_n^m(\cos\theta).
\end{aligned} \tag{1.85}$$

The total magnetic field on the sphere surface are expressed by the interpolation formulas

$$
\begin{aligned}
H_\phi &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[-(g_n^m + j_n^m)\sin m\phi + (h_n^m + k_n^m)\cos m\phi\big]\frac{mP_n^m(\cos\theta)}{\sin\theta},\\
H_\theta &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(g_n^m + j_n^m)\cos m\phi + (h_n^m + k_n^m)\sin m\phi\big]\frac{\partial P_n^m(\cos\theta)}{\partial\theta}, \qquad (1.86)\\
H_r &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[\big((n+1)g_n^m - nj_n^m\big)\cos m\phi + \big((n+1)h_n^m - nk_n^m\big)\cos m\phi\big]P_n^m(\cos\theta).
\end{aligned}
$$

Here the separating equations will take the form

$$
\begin{cases} g_n^m + j_n^m = p_n^m,\\ (n+1)g_n^m - nj_n^m = p_n'^m,\end{cases}
\qquad
\begin{cases} h_n^m + k_n^m = q_n^m,\\ (n+1)h_n^m - nk_n^m = q_n'^m.\end{cases} \tag{1.87}
$$

A determinant of any pair of equations from (1.87) equals $-(2n+1)$, therefore separation of coefficients of the fields from internal and external sources proceed uniquely.

Interpolation formulas (1.86) have been used up till now when reconstructing the MGF from its measurements at separate points of the Earth's surface on the world network of stations as well as when interpolating satellite data.

Interpolation formulas (1.86) have been obtained under the assumption of the MGF potentiality in the Earth's atmosphere. In connection with this sufficiently strict assumption, the interpolation decompositions of a magnetic field appeared to be rather simple.

If one takes into account the toroidal part of the MGF, then according to Theorem 1, the MGF can also be reconstructed using one scalar potential, however interpolation formulas will be different. There will be terms referring to the toroidal part of the MGF.

1.5.2. Interpolation decompositions of the MGF with allowance for its toroidal part

Let us now obtain interpolation decompositions of the MGF using the potential Q from Theorem 1 and the experiment discussed in Section 1.5.1. Let a source be inside and the field be outside the sphere, then the potential outside the sphere will have the form

$$
Q^i = \sum_{n=1}^{\infty}\sum_{m=0}^{n}(a_n^m\cos m\phi + b_n^m\sin m\phi)P_n^m(\cos\theta)\frac{R^{n+2}}{r^{n+1}}, \tag{1.88}
$$

where $a_n^m = \tilde{a}_n^m / R^{n+2}$, $b_n^m = \tilde{b}_n^m / R^{n+2}$, the index i is a source located inside the sphere, the index e is a source outside the sphere.

On the sphere surface we obtain

$$Q^i = R \sum_{n=1}^{\infty} \sum_{m=0}^{n} (a_n^m \cos m\phi + b_n^m \sin m\phi) P_n^m(\cos\theta). \tag{1.89}$$

Let a source locate outside the Earth, a magnetic field inside the sphere, formed by an external source, then the function Q will be equal to

$$Q^e = \sum_{n=1}^{\infty} \sum_{m=0}^{n} (c_n^m \cos m\phi + d_n^m \sin m\phi) P_n^m(\cos\theta) \frac{r^n}{R^{n-1}}, \tag{1.90}$$

where $c_n^m = \tilde{c}_n^m \cdot R^{n-1}$, $d_n^m = \tilde{d}_n^m \cdot R^{n-1}$. The function Q^e on the Earth's surface will be equal to

$$Q^e = R \sum_{n=1}^{\infty} \sum_{m=0}^{n} (c_n^m \cos m\phi + d_n^m \sin m\phi) P_n^m(\cos\theta). \tag{1.91}$$

The toroidal fields from external and internal sources at any point, according to (1.8), are equal to:

- From internal sources—

$$\begin{aligned} H_{T\theta}^i &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-a_n^m \sin m\phi + b_n^m \cos m\phi) \frac{m P_n^m(\cos\theta)}{\sin\theta} \frac{R^{n+2}}{r^{n+1}}, \\ H_{T\phi}^i &= -\sum_{n=1}^{\infty} \sum_{m=0}^{n} (a_n^m \cos m\phi + b_n^m \sin m\phi) \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \frac{R^{n+2}}{r^{n+1}}; \end{aligned} \tag{1.92}$$

- From external sources—

$$\begin{aligned} H_{T\theta}^e &= \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-c_n^m \sin m\phi + d_n^m \cos m\phi) \frac{m P_n^m(\cos\theta)}{\sin\theta} \frac{r^n}{R^{n-1}}, \\ H_{T\phi}^e &= -\sum_{n=1}^{\infty} \sum_{m=0}^{n} (c_n^m \cos m\phi + d_n^m \sin m\phi) \frac{\partial P_n^m(\cos\theta)}{\partial\theta} \frac{r^n}{R^{n-1}}. \end{aligned} \tag{1.93}$$

The total field on the Earth's surface for $r = R$ and with notations $i_n^m := a_n^m + c_n^m$, $j_n^m := b_n^m + d_n^m$ will be

$$\begin{aligned} H_{T\theta} &= R \sum_{n=1}^{\infty} \sum_{m=0}^{n} (-i_n^m \sin m\phi + j_n^m \cos m\phi) \frac{m P_n^m(\cos\theta)}{\sin\theta}, \\ H_{T\phi} &= -R \sum_{n=1}^{\infty} \sum_{m=0}^{n} (i_n^m \cos m\phi + j_n^m \sin m\phi) \frac{\partial P_n^m(\cos\theta)}{\partial\theta}. \end{aligned} \tag{1.94}$$

The poloidal magnetic field from the internal sources outside the sphere will take the form

$$\begin{aligned}
H^i_{P\theta} &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(a_n^m \cos m\phi + b_n^m \sin m\phi)\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\frac{nR^{n+2}}{r^{n+2}}, \\
H^i_{P\phi} &= -\sum_{n=1}^{\infty}\sum_{m=0}^{n}(-a_n^m \sin m\phi + b_n^m \cos m\phi)\frac{mP_n^m(\cos\theta)}{\sin\theta}\frac{nR^{n+2}}{r^{n+2}}, \\
H^i_{Pr} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(a_n^m \cos m\phi + b_n^m \sin m\phi)\Bigg(\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\operatorname{ctg}\theta + \\
&\quad \frac{\partial^2 P_n^m(\cos\theta)}{\partial\theta^2} - \frac{m^2 P_n^m(\cos\theta)}{\sin^2\theta}\Bigg)\frac{R^{n+2}}{r^{n+2}}.
\end{aligned} \tag{1.95}$$

The poloidal magnetic field from the external sources inside the sphere will take the form:

$$\begin{aligned}
H^e_{P\theta} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(c_n^m \cos m\phi + d_n^m \sin m\phi)\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\frac{(n+1)r^{n-1}}{R^{n-1}}, \\
H^e_{P\phi} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(-c_n^m \sin m\phi + d_n^m \cos m\phi)\frac{mP_n^m(\cos\theta)}{\sin\theta}\frac{(n+1)r^{n-1}}{R^{n-1}}, \\
H^e_{Pr} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(c_n^m \cos m\phi + d_n^m \sin m\phi)\times \\
&\quad \Bigg(\operatorname{ctg}\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \frac{\partial^2 P_n^m(\cos\theta)}{\partial\theta^2} - \frac{m^2 P_n^m(\cos\theta)}{\sin^2\theta}\Bigg)\frac{r^{n-1}}{R^{n-1}}.
\end{aligned} \tag{1.96}$$

If we introduce the notation

$$\bar{i}_n^m := -na_n^m + (n+1)c_n^m, \qquad \bar{j}_n^m := -nb_n^m + (n+1)d_n^m, \tag{1.97}$$

the total poloidal magnetic field with $r = R$ will take the form:

$$\begin{aligned}
H_{P\theta} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(\bar{i}_n^m \cos m\phi + \bar{j}_n^m \sin m\phi)\frac{\partial P_n^m(\cos\theta)}{\partial\theta}, \\
H_{P\phi} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(-\bar{i}_n^m \sin m\phi + \bar{j}_n^m \cos m\phi)\frac{mP_n^m(\cos\theta)}{\sin\theta}, \\
H_{Pr} &= \sum_{n=1}^{\infty}\sum_{m=0}^{n}(i_n^m \cos m\phi + j_n^m \sin m\phi)\Bigg(\operatorname{ctg}\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \\
&\quad \frac{\partial^2 P_n^m(\cos\theta)}{\partial\theta^2} - \frac{m^2 P_n^m(\cos\theta)}{\sin^2\theta}\Bigg).
\end{aligned} \tag{1.98}$$

From formulas (1.98) it follows that it is sufficient to measure one normal component of the magnetic field, i.e., H_{Pr} on the Earth's surface for reconstructing, in addition, the toroidal magnetic field (1.94) as coefficients in them are the same.

The total magnetic field (the poloidal and toroidal fields) looks like

$$
\begin{aligned}
H_\theta = & \sum_{n=1}^{\infty}\sum_{m=0}^{n}(\bar{i}_n^m \cos m\phi + \bar{j}_n^m \sin m\phi)\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \\
& (-i_n^m \sin m\phi + j_n^m \cos m\phi)\frac{RmP_n^m(\cos\theta)}{\sin\theta}, \\
H_\phi = & \sum_{n=1}^{\infty}\sum_{m=0}^{n}(-\bar{i}_n^m \sin m\phi + \bar{j}_n^m \cos m\phi)\frac{mP_n^m(\cos\theta)}{\sin\theta} - \\
& (i_n^m \cos m\phi + j_n^m \sin m\phi)\frac{R\partial P_n^m(\cos\theta)}{\partial\theta}, \\
H_r = & \sum_{n=1}^{\infty}\sum_{m=0}^{n}(i_n^m \cos m\phi + j_n^m \sin m\phi)\Big(\operatorname{ctg}\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \\
& \frac{\partial^2 P_n^m(\cos\theta)}{\partial\theta^2} - \frac{m^2 P_n^m(\cos\theta)}{\sin^2\theta}\Big).
\end{aligned}
\tag{1.99}
$$

Formulas (1.99) are evidence in favor of the conclusion from Theorem 9 that it is sufficient to have a normal component, in our case H_r, and one tangential component (H_ϕ or H_θ) for the reconstruction and separation of the whole total magnetic field to modifications and to the fields from external and internal sources. For separation of magnetic fields it is required to compose from definable decomposition coefficients (1.99) the following equations:

$$
\begin{gathered}
\begin{cases}
-na_n^m + (n+1)c_n^m = \bar{i}_n^m, \\
a_n^m + c_n^m = i_n^m,
\end{cases} \\
\begin{cases}
-nb_n^m + (n+1)d_n^m = \bar{j}_n^m, \\
b_n^m + d_n^m = j_n^m.
\end{cases}
\end{gathered}
\tag{1.100}
$$

The determinant of these pairs of equations always differs from zero:

$$
\begin{vmatrix} -n & n+1 \\ 1 & 1 \end{vmatrix} = -(2n+1), \tag{1.101}
$$

therefore equations (1.99) allow the calculation of each decomposition coefficient with the known right-hand side in (1.100). In this case, as noted above, it is sufficient to measure (to consider to be known) two components of the total field: one vertical and one of the horizontal components.

Interpolation formulas (1.99) essentially differ from the Gauss-Schmidt interpolation formulas (1.86) obtained from the assumption of the MGF potentiality in the Earth's atmosphere. In the tangential components of formulas (1.99), there appear terms, responsible for the toroidal magnetic field on the Earth's surface. That is why formulas (1.99) make possible to reconstruct (on the Earth's surface) not only a poloidal but also a toroidal field, being its important part. Monograph [6] deals with interpreting the MGF with allowance for a toroidal magnetic field.

1.5.3. Interpolation decompositions of the MGF at instants of its current system

In the late seventies of the last century, the author proposed that if scalar sources of the MGF and its variations in (1.86) be changed to vector sources (magnetic masses to electric currents), then earlier elucidated unclear difficulties will become clear, on the one hand, and on the other hand, interpolation at points of the Earth's surface of magnetic fields will improve both qualitatively and quantitatively. This will happen, in the author's opinion, due to the fact that unknown coefficients to be calculated in more complicated decompositions will acquire the physical sense of moments of an arbitrary current system, which is essentially the MGF source and its variations [6]. Analytically, the formulas in this case are based on a solenoidal field instead of a potential magnetic field:

$$
\begin{gathered}
\nabla\times\boldsymbol{H}=j',\ \ \nabla\cdot\boldsymbol{H}=0,\ \ \boldsymbol{H}=\nabla\times\boldsymbol{A},\ \ \Delta\boldsymbol{A}=j',\ \ \nabla\cdot\boldsymbol{A}=0,\ \ \boldsymbol{A}=\int\limits_W\frac{j'}{R_0}\,dw',\\
A_x=\int\limits_W\frac{j'_x}{R_0}\,dw',\quad A_y=\int\limits_W\frac{j'_y}{R_0}\,dw',\quad A_z=\int\limits_W\frac{j'_z}{R_0}\,dw',\\
A_\theta=A_x\cos\theta\cos\phi+A_y\cos\theta\sin\phi-A_z\sin\theta,\\
A_\phi=-A_x\sin\phi+A_y\cos\phi,\\
A_r=A_x\sin\theta\cos\phi+A_y\sin\theta\sin\phi+A_z\cos\theta.
\end{gathered}
\tag{1.102}
$$

Here x, y, z are rectangular coordinates fixed at the Earth's center, R_0 is a distance between points outside and inside the sphere, $\boldsymbol{j'}$ is the vector of the current density in the Earth, $\Delta=\nabla\nabla\cdot-\nabla\times\nabla\times$ is the vector Laplace operator, $\boldsymbol{A}$ is a vector potential, $\nabla\cdot A=0$ is the Coulomb calibration, R is the Earth's radius.

Expanding the function $1/R_0$ from (1.102) in spherical and trigonometric functions and integrating over the dotted coordinates, we obtain new interpolation series, whose coefficients in terms of physics are moments of an arbitrary current system, located both inside and outside the Earth [6].

If a source is located inside the sphere, we obtain

$$
\begin{aligned}
A^i_\theta=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[&(\varkappa^m_n\cos m\phi+\rho^m_n\sin m\phi)\cos\theta\cos\phi+\\
&(\mu^m_n\cos m\phi+\nu^m_n\sin m\phi)\cos\theta\sin\phi-\\
&(u^m_n\cos m\phi+v^m_n\sin m\phi)\sin\theta\big]\frac{R^{n+2}}{r^{n+1}}P^m_n(\cos\theta),
\end{aligned}
$$

$$A_\phi^i = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[-(\varkappa_n^m \cos m\phi + \rho_n^m \sin m\phi)\sin\phi +$$

$$(\mu_n^m \cos m\phi + \nu_n^m \sin m\phi)\cos\phi\big]\frac{R^{n+2}}{r^{n+1}}P_n^m(\cos\theta), \tag{1.103}$$

$$A_r^i = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(\varkappa_n^m \cos m\phi + \rho_n^m \sin m\phi)sin\theta\cos\phi +$$
$$(\mu_n^m \cos m\phi + \nu_n^m \sin m\phi)\sin\theta\sin\phi +$$
$$(u_n^m \cos m\phi + v_n^m \sin m\phi)\cos\theta\big]\frac{R^{n+2}}{r^{n+1}}P_n^m(\cos\theta).$$

In formula (1.103), complex coefficients (due to complex features of the current density components) acquire a physical sense to be found in concrete formulas:

$$\begin{aligned}
\varkappa_n^m &= \frac{1}{R^{n+2}}\int_W r'^n j'_x \cos m\phi' P_n^m(\cos\theta')dw',\\
\rho_n^m &= \frac{1}{R^{n+2}}\int_W r'^n j'_x \sin m\phi' P_n^m(\cos\theta')dw',\\
\mu_n^m &= \frac{1}{R^{n+2}}\int_W r'^n j'_y \cos m\phi' P_n^m(\cos\theta')dw',\\
\nu_n^m &= \frac{1}{R^{n+2}}\int_W r'^n j'_y \sin m\phi' P_n^m(\cos\theta')dw',\\
u_n^m &= \frac{1}{R^{n+2}}\int_W r'^n j'_z \cos m\phi' P_n^m(\cos\theta')dw',\\
v_n^m &= \frac{1}{R^{n+2}}\int_W r'^n j'_z \sin m\phi' P_n^m(\cos\theta')dw'.
\end{aligned} \tag{1.104}$$

As is known, the expression $r'\cos\theta'$ represents a projection of a vector radius $\boldsymbol{r}'$ onto the axis y', and $r'\sin\phi'\sin\theta'$ is a projection onto the axis y', etc. In this connection, constant complex coefficients (1.104) can to a first approximation be written down in the following way:

$$\varkappa_1^0 = \frac{1}{R^3}\int_W z' j'_x dw' = \frac{M'_y}{R^3}, \quad \rho_1^1 = \frac{1}{R^3}\int_W y' j'_x dw' = -\frac{M'_z}{R^3}. \tag{1.105}$$

From (1.105) follows that complex coefficients represent projections of moments of different orders of an arbitrary current system on an axis of the rectangular coordinate system. Each component of the current density generates certain projections of moments of current systems onto rectangular axes of the Cartesian coordinate system. Thus, the component j'_x yields only the projections $M^m_{ny'}$ and $M^m_{nz'}$, the component j'_y yields only the projections $M^m_{nx'}$ and $M^m_{nz'}$, etc. In this connection coefficients (1.104) are similar to the Gauss decomposition coefficients (1.86) that are projections of the dipole moments of different orders onto rectangular axes of the Cartesian coordinate system.

Based on the above definitions of the toroidal and poloidal magnetic fields and on formulas (1.103), let us write down decompositions of these fields for interior sources, omitting cumbersome intermediate formulas and choosing the notations for a magnetic field and its components as in the previous case. Decompositions of a toroidal field on the Earth's surface for $r = R$ will look like

$$\begin{aligned} H^i_{T\theta} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[&- \varkappa^m_n(\cos m\phi \sin m\phi + m \sin m\phi \cos\phi) + \\ &\rho^m_n(m\cos m\phi\cos\phi - \sin m\phi \sin\phi) + \\ &\mu^m_n(\cos m\phi \cos\phi - m\sin m\phi\sin\phi) + \\ &\nu^m_n(m\cos m\phi\sin\phi + \sin m\phi\cos\phi) + \\ &(-u^m_n \sin m\phi + v^m_n \cos m\phi) m ctg\theta\big] P^m_n(\cos\theta), \end{aligned} \tag{1.106}$$

$$\begin{aligned} H^i_{T\phi} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n} &(\varkappa^m_n \cos m\phi\cos\phi + \rho^m_n \sin m\phi\cos\phi) \times \\ &\Big(\cos\theta P^m_n(\cos\theta) + \sin\theta \frac{\partial P^m_n(\cos\theta)}{\partial\theta}\Big) + \\ &(\mu^m_n \cos m\phi\sin\phi + \nu^m_n \sin m\phi \sin\phi) \times \\ &\Big(\cos\theta P^m_n(\cos\theta) + \sin\theta \frac{\partial P^m_n(\cos\theta)}{\partial\theta}\Big) + \\ &(u^m_n\cos m\phi + v^m_n \sin m\phi)\Big(\cos\theta\frac{\partial P^m_n(\cos\theta)}{\partial\theta} - \sin\theta P^m_n(\cos\theta)\Big). \end{aligned}$$

It is necessary to add to components of toroidal magnetic field (1.106) three components of the poloidal magnetic field for $r = R$:

$$\begin{aligned} H^i_{P\theta} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} &[-n\varkappa^m_n \cos m\phi\sin\phi - n\rho^m_n \sin m\phi\sin\phi + \\ &n\mu^m_n \cos m\phi\cos\phi + n\nu^m_n \sin m\phi\cos\phi] P^m_n(\cos\theta), \\ H^i_{P\phi} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} &[-n\varkappa^m_n \cos m\phi\cos\phi\cos\theta - n\rho^m_n \sin m\phi\cos\phi\cos\theta - \\ &n\mu^m_n \cos m\phi\sin\phi\cos\theta - n\nu^m_n \sin m\phi\sin\phi\cos\theta + \\ &nu^m_n \cos m\phi\sin\theta + nv^m_n \sin m\phi\sin\theta] P^m_n(\cos\theta), \end{aligned} \tag{1.107}$$

$$H^i_{Pr} = \sum_{n=1}^{\infty} \varkappa_n^m \Big(\cos m\phi \sin\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} -$$
$$m \sin m\phi \cos\phi \operatorname{ctg}\theta P_n^m(\cos m\phi)\Big) +$$
$$\rho_n^m \Big(\sin m\phi \sin\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m\cos m\phi \cos\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) -$$
$$\mu_n^m \Big(\cos m\phi \cos\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m\sin m\phi \sin\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) -$$
$$\nu_n^m \Big(\sin m\phi \cos\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - m\cos m\phi \sin\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) +$$
$$(u_n^m \sin m\phi - v_n^m \cos m\phi) m P_n^m(\cos\theta).$$

Decompositions (1.106) and (1.107) represent a set of components of a toroidal and a poloidal fields from the Earth's interior sources. In addition, (1.106) and (1.107) realize the principle laid in Theorem 1: the normal component H^i_{Pr} contains the whole variety of coefficients (moments of the current system) that are needed for the calculation of toroidal field components (1.106). The presence of H^i_{Pr} on the Earth's surface uniquely resolves the problem of reconstruction both of a poloidal and a toroidal magnetic fields by one normal component.

Now we need to obtain similar decompositions for components of the fields from the external sources, i.e., from those outside the Earth. First write down decompositions for components of the external sources potential:

$$\begin{aligned}
A^e_\theta = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[&(a_n^m \cos m\phi + b_n^m \sin m\phi)\cos\theta\cos\phi + \\
&(c_n^m \cos m\phi + d_n^m \sin m\phi)\cos\theta\sin\phi - \\
&(e_n^m \cos m\phi + f_n^m \sin m\phi)\sin\theta\big] \frac{r^n}{R^{n-1}} P_n^m(\cos\theta), \\
A^e_\phi = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[&-(a_n^m \cos m\phi + b_n^m \sin m\phi)\sin\phi + \\
&(c_n^m \cos m\phi + d_n^m \sin m\phi)\cos\phi\big] \frac{r^n}{R^{n-1}} P_n^m(\cos\theta), \\
A^e_r = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[&(a_n^m \cos m\phi + b_n^m \sin m\phi)\sin\theta\cos\phi + \\
&(c_n^m \cos m\phi + d_n^m \sin m\phi)\sin\theta\sin\phi + \\
&(e_n^m \cos m\phi + f_n^m \sin m\phi)\cos\theta\big] \frac{r^n}{R^{n-1}} P_n^m(\cos\theta).
\end{aligned} \tag{1.108}$$

Decompositions of a toroidal field from exterior sources for $r = R$ can be written as

$$
\begin{aligned}
H^e_{T\theta} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[& - a_n^m(\cos m\phi \sin\phi + m \sin m\phi \cos\phi) + \\
& b_n^m(-\sin m\phi \sin\phi + m\cos m\phi \cos\phi) + \\
& c_n^m(\cos m\phi \cos\phi - m \sin m\phi \sin\phi) + \\
& d_n^m(\sin m\phi \cos\phi + m\cos m\phi \sin\phi) + \\
& (-e_n^m \sin m\phi + f_n^m \cos m\phi) m \operatorname{ctg}\theta\big] P_n^m(\cos\theta), \qquad (1.109)\\
H^e_{T\phi} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n} & (a_n^m \cos m\phi + b_n^m \sin m\phi)\cos\phi \times \\
& \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) + \\
(c_n^m \cos m\phi + d_n^m & \sin m\phi)\sin\phi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) + \\
(e_n^m \cos m\phi + f_n^m & \sin m\phi)\Big(-sin\theta P_n^m(\cos\theta) + \cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big).
\end{aligned}
$$

Decompositions of a poloidal magnetic field from exterior sources look like

$$
\begin{aligned}
H^e_{P\theta} = & \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[(n+1)a_n^m \cos m\phi \sin\phi + (n+1) b_n^m \sin m\phi \sin\phi - \\
& (n+1)c_n^m \cos m\phi\cos\phi - (n+1)d_n^m \sin m\phi \cos\phi\big] P_n^m(\cos\theta), \\
H^e_{P\phi} = & \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[\big((n+1)a_n^m \cos m\phi \cos\phi + (n+1) b_n^m \sin m\phi \times \\
& \cos\phi + (n+1)c_n^m \cos m\phi \sin\phi + (n+1) d_n^m \sin m\phi \sin\phi\big)\cos\theta - \\
& \big((n+1)e_n^m \cos m\phi + (n+1) f_n^m \sin m\phi\big)\sin\theta\big] P_n^m(\cos\theta), \qquad (1.110)\\
H^e_{Pr} = & \sum_{n=1}^{\infty}\sum_{m=0}^{n} a_n^m \Big(\cos m\phi \sin\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \\
& \qquad\qquad m \sin m\phi \cos\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) + \\
& b_n^m \Big(\sin m\phi \sin\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m \cos m\phi \cos\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& c_n^m \Big(\cos m\phi \cos\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m \sin m\phi \sin\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& d_n^m \Big(\sin m\phi \cos\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - m \cos m\phi \sin\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) + \\
& (e_n^m \sin m\phi - f_n^m \cos m\phi) m P_n^m(\cos\theta).
\end{aligned}
$$

Based on decompositions for external and internal sources it is possible to obtain decompositions for the total field (a toroidal plus a poloidal fields). Toroidal components of the total field are of the form

$$H_{T\theta} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[-\bar{i}_n^m(\cos m\phi \sin\phi + m \sin m\phi \cos\phi) +$$
$$\bar{j}_n^m(m\cos m\phi\cos\phi - \sin m\phi \sin\phi)\bar{k}_n^m(\cos m\phi \cos\phi -$$
$$m \sin m\phi \sin\phi) + \bar{l}_n^m(m\cos m\phi \sin\phi + \sin m\phi\cos\phi) +$$
$$(-\bar{q}_n^m \sin m\phi + \bar{p}_n^m \cos m\phi) m \operatorname{ctg}\theta\big] P_n^m(\cos\theta), \tag{1.111}$$

$$H_{T\phi} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n} (\bar{i}_n^m \cos m\phi + \bar{j}_n^m \sin m\phi)\cos\phi \times$$
$$\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$(\bar{k}_n^m \cos m\phi + \bar{l}_n^m \sin m\phi)\sin\phi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$(\bar{q}_n^m \cos m\phi + \bar{p}_n^m \sin m\phi)\Big(\cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big).$$

Components of the total poloidal magnetic field look like

$$H_{P\theta} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[-(\tilde{i}_n^m - \bar{i}_n^m)\cos m\phi \sin\phi - (\tilde{j}_n^m - \bar{j}_n^m)\sin m\phi\sin\phi +$$
$$(\tilde{k}_n^m - \bar{k}_n^m)\cos m\phi\cos\phi + (\tilde{l}_n^m - \bar{l}_n^m)\sin m\phi\cos\phi\big] P_n^m(\cos\theta),$$
$$H_{P\phi} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[(\bar{i}_n^m - \tilde{i}_n^m)\cos m\phi\cos\phi + (\bar{j}_n^m - \tilde{j}_n^m)\sin m\phi\cos\phi +$$
$$(\bar{k}_n^m - \tilde{k}_n^m)\cos m\phi \sin\phi + (\bar{l}_n^m - \tilde{l}_n^m)\sin m\phi\sin\phi\big] \cos\theta P_n^m(\cos\theta) -$$
$$\big[(\bar{q}_n^m + \tilde{q}_n^m)\cos m\phi + (\bar{p}_n^m + \tilde{p}_n^m)\sin m\phi\big] \sin\theta P_n^m(\cos\theta), \tag{1.112}$$
$$H_{Pr} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \bar{i}_n^m\Big(\cos m\phi \sin\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} -$$
$$m \sin m\phi\cos\phi \operatorname{ctg}\theta P_n^m(\cos\theta)\Big) +$$
$$\bar{j}_n^m\Big(\sin m\phi\sin\phi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m\cos m\phi\cos\phi\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) -$$
$$\bar{k}_n^m\Big(\cos m\phi cos\phi \frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m\sin m\phi\sin\phi\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) -$$
$$\bar{l}_n^m\Big(\sin m\phi\cos\phi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} - m\cos m\phi\sin\phi\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) +$$
$$\bar{q}_n^m m \sin m\phi P_n^m(\cos\theta) - \bar{p}_n^m m\cos m\phi P_n^m(\cos\theta).$$

Based on formulas (1.111) and (1.112) it is possible to obtain components of the observed field tangential to the Earth's surface. In the observed magnetic field, the vertical component coincides with that of the poloidal magnetic field (1.112), therefore by virtue of the equality $H_{Pr} \equiv H_r$, it is not presented here. Thus, components of the magnetic field observed on the Earth are of the form

$$
\begin{aligned}
H_\theta = {} & \sum_{n=1}^{\infty}\sum_{m=0}^{n}[-\bar{i}_n^m m\sin m\phi\cos\phi + \bar{j}_n^m m\cos m\phi\cos\phi - \\
& \bar{k}_n^m m\sin m\phi\sin\phi + \bar{l}_n^m m\cos m\phi\sin\phi]\,P_n^m(\cos\theta) - \\
& (\bar{q}_n^m \sin m\phi - \bar{p}_n^m \cos m\phi) m \ \operatorname{ctg}\theta P_n^m(\cos\theta) + \\
& [-\tilde{i}_n^m \cos m\phi\sin\phi - \tilde{j}_n^m \sin m\phi\sin\phi + \tilde{k}_n^m \cos m\phi\cos\phi + \\
& \tilde{l}_n^m \sin m\phi\cos\phi] P_n^m(\cos\theta), \qquad (1.113)\\
H_\phi = {} & -\sum_{n=1}^{\infty}\sum_{m=0}^{n}[\bar{i}_n^m \cos m\phi\cos\phi + \bar{j}_n^m \sin m\phi\cos\phi + \\
& \bar{k}_n^m \cos m\phi\sin\phi + \bar{l}_n^m \sin m\phi\sin\phi]\,\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \\
& (\bar{q}_n^m \cos m\phi + \bar{p}_n^m \sin m\phi)\cos\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \\
& [\tilde{i}_n^m \cos m\phi\cos\phi + \tilde{j}_n^m \sin m\phi\cos\phi + \tilde{k}_n^m \cos m\phi\sin\phi + \\
& \tilde{l}_n^m \sin m\phi\sin\phi]\cos\theta P_n^m(\cos\theta) + \\
& (\tilde{q}_n^m \cos m\phi + \tilde{p}_n^m \sin m\phi)\sin\theta P_n^m(\cos\theta), \\
& H_r \equiv H_{Pr}.
\end{aligned}
$$

Formulas (1.113) should be supplemented by the equations to which all the coefficients entering the decompositions obey. The coefficients of the observed magnetic field are related to those of external and internal fields observed by the following equations:

$$
\begin{aligned}
&\begin{cases} \varkappa_n^m + a_n^m = \bar{i}_n^m, \\ (n+1)\varkappa_n^m - n a_n^m = \tilde{i}_n^m, \end{cases} &&
\begin{cases} \rho_n^m + b_n^m = \bar{j}_n^m, \\ (n+1)\rho_n^m - n b_n^m = \tilde{j}_n^m, \end{cases} \\
&\begin{cases} \mu_n^m + c_n^m = \bar{k}_n^m, \\ (n+1)\mu_n^m - n c_n^m = \tilde{k}_n^m, \end{cases} &&
\begin{cases} \nu_n^m + d_n^m = \bar{l}_n^m, \\ (n+1)\nu_n^m - n d_n^m = \tilde{l}_n^m, \end{cases} \qquad (1.114)\\
&\begin{cases} u_n^m + e_n^m = \bar{q}_n^m, \\ -(n+1)u_n^m + n e_n^m = \tilde{q}_n^m, \end{cases} &&
\begin{cases} v_n^m + f_n^m = \bar{p}_n^m, \\ -(n+1)v_n^m + n f_n^m = \tilde{p}_n^m. \end{cases}
\end{aligned}
$$

Equations (1.114) permit writing down formulas for the internal coefficients:

$$
\begin{aligned}
\varkappa_n^m &= \frac{\tilde{i}_n^m + n\bar{i}_n^m}{2n+1}, & \rho_n^m &= \frac{\tilde{j}_n^m + n\bar{j}_n^m}{2n+1}, \\
\mu_n^m &= \frac{\tilde{k}_n^m + n\bar{k}_n^m}{2n+1}, & \nu_n^m &= \frac{\tilde{l}_n^m + n\bar{l}_n^m}{2n+1}, \qquad (1.115)\\
u_n^m &= \frac{n\bar{q}_n^m - \tilde{q}_n^m}{2n+1}, & v_n^m &= \frac{n\bar{p}_n^m - \tilde{p}_n^m}{2n+1}.
\end{aligned}
$$

For the coefficients of the external magnetic field, appropriate formulas are sought for in a similar way:

$$
\begin{aligned}
a_n^m &= \frac{(n+1)\bar{i}_n^m - \tilde{i}_n^m}{2n+1}, \qquad & b_n^m &= \frac{(n+1)\bar{j}_n^m - \tilde{j}_n^m}{2n+1}, \\
c_n^m &= \frac{(n+1)\bar{k}_n^m - \tilde{k}_n^m}{2n+1}, \qquad & d_n^m &= \frac{(n+1)\bar{l}_n^m - \tilde{l}_n^m}{2n+1}, \\
e_n^m &= \frac{(n+1)\bar{q}_n^m + \tilde{q}_n^m}{2n+1}, \qquad & f_n^m &= \frac{(n+1)\bar{p}_n^m + \tilde{p}_n^m}{2n+1}.
\end{aligned} \tag{1.116}
$$

Table 1.1

$\theta°$	$\varphi°$	Field										
		toroidal		poloidal			summarized			observed		
		$B_{T\theta}$	$B_{T\varphi}$	$B_{P\theta}$	$B_{P\varphi}$	B_{Pr}	B_θ	B_φ	B_r	B_θ	B_φ	B_r
40	0	1392	3235	−20383	−3977	40222	−18991	−741	40222	−19200	−2330	42810
	60	−550	3411	−19184	128	47319	−19734	3539	47319	−19840	3210	49580
	120	−181	−253	−21856	−1773	51498	−22037	−2026	51498	−21820	−3500	54060
	180	−922	−1163	−21652	4156	37986	−22574	2994	37986	−22540	2650	41500
	240	1011	−2506	−16791	9498	51495	−15780	6992	51495	−15950	6700	55150
	300	−583	−1349	−12286	−5704	50467	−12869	−7053	50467	−12840	−7150	54040
60	0	−1431	2019	−27897	−3554	24868	−29328	−1535	24868	−29150	−2960	26640
	60	569	3695	−33264	−2632	30984	−32695	1063	30989	−32280	1130	33030
	120	971	−3257	−35388	1654	30807	−34417	−1602	30807	−34730	−2190	32330
	180	−683	−43	−25730	4556	25898	−26413	4513	25898	−26540	4390	25690
	240	117	47	−26555	6194	39796	−26438	6240	39796	−26570	6680	38680
	300	421	−2178	−23133	−4333	43467	−22712	−6512	43467	−22660	−6580	42940
80	0	−1283	−792	−30557	−1196	5425	−31840	−1988	5425	−31680	−4510	5080
	60	2176	589	−38157	−1906	11319	−35981	−1316	11319	−37980	−1130	3900
	120	−414	−2531	−39104	3157	6783	−39518	626	6783	−39590	690	2820
	180	−30	2872	−31653	3040	7647	−31683	5913	7647	−31590	5680	9230
	240	−486	3591	−31073	1821	13381	−31560	5412	13381	−31670	5240	16070
	300	825	−3427	−29095	−2185	20483	−28271	−5612	20483	−28330	−5200	23390
100	0	−370	−231	−21372	−2171	−14996	−21742	−2402	−14996	−21880	−6400	−19800
	60	1302	−897	−27380	−3581	−16575	−26078	−4477	−16575	−28200	−4110	−24060
	120	−1459	−1739	−34874	3557	−22307	−36333	1818	−22307	−36460	1660	−27330
	180	144	3250	−34635	3460	−17231	−34491	6710	−17231	−34650	6740	−16340
	240	−284	3020	−30377	2609	−9072	−30661	5629	−9072	−30700	5580	−6880
	300	1353	−1624	−28300	−1841	−2215	−26947	−3466	−2215	−26820	−3150	−1640
120	0	2238	2123	−15298	−5151	−25949	−13061	−3028	−25949	−13120	6400	−24880
	60	−1239	−1175	−15400	−8049	−35003	−16639	−9223	−35003	−16430	−9030	−33680
	120	−2267	−2176	−23137	2266	−53236	−25405	90	−53236	−25600	−40	−51820
	180	999	1174	−29157	6874	−40264	−28158	8049	−40264	−28190	7980	−41130
	240	−67	1353	−27003	6245	−24132	−27070	7598	−24132	−27050	7960	−25320
	300	91	−865	−22265	412	−9031	−22175	−454	−9031	−22200	−390	−10830
140	0	3829	2362	−16869	−6250	−29185	−13040	−3888	−29185	−13220	−6420	−26720
	60	−1170	1599	−11911	−14500	−41777	−13081	−12901	−41777	−13060	−13100	−39590
	120	−1023	−896	−9873	−794	−66736	−10897	−1690	−66736	−10820	−3250	−64620
	180	−291	−73	−17082	8848	−60954	−17373	8774	−60954	−17320	8300	−57790
	240	−288	−352	−20515	11702	−46824	−20803	11350	−46824	−20810	11350	−43460
	300	−686	−1079	−21325	4055	−25760	−22011	2976	−25760	−22070	2830	−22300

Unknown coefficients entering decompositions (1.112) and (1.113) can be determined using experimental data to be obtained for a preset year, called "epoch". The following data are available: LOIZMIRAN (Saint-Petersburg), referring to the epoch of 1965. With these data the MGF was spherically analyzed by formulas (1.113). The detailed results of this analysis are presented in [6]. Here as illustration we present Table 1.1 listing synthesized after the spherical analysis values of the toroidal and poloidal magnetic fields on the Earth's surface for the epoch of 1965.

Analysis of Table 1.1 shows a clear existence on the Earth's surface both of a poloidal magnetic field, usually well elucidated by the Gauss algorithm for the spherical analysis, and the toroidal part of the MGF. The presence of the toroidal magnetic field on the Earth's surface made possible to solve the problem of its determination in the zone F of the Earth's liquid core (Section 1.3.3), thus predetermining the author's opinion about the hypothesis of dynamo-excitation of the MGF and enabled him to propose a developed version of excitation of the MGF by toroidal currents in the zone F of the Earth's liquid core.

1.5.4. On relationship of the MGF decompositions of different types

Sections 1.5.2 and 1.5.3 present the MGF decompositions obtained by the two different techniques: with one scalar function and with three scalar functions indicating to the vector character of the vector potential $\boldsymbol{A}$. It is natural to expect the presence of connection between these decompositions. This connection results from formulas (1.30). Really, the poloidal magnetic field is expressed through the function Q and the vector potential component as follows:

$$\begin{aligned} H_{P\theta} &= \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}(Qr) = -\frac{1}{r}\frac{\partial}{\partial r} rA_\phi, \\ H_{P\phi} &= \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(Qr) = \frac{1}{r}\frac{\partial}{\partial r} rA_\theta, \\ H_{Pr} &= -\frac{1}{r^2\sin\theta}\Big(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial(Qr)}{\partial\theta} + \frac{\partial}{\partial\phi}\frac{1}{\sin\theta}\frac{\partial(Qr)}{\partial\phi}\Big) \\ &= -\frac{1}{r\sin\theta}\Big(-\frac{\partial}{\partial\theta}\sin\theta A_\phi + \frac{\partial}{\partial\phi}A_\theta\Big). \end{aligned} \tag{1.117}$$

The toroidal magnetic field is similarly expressed:

$$\begin{aligned} H_{T\theta} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(Qr) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r, \\ H_{T\phi} &= -\frac{1}{r}\frac{\partial}{\partial\theta}(Qr) = -\frac{1}{r}\frac{\partial}{\partial\theta}A_r. \end{aligned} \tag{1.118}$$

Formulas (1.117) and (1.118) assign the desired magnetic fields of the Earth with the help of the scalar function Q and the vector components A_θ, A_ϕ, A_r, defined through the toroidal currents. In formulas (1.102), no divergence of the vector potential $\boldsymbol{A}$ was fixed. It would be reasonable to expect its presence in the vector potential $\boldsymbol{A}$, presented by formula (1.5) The following theorem gives the answer to this question:

Theorem 10. *The calibration conditions of Coulomb $\nabla \cdot \boldsymbol{A} = 0$ or of Lorentz $\nabla \cdot \boldsymbol{A} = \sigma\bar{\phi}$ for an auxiliary vector field $\boldsymbol{A}$ are uniquely fulfilled provided that*

$$Q(r, \theta, \phi) = Q(\theta, \phi)/r^3. \tag{1.119}$$

Actually, from formula (1.9) follows

$$\boldsymbol{A} = (Q\boldsymbol{r}) + \nabla \times (Q\boldsymbol{r}) + \nabla\bar{\phi}. \tag{1.120}$$

Calculate the divergence from (1.120):

$$\nabla \cdot \boldsymbol{A} = \nabla \cdot (Q\boldsymbol{r}) + \nabla \cdot \nabla \times (Q\boldsymbol{r}) + \nabla \cdot \nabla\bar{\phi}. \tag{1.121}$$

In (1.121), the second term by definition equals zero, the first term being

$$\nabla \cdot (Q\boldsymbol{r}) = Q\nabla \cdot \boldsymbol{r} + \boldsymbol{r} \cdot \nabla Q = \frac{3}{r^3}Q(\theta, \phi) - \frac{3}{r^3}Q(\theta, \phi) \equiv 0. \tag{1.122}$$

In the third term, we consider variants

$$\Delta\bar{\phi} = 0 \quad \text{or} \quad \Delta\bar{\phi} = \sigma\bar{\phi}, \tag{1.123}$$

where $\sigma = \text{const}$.

From (1.119)–(1.123) follow two above-mentioned versions of calibration of the auxiliary vector field $\boldsymbol{A}$, namely, the Coulomb calibration

$$\nabla \cdot \boldsymbol{A} = 0, \tag{1.124}$$

or the Lorentz calibration

$$\nabla \cdot \boldsymbol{A} = \sigma\bar{\phi}. \tag{1.125}$$

The physical sense of Theorem 10 is in that orthogonal decomposition (1.5) does not seriously change physical properties of the solenoidal vector magnetic field $\boldsymbol{H}$ and does not change mathematical calibration conditions of the vector potential according to Coulomb or according to Lorentz. However, it imposes a supplementary condition on the scalar function Q, namely, its dependence on the coordinate r, which, generally speaking is due to the property of the total magnetic field $\boldsymbol{H}$. The latter generally decreases with distance as $1/r^3$. The toroidal magnetic field must decrease as well. And this dependence for the toroidal magnetic field is laid by formula (1.119).

1.5.5. On inversion of original matrices in calculating unknown coefficients of the MGF decompositions

The Gauss method of spherical analysis is, as known, in that for defining unknown coefficients, for example, in decomposition (1.86) or (1.113), the MGF observations at points of the Earth's surface are used, to be exact, observations at the world network of magnetic stations, which currently range up to 200, geomagnetic maps and satellite observations being applied as well. Therewith, in decompositions (1.86) or (1.113) one needs to restrict himself to a certain number of terms per series. This number varies in literature. In this paper, interpolation calculations by formulas (1.113) up to $n = 10$ are carried out. It turned out that properties of the original matrices, obtained with the points with assigned coordinates, do not meet the requirements of methods of their precise inversion. They do not possess an explicit diagonal predominance, and their determinators are sufficiently small resulting in large conditioning numbers. It is practically impossible to convert ill-conditioned matrices of high order with the use of exact methods. Therefore in our calculations we made use of the Moore [25] and the Penrose [26] pseudo-inversion technique with the Tokhonov regularization. One of the algorithms of such a strategy could be the following.

Let a matrix A of the spherical analysis by formulas (1.113) be of $n \times m$ and $n > m$ dimension; X be the vector of solutions with the coordinates $(x_1, \ldots, x_m)^T$, b be the vector of the right-hand sides with the coordinates $(b_1, \ldots, b_n)^T$. In this case, by formulas (1.113)coordinates of the vector x_i are coefficients of decompositions of the total magnetic field on a sphere. Entries of the matrix A are basis functions in the field components decompositions. These entries depend on coordinates of observations points, therefore, they can be calculated with rather a high accuracy. The right-hand sides b_i are values of intensity of components of the magnetic field $\boldsymbol{H}$ complicated by measurements errors. For decreasing the influence of observations errors, the matrix A is strongly overdetermined. However, when calculating the unknowns x_i with a strongly redetermined matrix A, the following unstandard situations can occur.

The rank r of the matrix A can be equal to or less than the number of unknowns m. If $r = m < n$, there is no solution, exactly satisfying the original equation $Ax = b$. Therewith to solve the equation, the least squares method is applied that minimizes the mean square of the residual of the form

$$\varepsilon^T \cdot \varepsilon = (Ax - b)^T (Ax - b). \tag{1.126}$$

Here T denotes transposition.

If the rank $r = m = n$, then the matrix A has its inverse. The solution will be $x = A^{-1}b$. The generalized solution coincides with it.

If $r < m < n$, then there may be infinitely many solutions by the least squares method. The system of normal equations is degenerate, its matrix $A^T A$ has no inverse. In this case, we introduce a solution $\tilde{x}$, obtained with a pseudo-inverse matrix H in the following manner: $\tilde{x} = Hb$. Therewith, the solution $\tilde{x}$, approximately satisfies the equation. In other words, if the solution $\tilde{x} = Hb$ is substituted into the system $Ax = b$, the result must not strongly differ from the right-hand side b. This means that the product AH should be a unit matrix of order n, i.e., I_n. Thus, for a solution be close to the true one, the product AH should be made not strongly different from I_n. On the other hand, the more HA differs from I_m, the smoother solution is obtained.

As was noted above, the right-hand sides of the equation $Ax = b$ contain observations errors. Let R_b be a covariance matrix of observations, then the covariance matrix of estimations of the solution will be $R_{\tilde{x}} = HR_bH^T$. Hence follows that the dispersion of the solution should not be too great. Since the matrix H is complicated by observations errors, the demand for smallness of dispersion imposes constraints on the operator H. In other words, on H one should impose a requirement of regularizing the solution. Below we present one of possible ways of such a regularization.

The algorithm of finding a pseudo-inverse matrix H is based on a singular decomposition of the matrix A. Any rectangular matrix A, whose rank $r \leq m, n$, can be presented as the following product: $A = U\Sigma V^T$, where Σ is a diagonal $r \times r$ matrix and U, V are rectangular $n \times r$ and $m \times r$ matrices, respectively. Columns of the rotation matrices U and V are vectors u_i and v_i defined from the system of equations

$$Av_i = \lambda_i u_i, \qquad A^T u_i = \lambda_i v_i. \tag{1.127}$$

Diagonal entries λ_i of the matrix Σ are eigenvalues of the matrix A. Systems of equations for defining U and V can be written down as

$$\begin{aligned} AV &= U\Sigma, & A^T U &= V\Sigma, \\ A^T AV &= V\Sigma^2, & AA^T U &= U\Sigma^2. \end{aligned} \tag{1.128}$$

Clearly, any vector v_i is an eigenvector corresponding to non-zero eigenvalues of the symmetric matrix $A^T A$, and u_i — to the eigenvector of the matrix AA^T. In this connection, the matrices U and V are orthonormal:

$$U^T U = I_r, \qquad V^T V = I_r. \tag{1.129}$$

In this case, a pseudo-inverse matrix H is defined as follows:

$$H = V\Sigma^{-1}U^T. \tag{1.130}$$

A regularizing matrix $\bar{H}$ can be calculated by the formula

$$\bar{H} = V\Sigma(\Sigma^2 + \alpha I)^{-1}U^T, \tag{1.131}$$

where $I = A^{-1}A$; α is a regularization parameter that is equal, for example, to the relation between the dispersion of noise and a priori dispersion of the solution.

As was noted above, when $r = m = n$, and $A = V\Sigma V^T$, then

$$H = V\Sigma^{-1}V^T = A^{-1}. \tag{1.132}$$

Here a pseudo-inverse matrix coincides with the inverse one and the system has a unique solution $x = A^{-1}b$, the generalized solution coinciding with it.

The case when $r = m < n$ was also mentioned above, therewith is of interest the fact of coincidence of the solution, obtained by the least squares method and of that by the pseudo-inversion. The method of the least squares (L.S.) yields the solution to equation

$$A^T A x_{\text{L.S.}} = A^T b. \tag{1.133}$$

If $r = m$, the matrix $A^T A$ has its inverse, and then

$$x_{\text{L.S.}} = (A^T A)^{-1} A^T b. \tag{1.134}$$

Because of the fact that the matrix A is equal to $A = U\Sigma V^T$, we can write down

$$x_{\text{L.S.}} = (V\Sigma U^T U\Sigma V^T)^{-1} Ab. \tag{1.135}$$

With allowance for the orthogonality of U and V, write down

$$x_{\text{L.S.}} = V\Sigma^{-2}V^T V\Sigma U^T b = V\Sigma^{-1}U^T b = Hb. \tag{1.136}$$

Thus, for $r = m < n$, the pseudo-inversion and the least squares method yield the same result.

If $r < m < n$, then there may be as many solutions by the least squares method as is wished. Decompose the solution and the vector of observations in terms of the system of eigenvectors

$$x = V\alpha + V_0\alpha_0, \qquad b = U\beta + U_0\beta_0. \tag{1.137}$$

Consider a solution minimizing the mean residual square

$$\varepsilon^2 = A^T A x - A^T b. \tag{1.138}$$

Substitute into it expansions (1.137) and obtain

$$\varepsilon^2 = V\Sigma^2\alpha + V\Sigma^2 V^T V_0\alpha_0 - V\Sigma\beta - V\Sigma U^T U_0\beta_0. \qquad (1.139)$$

From (1.139) it follows that a residual square minimum is attained when $\alpha = \Sigma^{-1}\beta$. It is also necessary to impose on the value α_0 the condition of the norm minimum of the solution $\|x\|$, where $\alpha_0 = 0$. Hence, the solution, corresponding to a residual square minimum and the norm of solution, will look like

$$x = V\Sigma^{-1}\beta. \qquad (1.140)$$

From (1.137) follows that the parameter $\beta = U^T b$, so the solution will be

$$x = V\Sigma^{-1}U^T b = Hb. \qquad (1.141)$$

Thus, in this important case, the generalized inverse operator H results in an approximate solution, coinciding with that by the method of least squares and possessing a minimum norm.

The method of pseudo-inversion is sometimes called the rotation method because the matrices U and V turn the space of columns of the matrix A up to its coincidence with the space of lines, so that the matrix A turns into the diagonal matrix Σ.

The results of the pseudo-inversion by the above algorithm were compared to those of the solution to normal equations with almost a special matrix by the conjugate gradients method. Both methods have brought about the coinciding results. The method of pseudo-inversion of matrices of the equation $Ax = b$ for the rang of matrices $r < m < n$ is extremely important for the spherical analysis of electromagnetic fields as the ranges of their expansion matrices are often lower/smaller than the number of unknown coefficients in expansions (1.86) and (1.113) to be determined.

The algorithm given here has been used without regularization since the late 1960s and with regularization since the 1990s to calculate the expansion coefficients by the spherical and the spatial MGF analysis and its long-period variations.

1.6. On the generalized electrodynamic equations for the main geomagnetic field

Consideration of the MGF toroidal part requires that the principal property of the toroidal magnetic field be determined, namely, the absence of the Lorentz force in its intensity. Indeed, the Lorentz force $\boldsymbol{F}_L$ is defined by the formula

$$\boldsymbol{F}_{\mathrm{L}} = [\boldsymbol{j} \times \boldsymbol{B}] = [\sigma\boldsymbol{E} \times \mu\boldsymbol{H}], \qquad (1.142)$$

where $\boldsymbol{E}$ is the intensity of the electric field in the source, σ is conductivity in it, and σ is magnetic permeability.

Since the toroidal magnetic field generates no electric currents owing to its main property $\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P$, the Lorentz force in its intensity is determined with allowance for formula (1.28), therefore it can be written down as

$$\boldsymbol{F}_{\mathrm{L}}^{T} = \left[\frac{\gamma}{\eta}\boldsymbol{H}_T \times \mu\boldsymbol{H}_T\right] \equiv 0. \tag{1.143}$$

In (1.143), the Lorentz force $\boldsymbol{F}_{\mathrm{L}}^{T}$ is identically equal to zero because the vector product of the toroidal magnetic field by itself equals zero due to the coinciding direction. Therefore, the toroidal field intensity should be measured by magnetometers, immediately responding to the intensity $\boldsymbol{H}_T$.

The derivation of general equations, to which the MGF is subject both in a source and in the Earth's atmosphere is of interest, also, in terms of physical applications. In this paper, one can come across such equations. Now we can write them down in the general form. In the area of a source, the equations take the form

$$\begin{aligned} &\nabla \times \boldsymbol{H}_P = \boldsymbol{j} + \chi\boldsymbol{H}_T, \qquad &&\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P,\\ &\nabla \cdot (\boldsymbol{H}_T, \boldsymbol{H}_P) = 0, &&\boldsymbol{B}_{T,P} = \mu\boldsymbol{H}_{T,P}. \end{aligned} \tag{1.144}$$

Outside the source, in the Earth's atmosphere, the form of equations is reduced to

$$\begin{aligned} &\nabla \times \boldsymbol{H}_P = 0, \qquad &&\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P,\\ &\nabla \cdot (\boldsymbol{H}_T, \boldsymbol{H}_P) = 0, &&\boldsymbol{B}_{T,P} = \mu\boldsymbol{H}_{T,P}. \end{aligned} \tag{1.145}$$

The physical sense of formulas (1.144) is clear. In the source, a mutual generation of $\boldsymbol{H}_P$ and $\boldsymbol{H}_T$ according to Theorem 6 is possible, because equations (1.145) are closed with respect to the fields they include. In the atmosphere, according to (1.145) and Theorems 2 and 4, a toroidal magnetic field is present due to the effect of boundary conditions (1.28). This field is not potential, but does not generate electric currents through the Atmosphere. This explains the occurrence of a non-potential part of the magnetic field in the essentially unconducting atmosphere of the Earth, revealed in [7, 28]. The Maxwell equations, generalized in (1.144) and (1.145) to the toroidal magnetic fields in the atmosphere is based on the experiment conducted at the world network of magnetic stations, on the one hand, and on the unclosed character of Maxwell's equations as they are, on the other hand.

Thus, the outlined theory of the concept proposed of the MGF, observed by the world network of geomagnetic stations, explains the existence of the toroidal and poloidal magnetic fields in the Earth's atmosphere and unambiguously indicates to the source of such fields as toroidal electric currents.

Chapter 2

Varying electromagnetic fields of the Earth's electromagnetic variations

Introduction

To the alternating part of the Earth's electromagnetic field we refer the MGF variations of different periods: starting with ages and finishing with short-time periods. For rather a long time, the long-period variations, observed by the world network of stations on the Earth's surface, were investigated with the use of the Gauss–Schmidt expansions for interpolation of their intensity. As is mentioned in Chapter 1, such expansions are based on the assumption of potentiality of the field of variations in the air. This automatically results in a one-modal interpretation scheme of the magnetic fields obtained after interpolation.

As a result of one-modal interpretation of only magnetic components, there arose problems associated with the existence in the Earth's atmosphere of a significant non-potential part in its intensity. This phenomenon was first detected in [28] and later confirmed in [7]. Much later Chetaev [20] with his experiments with short-period variations, their vertical electric field to be exact, discovered in the atmosphere significant values in its intensity, which should not take place according to standard Maxwell's equations.

The above-mentioned data prompted the author to initiate a research of electrodynamics of geoelectromagnetic variations with allowance for their possible two-modality.

In order not to introduce in advance the principle of variations potentiality in the air, the atmosphere conductivity when constructing the electrodynamics of an alternating field of variations was taken into account. This considerably complicated the mathematical part of the research, but directly revealed the above-mentioned problems.

2.1. On two-modal presentation of electromagnetic variations fields in the air

The research into geoelectromagnetic variations fields in the atmosphere with a weak conductivity begins with proving the following

Theorem 11. *The electromagnetic field of variations in the air is two-modal.*

Really, let an alternating magnetic field $\boldsymbol{H}$ and an electric field $\boldsymbol{E}$ of geoelectromagnetic variations be time-dependent as $e^{i\omega t}$, and external sources of variations be denoted by $\boldsymbol{j}^{\mathrm{CT}}$ and depend on time in the same manner as fields. Further let us omit the time dependence in the form of $e^{i\omega t}$ everywhere, however keeping it in mind when calculating temporal variables. Then Maxwell's equations connecting sources and fields in the air can be written down as

$$\begin{aligned} \nabla \times \boldsymbol{H} &= \sigma_0 \boldsymbol{E} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{j}^{\mathrm{CT}}, \qquad && \nabla \cdot \boldsymbol{E} = 0, \\ \nabla \times \boldsymbol{E} &= -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, && \nabla \cdot \boldsymbol{H} = 0, \end{aligned} \tag{2.1}$$

where σ_0, ε_0, μ_0 are electromagnetic constants of the air.

Formulas (2.1) allow us to introduce the vector potential $\boldsymbol{A}$ and to conventially express the electric and magnetic fields by the vector-potential:

$$\boldsymbol{H} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -i\omega\mu_0 \boldsymbol{A} + \nabla\phi. \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain

$$\begin{aligned} \nabla \times \nabla \times \boldsymbol{A} &= \sigma' \boldsymbol{E} + \boldsymbol{j}^{\mathrm{CT}}, \\ \nabla\nabla \cdot \boldsymbol{A} - \Delta \boldsymbol{A} &= -\bar{\varkappa}^2 \boldsymbol{A} + \sigma' \quad \nabla\phi + \boldsymbol{j}^{\mathrm{CT}}, \end{aligned} \tag{2.3}$$

where $\sigma' = \sigma_0 + i\omega\varepsilon_0$, $\bar{\varkappa}^2 = i\omega\mu_0\sigma_0 - \omega^2\varepsilon_0\mu_0$. Formulas (2.3) can be written down as

$$\Delta \boldsymbol{A} = \bar{\varkappa}^2 \boldsymbol{A} + \nabla(\nabla \cdot \boldsymbol{A} - \sigma'\phi) + \boldsymbol{j}^{\mathrm{CT}}. \tag{2.4}$$

It is required to introduce into (2.4) the Lorentz calibration

$$\nabla \cdot \boldsymbol{A} - \sigma'\phi = 0. \tag{2.5}$$

Then for the vector-potential $\boldsymbol{A}$ we obtain the equation

$$\Delta \boldsymbol{A} + \bar{\varkappa}^2 \boldsymbol{A} = \boldsymbol{j}^{\mathrm{CT}}, \tag{2.6}$$

which differs from (1.29) by the presence of one more term $\bar{\varkappa}^2 \boldsymbol{A}$ corresponding to the time-dependent structure of the electromagnetic field of variations. Equation (2.6) is the Helmholtz equation whose projections on the axis of the spherical coordinate system can be obtained in a similar to the previous way, keeping in mind that in (2.6) the Laplace operator is vectorial and equal to $\Delta = \nabla\nabla \cdot - \nabla \times \nabla \times$. Then

$$\Delta A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial \phi} + \bar{\varkappa}^2 A_\theta = j_\theta^{\mathrm{CT}},$$
$$\Delta A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi} + \bar{\varkappa}^2 A_\phi = j_\phi^{\mathrm{CT}}, \tag{2.7}$$
$$\Delta A_r - \frac{2}{r^2}A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2ctg\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi} + \bar{\varkappa}^2 A_r = j_r^{\mathrm{CT}}.$$

Here a scalar spherical Laplace operator should be applied to each of spherical components of the vector potential.

Equations (2.7) show that in the vector field, the vector-potential and the current, exciting variations, are subject (in a spherical case) to the property of each one of spherical components of the current to generate a respective potential component as well as derivatives of all the three potential components. Now let us turn to an auxiliary scalar function Q, introduced in Chapter 1 by formula (1.5), however this time it will be a function of four variables $Q(r,\theta,\phi,t) \in C^\infty$ and will depend on time as the magnetic and electric fields and the parameter χ from formula (1.28), for $t > 0$, will be equal to $\chi = -(i\omega\mu\sigma')^{1/2}$. Substituting (1.5) into (2.2) and taking into account the above-said about the function Q, will result in

$$\begin{aligned} \boldsymbol{H} &= \nabla\times(Q\boldsymbol{r}) + \nabla\times\nabla\times(Q\boldsymbol{r}), \\ \boldsymbol{E} &= -i\omega\mu_0(Q\boldsymbol{r}) - i\omega\mu_0\nabla\times(Q\boldsymbol{r}) + \frac{1}{\sigma'}\nabla\nabla\cdot(Q\boldsymbol{r}) + \\ &\quad \frac{1}{\sigma'}\nabla\nabla\cdot\nabla\times(Q\boldsymbol{r}). \end{aligned} \tag{2.8}$$

The latter term of the second formula from (2.8) automatically vanishes for any Q. The term

$$-i\omega\mu_0(Q\boldsymbol{r}) \tag{2.9}$$

is to be excluded.

Now, following [17], we need to introduce two modifications (modes) of the electromagnetic field: a field of the magnetic type (MT) and that of the electric type (ET). They can be introduced having formed the above modifications of the field by the rule of the source identity:

- MT field

$$\boldsymbol{H}^{\mathrm{MT}} = \nabla\times\nabla\times(Q\boldsymbol{r}), \quad \boldsymbol{E}^{\mathrm{MT}} = -i\omega\mu_0\nabla\times(Q\boldsymbol{r}); \tag{2.10}$$

- ET field

$$\boldsymbol{H}^{\mathrm{ET}} = \nabla\times(Q\boldsymbol{r}), \quad \boldsymbol{E}^{\mathrm{ET}} = -i\omega\mu_0(Q\boldsymbol{r}) + \frac{1}{\sigma'}\nabla\nabla\cdot(Q\boldsymbol{r}). \tag{2.11}$$

It is not difficult to understand that the fields defined in (2.10) and (2.11) are in agreement with the classical definition of the MT and the ET-fields [17]. At the same time it is easy to see that each of them consists of the two earlier introduced types of fields: the MT field (defined in (2.10) consists of the poloidal magnetic and the toroidal electric fields, while the ET-field—of the toroidal magnetic and poloidal electric fields. The first term from the second formula of (2.11) is compensated by the second one, which will be proved below.

Thus, the way of dividing variable fields into modifications (modes) differs from that generally accepted in the MGF. Nevertheless, the excitation of the MT and the ET modes is similar to the MGF revealed in the process of investigation. In order to detect toroidal components of the magnetic field from j_θ^{CT} and j_ϕ^{CT} components of the toroidal current in the sources, it is sufficient to compare formulas (2.7), (1.29) and (1.32).

Properties of a variable magnetic field are the same as of a constant field in that the vortices of a toroidal variable field in the air do not generate current but a poloidal field that can be essentially potential due to a weak conductivity of the atmosphere. Thus, the presence of the toroidal part of the magnetic field (the ET mode) in the air does not result in appearance of a radial current through the atmosphere. The current through the atmosphere does not essentially excite an observable magnetic field because of its minor density, i.e., of order $10^{-12} \div 10^{-14}$ A/m^2. On the other hand, a toroidal non-potential magnetic field in the air is generated by quite different current components, namely, j_θ^{CT}- and j_ϕ^{CT}-components of the toroidal current in sources of variations. How these current components occur in spherical layers: it makes no difference whether by the induction way as in the Earth, or due to the dynamo-excitation or the wind as in the ionosphere. Of importance is only the fact that the currents j_θ^{CT} and j_ϕ^{CT} flow along the spherical ionosphere surface or in the Earth's spherical layers. *A spherical* property of the toroidal current is the first cause of occurrence of the toroidal mode together with the ET-field in the air. Similarly, the spherical property is the first cause of generation of the toroidal magnetic field in the air.

Scalar components of the MT- and ET-fields can be expressed through the vector potential components and an auxiliary scalar function $Q(r, \theta, \phi, t)$ with the notation from (1.30). The MT field can be written down with the components A_θ, A_ϕ and the function Q:

$$H_\theta^{\mathrm{MT}} = \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial \theta}(Qr) = \frac{1}{r}\frac{\partial}{\partial r} r A_\phi,$$

$$H_\phi^{\mathrm{MT}} = \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}(Qr) = \frac{1}{r}\frac{\partial}{\partial r} r A_\theta,$$

$$
\begin{aligned}
H_r^{\mathrm{MT}} &= -\frac{1}{r^2 \sin\theta}\Big(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial(Qr)}{\partial\theta} + \frac{\partial}{\partial\phi}\frac{1}{\sin\theta}\frac{\partial(Qr)}{\partial\phi}\Big) \\
&\quad -\frac{1}{r\sin\theta}\Big(-\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}A_\phi + \frac{\partial A_\theta}{\partial\phi},\Big), \qquad (2.12) \\
E_\theta^{\mathrm{MT}} &= -i\omega\mu_0\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(Qr) = -i\omega\mu_0 A_\theta, \\
E_\phi^{\mathrm{MT}} &= -i\omega\mu_0\Big(-\frac{1}{r}\frac{\partial}{\partial\theta}(Qr)\Big) = -i\omega\mu_0 A_\phi,
\end{aligned}
$$

the ET-field— by the component A_r and the auxiliary function Q:

$$
\begin{aligned}
H_\theta^{\mathrm{ET}} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(Qr) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r, \\
H_\phi^{\mathrm{ET}} &= -\frac{1}{r}\frac{\partial}{\partial\theta}(Qr) = -\frac{1}{r}\frac{\partial}{\partial\theta}A_r, \\
E_\theta^{\mathrm{ET}} &= \frac{1}{\sigma'}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r^2}\frac{\partial}{\partial r}r^2(Qr) = \frac{1}{\sigma'}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r, \qquad (2.13) \\
E_\phi^{\mathrm{ET}} &= \frac{1}{\sigma'}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{1}{r^2}\frac{\partial}{\partial r}r^2(Qr) = \frac{1}{\sigma'}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r, \\
E_r^{\mathrm{ET}} &= -i\omega\mu_0(Qr) + \frac{1}{\sigma'}\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}r^2(Qr) \\
&= -i\omega\mu_0 A_r + \frac{1}{\sigma'}\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r.
\end{aligned}
$$

It is not difficult to understand that in the latter formula of (2.13) there is an induction term $-i\omega\mu_0 A_r$, corresponding to the radial current in the "Earth–Atmosphere–Ionosphere–Magnetosphere" model that is not to be found in the air due to the presence of an essentially unconducting atmosphere in the model. In [6], the compensation of the induction term by the second potential term in the case of a variable field is specially studied. To prove this fact, let us expand the internal and external functions Q in the series of the form

$$
\begin{aligned}
Q^i(r,\theta,\phi,t) &= \sum_{n=1}^{\infty} K_{n+1/2}(\varkappa r/R_0)S_n(\theta,\phi), \\
Q^e(r,\theta,\phi,7) &= \sum_{n=1}^{\infty} I_{n+1/2}(\varkappa r/R_0)S_n(\theta,\phi),
\end{aligned}
\qquad (2.14)
$$

where $\varkappa = i\omega\mu_0\sigma_0 R_0^2 - \varepsilon_0\omega^2\mu_0 R_0^2$, R_0 is the the Earth's radius. The factor $e^{i\omega t}$ in the right- and in the left-hand sides is omitted.

Formulas (2.14) as radial components include the Bessel functions of a half-integer index and the known function $S_n(\theta,\phi)$, presented by (2.14). The first and second derivatives, for example, of the function Q^i to be used later on, will have the form

$$
\begin{aligned}
\frac{\partial Q^i}{\partial r} &= \sum_{n=1}^{\infty} S_n(\theta,\phi) K'_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) = \sum_{n=1}^{\infty} S_n(\theta,\phi) \times \\
&\Big[-\frac{\varkappa}{R_0} K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{n+1/2}{r} K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big], \\
\frac{\partial^2 Q^i}{\partial r^2} &= \sum_{n=1}^{\infty} S_n(\theta,\phi) K''_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) = \sum_{n=1}^{\infty} S_n(\theta,\phi) \times \\
&\Big[\Big(\frac{(n+1/2)(n+3/2)}{r^2} + \frac{\varkappa^2}{R_0^2}\Big) K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) \frac{\varkappa}{r R_0} K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big].
\end{aligned}
\tag{2.15}
$$

Let us transform a radial component of the ET-field from (2.13):

$$
E_r^{\mathrm{ET}} = -i\omega\mu_0 Q r + \frac{1}{\sigma'}\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 Q r = \frac{1}{\sigma'}\Big[-\bar{\varkappa}^2 Q r + r\frac{\partial^2 Q}{\partial r^2} + 4\frac{\partial Q}{\partial r}\Big]. \tag{2.16}
$$

It is required to substitute in turn the expansions of the auxiliary functions Q^i and Q^e into formula (2.16) and to sum the results for obtaining the total electric field E_r^{ET}. Let us demonstrate this only with the function Q^i:

$$
\begin{aligned}
E_r^{i,\mathrm{ET}} &= \frac{1}{\sigma'} \sum_{n=1}^{\infty} S_n(\theta,\phi) \Big[-\bar{\varkappa}^2 r K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \\
& r\Big(\frac{(n+1/2)(n+3/2)}{r^2} + \frac{\varkappa^2}{R_0^2}\Big) K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \frac{\varkappa}{R_0} K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \\
& \frac{4\varkappa}{R_0} K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{4(n+1/2)}{r} K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big].
\end{aligned}
\tag{2.17}
$$

Let us present similar terms with allowance for $\varkappa^2 = \bar{\varkappa}^2 R_0^2$. As a result, we obtain

$$
\begin{aligned}
E_r^{i,\mathrm{ET}} &= \frac{1}{\sigma'} \sum_{n=1}^{\infty} S_n(\theta,\phi) \Big[-\bar{\varkappa}^2 r K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \bar{\varkappa}^2 r K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \\
& \frac{n(n-2) - 5/4}{r} K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{3\varkappa}{R_0} K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big].
\end{aligned}
\tag{2.18}
$$

It is not difficult to understand that in (2.18) the induction term disappears, only a potential part of the $E_r^{i,\mathrm{ET}}$-component remains, which for $r = R_0$ on the Earth's surface will have the form

$$E_r^{i,\mathrm{ET}} = \frac{1}{\sigma'}\sum_{n=1}^{\infty} S_n(\theta,\phi)\Big(\frac{n(n-2)-5/4}{R_0}K_{n+1/2}(\varkappa) - \frac{3\varkappa}{R_0}K_{n-1/2}(\varkappa)\Big). \quad (2.19)$$

Similar formulas can also be presented with the second auxiliary function Q^e. The result would be the same: the induction term disappears.

Thus, the EMF of the electric type consists of the toroidal magnetic and the poloidal electric fields. In this case, the ET will be

$$\boldsymbol{H}^{\mathrm{ET}} = \nabla \times (Q\boldsymbol{r}), \qquad \boldsymbol{E}^{\mathrm{ET}} = \frac{1}{\sigma'}\nabla\nabla \cdot (Q\boldsymbol{r}). \quad (2.20)$$

Formulas (2.10) and (2.20) prove Theorem 11 and completely correspond to Theorem 1 which, according to the above-said, realizes the possibility of reconstructing the poloidal and the toroidal electromagnetic fields with the help of one scalar function by a specified normal component of the magnetic field H_{Pr}^{MT}. According to (2.10) and (2.20), the electromagnetic fields of the MT- and the ET-modes are reconstructed on the Earth's surface also by means of one scalar function Q to be dependent in this case on the four variables: three spatial coordinates and time.

2.2. On a force and a non-force components of the field of variations in the air

In a variable field of electromagnetic variations, observed in the Earth's atmosphere, as follows from the previous section, two modifications of the electromagnetic field, namely, the MT- and the ET-fields, are realized.

Now let us ask which of these modifications is a force one, i.e., which one possesses a non-zero Lorentz force in a magnetic field, while an electric field possesses an electromotive induction force (e.m.f.). The answer to this question is given in the following theorem.

Theorem 12. *The electromagnetic field of the Earth's electromagnetic variations contains a force and a non-force components.*

Really, a force modification consists of a poloidal magnetic and a toroidal electromagnetic fields ($\boldsymbol{H}_P^{\mathrm{MT}}$, $\boldsymbol{E}_T^{\mathrm{MT}}$). The Lorentz force for this modification is not equal to zero, the e.m.f of induction also differs from zero. In our definition this is a MT-field, specified by formula (2.10):

$$\boldsymbol{F}_{\mathrm{L}} = [\sigma'\boldsymbol{E}_T^{\mathrm{MT}} \times \mu_0\boldsymbol{H}_P^{\mathrm{MT}}] \neq 0, \quad (2.21)$$

$$\text{e.m.f.} = \oint_L (\boldsymbol{E}_T^{\mathrm{MT}} \cdot \boldsymbol{dl}) = \int_W (\nabla \times \boldsymbol{E}_T^{\mathrm{MT}} \cdot \boldsymbol{ds}) = -\mu_0 \int_W \Big(\frac{\partial \boldsymbol{H}_P^{\mathrm{MT}}}{\partial t} \cdot \boldsymbol{ds}\Big) \neq 0.$$

A non-force modification consists of a toroidal magnetic and a poloidal electric fields ($\boldsymbol{H}_T^{ET}$, $\boldsymbol{E}_P^{ET}$). For this modification, the Lorentz force equals zero and the e.m.f of induction is also equal to zero [6]. In our definition this is an ET-mode, given by (2.20):

$$\boldsymbol{F}_{\mathrm{L}} = [\chi \boldsymbol{H}_T^{\mathrm{ET}} \times \mu_0 \boldsymbol{H}_T^{\mathrm{ET}}] \equiv 0,$$
$$\text{e.m.f.} = \int_L (\boldsymbol{E}_p^{\mathrm{ET}} \cdot \boldsymbol{dl}) = \int_W (\nabla \times \boldsymbol{E}_P^{\mathrm{ET}} \cdot \boldsymbol{ds}) \equiv 0. \tag{2.22}$$

In formulas (2.22), the Lorentz force $\boldsymbol{F}_{\mathrm{L}}$ identically equals zero because of the vectorial product of the toroidal magnetic field by itself equals zero due to the coinciding direction. The constants χ and μ_0 do not change the direction. The e.m.f of induction is also identically equal to zero because the rotor of the poloidal electric field equals zero as this field is a gradient of a certain scalar, specified by divergence of the vector ($Q\boldsymbol{r}$).

The two-modality of the electromagnetic fields observed on the Earth should be taken into consideration when applying them to geophysical prospecting with the use of natural electromagnetic fields. A non-force magnetic field does not possess the property to excite electric current in a medium, therefore a skin-effect is not characteristic of it. That is why a non-force ET-mode penetrates into the Earth essentially three times as deep as a force field [4, 5]. In the context of a non-force character of the second mode of the EMF, it is required to measure its magnetic and electric fields with *magnetometer* and *electrometer*, respectively.

Thus, Theorem 12 fixes the existence of force and non-force electromagnetic fields in a weakly conducting Earth's atmosphere. Examples of the solar-daily variations will be given below.

2.3. On boundary conditions for the fields of electromagnetic variations

When declining the idea of two-modality of an observed natural EMF, a natural factor was a known boundary condition for checking the circulation of a magnetic field along the Earth's surface around a closed contour. The supporters of such a verification usually present a well-known chain of equalities [15]:

$$\oint (\boldsymbol{H} \cdot \boldsymbol{dl}) = \int (\nabla \times \boldsymbol{H} \cdot \boldsymbol{ds}) = \int j_n \cdot ds|_{j_n=0} = 0, \tag{2.23}$$

And since current does not flow into an essentially unconducting atmosphere ($j_n \approx 0$), the product under the latter integral in (2.23) is equated to zero. This results in a zero rotor of the magnetic field under the second integral in (2.23), thus confirming the potentiality of the magnetic field observed in the atmosphere. Therefore, since a magnetic field in the air is considered to be potential, then it cannot contain a non-potential part.

These, seemingly convincing arguments resulted in the fact that toroidal magnetic components of the field in the air were deliberately excluded. However, following Chapter 1, a toroidal part of the magnetic field can nevertheless exist in the unconducting atmosphere owing to

$$\oint(\boldsymbol{H}_T^{\mathrm{ET}} \cdot \boldsymbol{dl}) = \int(\nabla \times \boldsymbol{H}_T^{\mathrm{ET}} \cdot \boldsymbol{ds}) = \int H_{Pn}^{\mathrm{MT}} \cdot ds|_{H_{Pn}^{\mathrm{MT}} \neq 0} \neq 0, \qquad (2.24)$$

because of $H_{Pn}^{\mathrm{MT}} \neq 0$ in the air. Formula (2.24), written for a toroidal magnetic field assigns a specific physical meaning to boundary conditions for the magnetic field variations. According to (1.7), a toroidal field satisfies the known property $\nabla \times \boldsymbol{H}_T = \boldsymbol{H}_P$, therefore a non-potential toroidal part of the magnetic field of variations in the air can exist. Its vortices transfer to a poloidal magnetic field, whose vortices, according to (2.23), are potential.

It is such a two-step transfer in the vortices of the observed magnetic field that explains a non-potential part of the EMF of variations in the air [7, 27] and the whole ET-mode completely.

As follows from this investigation, the source of the ET-mode is just these torodial electric currents in a spherical source, namely, j_θ^{ST}- and j_ϕ^{ST}-components. The latter (in the spherical case), in addition to the polodial part, also generate a non-potential torodial part of the magnetic field. The presence of a torodial component in the magnetic variations field brings about the ET-mode.

As was mentioned above, neither inside the Earth nor on its surface, because of a weak conductivity , thin, ideally conducting screens can exist and high-permeable masses on the Earth's surface are not observed. Therefore, boundary conditions for magnetic fields of both types on the Earth's surface are standard:

$$(\boldsymbol{H}_P^{1,\mathrm{MT}} - \boldsymbol{H}_P^{2,\mathrm{MT}})|_{r=R_0} = 0, \quad (\boldsymbol{H}_T^{1,\mathrm{ET}} - \boldsymbol{H}_T^{2,\mathrm{ET}})|_{r=R_0} = 0. \qquad (2.25)$$

Here 1 and 2 are numbers of the upper and lower parts of the surface S, R_0 is the Earth's radius.

Boundary conditions (2.25) and formula (2.24) permit the ET-modes to a toroidal magnetic field, provided it is present, to freely penetrate into the Earth's atmosphere and to be measured there by magnetometers both on the world network of magnetic observatories and at separate points of the Earth's surface in regional investigations.

Somewhat different are boundary conditions for an electric field of the ET-mode. First, in (2.19) an induction term in E_r^{ET}-component is proved to be absent. The current does not flow into the atmosphere neither from the

Earth nor from the ionosphere. Second, boundary conditions for the vertical and tangential components towards the Earth's surface in the electric field are known to be different [6]. The boundary conditions are formed based on the behavior of currents at the interface of conducting media. Let: j_n^0 be a current component that is normal from the air, j_n^{E} be a current component that is normal from the Earth to the interface, σ' be complex conductivities of the air, σ'_{E} be the Earth's complex conductivity, σ'_{I} be complex conductivity of the ionosphere, E_n^0, E_t^0 be normal and tangential components of the electric field in the air, respectively. Then

$$\begin{aligned} &j_n^0 = j_n^{\mathrm{E}}, \qquad \text{hence} \quad E_n^0 = \frac{1}{\sigma'} j_n^E, \quad \frac{1}{\sigma'_{\mathrm{E}}} E_{Pn}^{1,ET} = \frac{1}{\sigma'} E_{Pn}^{2,\mathrm{ET}}, \\ &\frac{j_t^0}{j_t^{\mathrm{E}}} = \frac{\sigma'}{\sigma'_E}, \quad \text{hence} \quad E_t^0 = \frac{1}{\sigma'_{\mathrm{E}}} j_t^{\mathrm{E}}, \quad E_{Pt}^{1,\mathrm{ET}} = E_{Pt}^{2,\mathrm{ET}}. \end{aligned} \tag{2.26}$$

Here indices 1 and 2 denote Earth and air, respectively.

Formulas (2.26) oblige choosing different conductivities as starting point of calculation of electric components of the ET-mode: in the radial component—conductivity of the air σ', in the horizontal components—conductivity of the Earth's upper layer σ'_E or conductivity of the ionosphere σ'_I for the external field. When calculating, this difference in levels of reading electric components of the ET-mode should be understood and taken into account.

From definition (2.10) for a torodial electric field of the MT-mode, automatically follow its boundary conditions:

$$(\boldsymbol{E}_T^{1,\mathrm{MT}} - \boldsymbol{E}_T^{2,\mathrm{MT}})|_{r=R_0} = 0, \qquad E_{Tn}^{\mathrm{MT}} = 0.$$

Thus, the formulated boundary conditions for electromagnetic fields make possible to construct with confidence the two-modality theories of a natural electromagnetic field of variations, bearing in mind that the spherical feature of the Earth along with that of the ionosphere as well as the spherical feature of the MGF sources and its variations allow the existence of a two-modality field in the air. A variable toroidal magnetic field in the air can be present, being non-potential, however, it does not generate electric currents neither at the expense of a magnetic nor at the expense of an electric field.

2.4. Interpolation decompositions of a two-modality electromagnetic field of variations at instants of its current system

Similar to the above, it is required to propose decompositions of a two-modality electromagnetic field of global electromagnetic variations in the air. Such decompositions are presented in [3]. Let us recall the main stages of their derivation.

One must bear in mind that sources of the field of variations can exist both in the Earth and outside it, that is why equation (2.6) decomposes into two equations (for internal and external sources):

$$\begin{aligned} \Delta \boldsymbol{A}^i + \bar{\varkappa}^2 \boldsymbol{A}^i &= \boldsymbol{j}_i^{\mathrm{EC}}, \\ \Delta \boldsymbol{A}^e + \bar{\varkappa}^2 \boldsymbol{A}^e &= \boldsymbol{j}_e^{\mathrm{EC}}. \end{aligned} \tag{2.27}$$

Solutions to these equations in the air are of the form

$$\begin{aligned} \boldsymbol{A}^i(p) &= \frac{1}{4\pi} \int\limits_W \boldsymbol{j}_i^{\mathrm{EC}}(q) \frac{e^{-i\bar{\varkappa}R(p,q)}}{R(p,q)} d\vartheta_q, \qquad r \geq R_0, \\ \boldsymbol{A}^e(q) &= \frac{1}{4\pi} \int\limits_G \boldsymbol{j}_e^{\mathrm{EC}}(p) \frac{e^{i\bar{\varkappa}R(p,q)}}{R(p,q)} d\vartheta_p, \qquad r \leq R_0 + h. \end{aligned} \tag{2.28}$$

Here p is a point outside the Earth, q is a point in the Earth, h is a height up to the ionosphere, $\boldsymbol{j}_i^{\mathrm{EC}}(q)$ are electric currents in the Earth, $\boldsymbol{j}_e^{\mathrm{EC}}(p)$ are electric currents in the ionosphere and outside it, R_0 is the Earth's radius.

The first integral represents a potential of internal sources. It is analytical at all the points p for $r \geq R_0$. At the internal points of the sphere W for $R \to 0$ a peculiarity of the form $1/R$ holds, which is excluded by an appropriate selection of a spherical Bessel function of a half-integer index tending to zero in zero. This is also in agreement with the physics of the event, as sources of most of variations are damping with depth due to the skin-effect since they are induced by the field of external origin. Integral (2.28) provides the fulfilment of the radiation condition for the fields of external sources at infinity. It is analytical at all the points q for $r \leq R_0 + h$ except for the points, where $R = 0$, having a peculiarity of the form $1/R$ to be excluded by an appropriate selection of the Bessel function. This integral has a peculiarity at infinity thus allowing one to provide a prescribed growth of the field when approaching an external source. Since the integral in question is applied to a limited domain, occupied with the atmosphere, the peculiarity occurring at infinity does not directly influence the calculation of the field in the atmosphere.

Similar to the known expansions of the function $1/R$, it is now needed to expand in spherical functions the fundamental solutions $\exp(\pm\, i\bar{\varkappa}R)/R$. Clearly, due to symmetry, these expansions should differ from the Gauss expansions only by the law of a change in the spherical coordinates r and r' in the manner required by the physics of sources. An important factor is the fulfilment of the condition of transferring the new expansions to the Gauss expansions for the fields of external and internal sources, respectively, for $\bar{\varkappa}^2 \to 0$.

Thus, let us write down expansions of required functions using as radial the spherical Bessel functions in the Morze definition:

$$\begin{aligned}\exp(-i\bar{\varkappa}R)/R = \sum_{n=1}^{\infty}\sum_{m=0}^{n} & \bar{c}_n^m \cos m\varphi \cos m\varphi' P_n^m(\cos\theta) P_n^m(\cos\theta') \times \\ & K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big) + \\ & \bar{c}_n^m \sin m\varphi \sin m\varphi' P_n^m(\cos\theta) P_n^m(\cos\theta') \times \\ & K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big), \qquad r \ge R_0. \qquad (2.29)\end{aligned}$$

Similarly, for the function with a positive exponent

$$\begin{aligned}\exp(i\bar{\varkappa}R)/R = \sum_{n=1}^{\infty}\sum_{m=0}^{n} & \bar{c}_n^m \cos m\varphi \cos m\varphi' P_n^m(\cos\theta) P_n^m(\cos\theta') \times \\ & I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big) + \\ & \bar{c}_n^m \sin m\varphi \sin m\varphi' P_n^m(\cos\theta) P_n^m(\cos\theta') \times \\ & I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big), \qquad r \le R_0 + h. \qquad (2.30)\end{aligned}$$

In formulas (2.29) and (2.30), the coefficient $\bar{c}_n^m$ means the number of combinations from n by m. If the wave number $\varkappa$ tends to zero, then due to the asymptotic behavior of the spherical Bessel functions with small arguments, products of these functions of the first and second kinds yield a needed relation of radii, and the formulas as a whole, transfer to the Gauss expansions for the function $1/R$.

Expansion (2.29) is used when integrating over the domain W (the Earth), therefore the functions of the coordinates with a prime, entering it, exclude the peculiarity in zero. Functions with coordinates without primes do not realize this peculiarity because the expansion in them is used only for $r \ge R_0$.

Expansion (2.30) is applied when integrating over the domain G (the ionosphere), therefore the functions of the coordinates with a prime used for integration exclude this peculiarity at infinity. The functions of coordinates without prime are given for $r \le R_0 + h$, therefore their peculiarity is not realized at infinity: they tend in zero to zero.

The convergence of expansions (2.29) and (2.30) for $r = R_0$ is not worse than that of Fourier series [17], which converge absolutely and uniformly, that is why their substitution into integrals (2.28) changing places of integration and summation are possible. Let us make use of this possibility without reserve.

In the rectangular coordinate system, fixed at the Earth's center, integrals (2.28) can be expanded in components along the coordinate axes:

$$
\begin{aligned}
\boldsymbol{A}_x^i(p) &= \frac{1}{4\pi}\int\limits_W j_{ix}^{\mathrm{EC}}(q)\frac{e^{-i\bar{\varkappa}R(p,q)}}{R(p,q)}dv_q,\\
\boldsymbol{A}_y^i(p) &= \frac{1}{4\pi}\int\limits_W j_{iy}^{\mathrm{EC}}(q)\frac{e^{-i\bar{\varkappa}R(p,q)}}{R(p,q)}dv_q,\\
\boldsymbol{A}_z^i(p) &= \frac{1}{4\pi}\int\limits_W j_{iz}^{\mathrm{EC}}(q)\frac{e^{-i\bar{\varkappa}R(p,q)}}{R(p,q)}dv_q, \qquad r \geq R_0,\\
\boldsymbol{A}_x^e(q) &= \frac{1}{4\pi}\int\limits_G j_{ex}^{\mathrm{EC}}(p)\frac{e^{i\bar{\varkappa}R(p,q)}}{R(p,q)}dv_p,\\
\boldsymbol{A}_y^e(q) &= \frac{1}{4\pi}\int\limits_G j_{ey}^{\mathrm{EC}}(p)\frac{e^{i\bar{\varkappa}R(p,q)}}{R(p,q)}dv_p,\\
\boldsymbol{A}_z^e(q) &= \frac{1}{4\pi}\int\limits_G j_{ez}^{\mathrm{EC}}(p)\frac{e^{i\bar{\varkappa}R(p,q)}}{R(p,q)}dv_p, \qquad r \leq R_0 + h.
\end{aligned}
\tag{2.31}
$$

It is now required to substitute expansions (2.29) and (2.30) into (2.31), to gather under integrals the functions of coordinates with a prime. This allows us to denote integrals with constants. For constant internal sources, the following presentations are valid:

$$
\begin{aligned}
\varkappa_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_W j_{ix}^{\mathrm{EC}}(q)\cos m\varphi' P_n^m(\cos\theta') I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
\rho_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_W j_{ix}^{\mathrm{EC}}(q)\sin m\varphi' P_n^m(\cos\theta') I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
\mu_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_W j_{iy}^{\mathrm{EC}}(q)\cos m\varphi' P_n^m(\cos\theta') I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
\nu_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_W j_{iy}^{\mathrm{EC}}(q)\sin m\varphi' P_n^m(\cos\theta') I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
u_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_W j_{iz}^{\mathrm{EC}}(q)\cos m\varphi' P_n^m(\cos\theta') I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
\upsilon_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_W j_{iz}^{\mathrm{EC}}(q)\sin m\varphi' P_n^m(\cos\theta') I_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv'.
\end{aligned}
\tag{2.32}
$$

For arbitrary constant external sources it is necessary to introduce similar presentations:

$$
\begin{aligned}
a_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_G j_{ex}^{\mathrm{EC}}(p)\cos m\varphi' P_n^m(\cos\theta')K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
b_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_G j_{ex}^{\mathrm{EC}}(p)\sin m\varphi' P_n^m(\cos\theta')K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
c_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_G j_{ey}^{\mathrm{EC}}(p)\cos m\varphi' P_n^m(\cos\theta')K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
d_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_G j_{ey}^{\mathrm{EC}}(p)\sin m\varphi' P_n^m(\cos\theta')K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
e_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_G j_{ez}^{\mathrm{EC}}(p)\cos m\varphi' P_n^m(\cos\theta')K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv',\\
f_n^m &= \frac{\bar{c}_n^m}{4\pi}\int\limits_G j_{ez}^{\mathrm{EC}}(p)\sin m\varphi' P_n^m(\cos\theta')K_{n+1/2}\Big(\frac{\varkappa r'}{R_0}\Big)dv'.
\end{aligned}
\tag{2.33}
$$

With allowance for the notations from (2.32) and (2.33) as well as for formulas of transferring the rectangular components to spherical ones (2.2) by substituting expansions (2.29) and (2.30) into (2.31), it is possible to obtain expansions for potentials of internal and external sources:

$$
\begin{aligned}
A_\theta^i(p) = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[&(\varkappa_n^m\cos m\varphi+\rho_n^m\sin m\varphi)\cos\theta\cos\varphi+\\
&(\mu_n^m\cos m\varphi+\nu_n^m\sin m\varphi)\cos\theta\cos\varphi-\\
&(u_n^m\cos m\varphi+v_n^m\sin m\varphi)\sin\theta\big]P_n^m(\cos\theta)K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),\\
A_\varphi^i(p) = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[&-(\varkappa_n^m\cos m\varphi+\rho_n^m\sin m\varphi)\sin\varphi+\\
&(\mu_n^m\cos m\varphi+\nu_n^m\sin m\varphi)\cos\varphi\big]P_n^m(\cos\theta)K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),\\
A_r^i(p) = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[&(\varkappa_n^m\cos m\varphi+\rho_n^m\sin m\varphi)\sin\theta\cos\varphi+\\
&(\mu_n^m\cos m\varphi+\nu_n^m\sin m\varphi)\sin\theta\sin\varphi+\\
&(u_n^m\cos m\varphi+v_n^m\sin m\varphi)\cos\theta\big]P_n^m(\cos\theta)K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),\quad r\geq R_0,
\end{aligned}
\tag{2.34}
$$

$$A_\theta^e(q) = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[(a_n^m \cos m\varphi + b_n^m \sin m\varphi)\cos\theta\cos\varphi +$$
$$(c_n^m \cos m\varphi + d_n^m \sin m\varphi)\cos\theta\cos\varphi -$$
$$(e_n^m \cos m\varphi + f_n^m \sin m\varphi)\sin\theta\big]P_n^m(\cos\theta) I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),$$

$$A_\varphi^e(q) = \sum_{n=1}^{\infty}\sum_{m=0}^{n} [-(a_n^m \cos m\varphi + b_n^m \sin m\varphi)\sin\varphi +$$
$$(c_n^m \cos m\varphi + d_n^m \sin m\varphi)\cos\varphi\big]P_n^m(\cos\theta) I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),$$

$$A_r^e(q) = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[(a_n^m \cos m\varphi + b_n^m \sin m\varphi)\sin\theta\cos\varphi +$$
$$(c_n^m \cos m\varphi + d_n^m \sin m\varphi)\sin\theta\sin\varphi +$$
$$(e_n^m \cos m\varphi + f_n^m \sin m\varphi)\cos\theta\big]P_n^m(\cos\theta) I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big), \quad r \le R_0 + h.$$

The analysis of expansions of the potentials of internal and external sources (2.34) has revealed that in the given approach all the three components of the potential both of internal and external sources are expressed through the same coefficients, which are integrals of the respective components of the current density, generating both the poloidal and the toroidal parts of the observed field. The obtained expansions of the potentials contain 12 types of constant coefficients subject to determination. For their determination, one reasonably needs more information about the observed field, i.e., the data obtained from a greater number of points. Below we can check that the lack of the number of points benefits in the components of the field needed for measurements. In our case, it is sufficient to measure two components of a magnetic field: one vertical and one tangential in order to reconstruct all the magnetic and electric components of the field, to separate fields from external and internal sources, as evidenced by the corresponding theorem. This fact is of importance for the current geophysical practice, because most of stations of the world network measure only magnetic components.

In order to obtain expansions of the EMF of the magnetic type of internal sources, it is necessary to substitute the first three components of the potential from (2.34) into (2.12). As a result, we come to

$$H_\theta^{i,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} \big[(\varkappa_n^m \cos m\varphi + \rho_n^m \sin m\varphi)\sin\varphi - \tag{2.35}$$
$$(\mu_n^m \cos m\varphi + \nu_n^m \sin m\varphi)\cos\varphi\big] \times$$
$$P_n^m(\cos\theta)\Big(\frac{n-1/2}{r}K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{\varkappa}{R_0}K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big),$$

$$
\begin{aligned}
H_\varphi^{i,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} & \big[(\varkappa_n^m \cos m\varphi + \rho_n^m \sin m\varphi)\cos\theta\cos\varphi + \\
& (\mu_n^m \cos m\varphi + \nu_n^m \sin m\varphi)\cos\theta\sin\varphi - \\
& (u_n^m \cos m\varphi + v_n^m \sin m\varphi)\sin\theta\big] \times \\
& P_n^m(\cos\theta)\Big(-\frac{n-1/2}{r}K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{\varkappa}{R_0}K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big),
\end{aligned}
$$

$$
\begin{aligned}
H_r^{i,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} & \Big[\varkappa_n^m\Big(\cos m\varphi\sin\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \\
& m\sin m\varphi\cos\varphi\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) + \\
& \rho_n^m\Big(\sin m\varphi\sin\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m\cos m\varphi\cos\varphi\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& \mu_n^m\Big(\cos m\varphi\cos\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + m\sin m\varphi\sin\varphi\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& \nu_n^m\Big(\sin m\varphi\cos\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} - m\cos m\varphi\sin\varphi\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& (v_n^m\cos m\varphi - u_n^m\sin m\varphi)mP_n^m(\cos\theta)\Big]\frac{1}{r}K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),
\end{aligned}
$$

$$
\begin{aligned}
E_\theta^{i,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} & -i\omega\mu_0\big[(\varkappa_n^m \cos m\varphi + \rho_n^m \sin m\varphi)\cos\theta\cos\varphi + \\
& (\mu_n^m \cos m\varphi + \nu_n^m \sin m\varphi)\cos\theta\sin\varphi - \\
& (u_n^m \cos m\varphi + v_n^m \sin m\varphi)\sin\theta\big]P_n^m(\cos\theta)K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),
\end{aligned}
$$

$$
\begin{aligned}
E_\varphi^{i,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} & i\omega\mu_0\big[(\varkappa_n^m \cos m\varphi + \rho_n^m \sin m\varphi)\sin\varphi - \\
& (\mu_n^m \cos m\varphi + \nu_n^m \sin m\varphi)\cos\varphi\big]P_n^m(\cos\theta)K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big).
\end{aligned}
$$

The ET-field of internal sources has the following expansions

$$
\begin{aligned}
H_\theta^{i,\mathrm{ET}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n} & \big[-\varkappa_n^m(m\sin m\varphi\cos\varphi + \cos m\varphi\sin\varphi) + \\
& \rho_n^m(m\cos m\varphi\cos\varphi - \sin m\varphi\sin\varphi) - \\
& \mu_n^m(m\sin m\varphi\sin\varphi - \cos m\varphi\cos\varphi) + \\
& \nu_n^m(m\cos m\varphi\sin\varphi + \sin m\varphi\cos\varphi) + \\
& (v_n^m\cos m\varphi - u_n^m\sin m\varphi)m\operatorname{ctg}\theta\big]P_n^m(\cos\theta)\frac{1}{r}K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),
\end{aligned}
\tag{2.36}
$$

$$
\begin{aligned}
H_\varphi^{i,\mathrm{ET}} = &-\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[\varkappa_n^m \cos m\varphi \cos\varphi \times\\
&\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&\rho_n^m \sin m\varphi \cos\varphi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&\mu_n^m \cos m\varphi \sin\varphi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&\nu_n^m \sin m\varphi \sin\varphi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&u_n^m \cos m\varphi \Big(\cos\theta \frac{\partial}{\partial\theta} P_n^m(\cos\theta) - \sin\theta P_n^m(\cos\theta)\Big) +\\
&v_n^m \sin m\varphi \Big(\cos\theta \frac{\partial}{\partial\theta} P_n^m(\cos\theta) - \sin\theta P_n^m(\cos\theta)\Big)\Big] \frac{1}{r} K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),\\
E_\theta^{i,\mathrm{ET}} = &\frac{-1}{\sigma' r}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[\varkappa_n^m \cos m\varphi \cos\varphi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&\rho_n^m \sin m\varphi \cos\varphi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&\mu_n^m \cos m\varphi \sin\varphi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&\nu_n^m \sin m\varphi \sin\varphi \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +\\
&u_n^m \cos m\varphi \Big(\cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big) +\\
&v_n^m \sin m\varphi \Big(\cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big)\Big] \times\\
&\Big(\frac{n-3/2}{r} K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \frac{\varkappa}{R_0} K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big),\\
E_\varphi^{i,\mathrm{ET}} = &\frac{-1}{\sigma' r}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big\{\big[-\varkappa_n^m (m \sin m\varphi \cos\varphi + \cos m\varphi \sin\varphi) +\\
&\rho_n^m (m \cos m\varphi \cos\varphi - \sin m\varphi \sin\varphi) -\\
&\mu_n^m (m \sin m\varphi \sin\varphi - \cos m\varphi \cos\varphi) +\\
&\nu_n^m (m \cos m\varphi \sin\varphi + \sin m\varphi \cos\varphi)\big] P_n^m(\cos\theta) -\\
&(u_n^m \sin m\varphi - v_n^m \cos m\varphi) m \operatorname{ctg}\theta P_n^m(\cos\theta)\Big\} \times\\
&\Big(\frac{n-3/2}{r} K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \frac{\varkappa}{R_0} K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big),
\end{aligned}
$$

$$E_r^{i,\mathrm{ET}} = \frac{1}{\sigma' r}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(\varkappa_n^m \cos m\varphi + \rho_n^m \sin m\varphi)\sin\theta\cos\varphi +$$
$$(\mu_n^m \cos m\varphi + \nu_n^m \sin m\varphi)\sin\theta\sin\varphi +$$
$$(u_n^m \cos m\varphi + v_n^m \sin m\varphi)\cos\theta\big]P_n^m(\cos\theta)\times$$
$$\Big[\Big(-\bar{\varkappa}^2 r^2 + n^2 - 9/4 + \frac{\varkappa^2 r^2}{R_0^2}\Big)\frac{1}{r}K_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{\varkappa}{R_0}K_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big].$$

The analysis of the latter formulas has revealed that both types of the fields are expressed through the same coefficients. In addition, the coefficients of expansions (2.35) and (2.36) testify to the fact that it is sufficient to measure the vertical component of the magnetic field $H_r^{i,\mathrm{MT}}$ for reconstructing the whole electromagnetic field of the MT- and ET-modes of internal sources. This essentially amplifies the value of Theorem 1, as it solves, in addition, the problem of reconstructing variables of electromagnetic fields that are regular with respect to time. On the Earth's surface, the electric field of the ET-mode is potential for $r = R_0$. The induction part of the radial component $E_r^{i,ET}$ has been compensated.

The fields of the magnetic and electric types from external sources in expansions are written down as follows.

The MT-field for $r \le R_0 + h$:

$$H_\theta^{e,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m \cos m\varphi + b_n^m \sin m\varphi)\sin\varphi - \tag{2.37}$$
$$(c_n^m \cos m\varphi + d_n^m \sin m\varphi)\cos\varphi\big]\times$$
$$P_n^m(\cos\theta)\Big(\frac{\varkappa}{R_0}I_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{n-1/2}{r}I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big),$$
$$H_\varphi^{e,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m \cos m\varphi + b_n^m \sin m\varphi)\cos\theta\cos\varphi +$$
$$(c_n^m \cos m\varphi + d_n^m \sin m\varphi)\cos\theta\sin\varphi -$$
$$(e_n^m \cos m\varphi + f_n^m \sin m\varphi)\sin\theta\big]\times$$
$$P_n^m(\cos\theta)\Big(\frac{\varkappa}{R_0}I_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big) - \frac{n-1/2}{r}I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big),$$
$$H_r^{e,\mathrm{MT}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[a_n^m\Big(\cos m\varphi\sin\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} -$$
$$\sin m\varphi\cos\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) +$$
$$b_n^m\Big(\sin m\varphi\sin\varphi\frac{\partial}{\partial\theta}P_n^m(\cos\theta) + \cos m\varphi\cos\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) -$$

$$
\begin{aligned}
& c_n^m\Big(\cos m\varphi\cos\varphi\frac{\partial}{\partial\theta}P_n^m(\cos\theta)+\sin m\varphi\sin\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big)-\\
& d_n^m\Big(\sin m\varphi\cos\varphi\frac{\partial}{\partial\theta}P_n^m(\cos\theta)-\cos m\varphi\sin\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big)-\\
& (f_n^m\cos m\varphi-e_n^m\sin m\varphi)mP_n^m(\cos\theta)\Big]\frac{1}{r}I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),
\end{aligned}
$$

$$
\begin{aligned}
E_\theta^{e,\mathrm{MT}} = & -i\omega\mu_0\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m\cos m\varphi+b_n^m\sin m\varphi)\cos\theta\cos\varphi+\\
& (c_n^m\cos m\varphi+d_n^m\sin m\varphi)\cos\theta\sin\varphi-\\
& (e_n^m\cos m\varphi+f_n^m\sin m\varphi)\sin\theta\big]P_n^m(\cos\theta)I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),\\
E_\varphi^{e,\mathrm{MT}} = & \, i\omega\mu_0\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m\cos m\varphi+b_n^m\sin m\varphi)\sin\varphi-\\
& (c_n^m\cos m\varphi+d_n^m\sin m\varphi)\cos\varphi\big]P_n^m(\cos\theta)I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big).
\end{aligned}
$$

The ET-field of external sources for $r \le R_0 + h$ is calculated in a similar way:

$$
\begin{aligned}
H_\theta^{e,\mathrm{ET}} = & \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[-a_n^m(m\sin m\varphi\cos\varphi+\cos m\varphi\sin\varphi)+\\
& b_n^m(m\cos m\varphi\cos\varphi-\sin m\varphi\sin\varphi)-\\
& c_n^m(m\sin m\varphi\sin\varphi-\cos m\varphi\cos\varphi)+\\
& d_n^m(m\cos m\varphi\sin\varphi+\sin m\varphi\cos\varphi)-\\
& (e_n^m\sin m\varphi-f_n^m\cos m\varphi)m\operatorname{ctg}\theta\big]P_n^m(\cos\theta)\frac{1}{r}I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),
\end{aligned}
\tag{2.38}
$$

$$
\begin{aligned}
H_\varphi^{e,\mathrm{ET}} = & -\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[a_n^m\cos m\varphi\cos\varphi\times\\
& \Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\Big)+\\
& b_n^m\sin m\varphi\cos\varphi\Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\Big)+\\
& c_n^m\cos m\varphi\sin\varphi\Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\Big)+\\
& d_n^m\sin m\varphi\sin\varphi\Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\Big)+\\
& e_n^m\cos m\varphi\Big(\cos\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)-\sin\theta P_n^m(\cos\theta)\Big)+\\
& f_n^m\sin m\varphi\Big(\cos\theta\frac{\partial}{\partial\theta}P_n^m(\cos\theta)-\sin\theta P_n^m(\cos\theta)\Big)\Big]\frac{1}{r}I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big),
\end{aligned}
$$

$$
\begin{aligned}
E_\theta^{e,\mathrm{ET}} &= \frac{-1}{\sigma' r}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[a_n^m \cos m\varphi \cos\varphi \times \\
&\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) + \\
&b_n^m \sin m\varphi \cos\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) + \\
&c_n^m \cos m\varphi \sin\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) + \\
&d_n^m \sin m\varphi \sin\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) + \\
&e_n^m \cos m\varphi\Big(\cos\theta \frac{\partial}{\partial\theta} P_n^m(\cos\theta) - \sin\theta P_n^m(\cos\theta)\Big) + \\
&f_n^m \sin m\varphi\Big(\cos\theta \frac{\partial}{\partial\theta} P_n^m(\cos\theta) - \sin\theta P_n^m(\cos\theta)\Big)\Big] \times \\
&\Big(-\frac{\varkappa}{R_0} I_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \frac{n-3/2}{r} I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big), \\
E_\varphi^{e,\mathrm{ET}} &= \frac{-1}{\sigma' r}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[-a_n^m(m\sin m\varphi\cos\varphi + \cos m\varphi \sin\varphi) + \\
&b_n^m(m\cos m\varphi\cos\varphi - \sin m\varphi\sin\varphi) - \\
&c_n^m(m\sin m\varphi\sin\varphi - \cos m\varphi\cos\varphi) + \\
&d_n^m(m\cos m\varphi\sin\varphi + \sin m\varphi\cos\varphi) - \\
&(e_n^m \sin m\varphi - f_n^m \cos m\varphi) m \operatorname{ctg}\theta\big] P_n^m(\cos\theta) \times \\
&\Big(-\frac{\varkappa}{R_0} I_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \frac{n-3/2}{r} I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big), \\
E_r^{e,\mathrm{ET}} &= \frac{1}{\sigma' r}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m \cos m\varphi + b_n^m \sin m\varphi)\sin\theta\cos\varphi + \\
&(c_n^m \cos m\varphi + d_n^m \sin m\varphi)\sin\theta\sin\varphi + \\
&(e_n^m \cos m\varphi + f_n^m \sin m\varphi)\cos\theta\big] P_n^m(\cos\theta) \times \\
&\Big[\Big(-\bar{\varkappa}^2 r^2 + n^2 - 9/4 + \frac{\varkappa^2 r^2}{R_0^2}\Big)\frac{1}{r} I_{n+1/2}\Big(\frac{\varkappa r}{R_0}\Big) + \frac{\varkappa}{R_0} I_{n-1/2}\Big(\frac{\varkappa r}{R_0}\Big)\Big].
\end{aligned}
$$

The ET-field from external sources is also expressed through the same coefficients. The ET-field is completely poloidal, the induction part of the radial component of the electric field on the Earth's surface being compensated for $r = R_0$, as $\bar{\varkappa}^2 R_0^2 = \varkappa^2$.

The total field of the MT- and the ET-modes is obtained by superposition of fields of internal and external sources. As a result, we arrive at the equations, separating unknown coefficients:

$$\begin{cases} \varkappa_n^m \dfrac{1}{r}\, K_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + a_n^m \dfrac{1}{r} I_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = i_n^m, \\ -\varkappa_n^m \dfrac{\varkappa}{R_0} K_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + a_n^m \dfrac{\varkappa}{R_0} I_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = \bar{i}_n^m, \end{cases}$$

$$\begin{cases} \rho_n^m \dfrac{1}{r} K_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + b_n^m \dfrac{1}{r} I_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = j_n^m, \\ -\rho_n^m \dfrac{\varkappa}{R_0} K_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + b_n^m \dfrac{\varkappa}{R_0} I_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = \bar{j}_n^m, \end{cases}$$

$$\begin{cases} \mu_n^m \dfrac{1}{r} K_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + c_n^m \dfrac{1}{r} I_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = k_n^m, \\ -\mu_n^m \dfrac{\varkappa}{R_0} K_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + c_n^m \dfrac{\varkappa}{R_0} I_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = \bar{k}_n^m, \end{cases} \tag{2.39}$$

$$\begin{cases} \nu_n^m \dfrac{1}{r} K_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + d_n^m \dfrac{1}{r} I_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = l_n^m, \\ -\nu_n^m \dfrac{\varkappa}{R_0} K_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + d_n^m \dfrac{\varkappa}{R_0} I_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = \bar{l}_n^m, \end{cases}$$

$$\begin{cases} u_n^m \dfrac{1}{r} K_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + e_n^m \dfrac{1}{r} I_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = q_n^m, \\ -u_n^m \dfrac{\varkappa}{R_0} K_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + e_n^m \dfrac{\varkappa}{R_0} I_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = \bar{q}_n^m, \end{cases}$$

$$\begin{cases} v_n^m \dfrac{1}{r} K_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + f_n^m \dfrac{1}{r} I_{n+1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = p_n^m, \\ -v_n^m \dfrac{\varkappa}{R_0} K_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) + f_n^m \dfrac{\varkappa}{R_0} I_{n-1/2}\Big(\dfrac{\varkappa r}{R_0}\Big) = \bar{p}_n^m. \end{cases}$$

The total field of the MT-mode on the Earth's surface for $r = R_0$ looks like

$$\begin{aligned} H_\theta^{\mathrm{MT}} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \big[& \big(\bar{i}_n^m - (n-1/2) i_n^m\big) \cos m\varphi \sin\varphi + \\ & \big(\bar{j}_n^m - (n-1/2) j_n^m\big) \sin m\varphi \sin\varphi - \\ & \big(\bar{k}_n^m - (n-1/2) k_n^m\big) \cos m\varphi \cos\varphi - \\ & \big(\bar{l}_n^m - (n-1/2) l_n^m\big) \sin m\varphi \cos\varphi \big] P_n^m(\cos\theta), \end{aligned} \tag{2.40}$$

$$\begin{aligned} H_\varphi^{\mathrm{MT}} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \big[& \big(\bar{i}_n^m - (n-1/2) i_n^m\big) \cos m\varphi \cos\varphi \cos\theta + \\ & \big(\bar{j}_n^m - (n-1/2) j_n^m\big) \sin m\varphi \cos\varphi \cos\theta + \\ & \big(\bar{k}_n^m - (n-1/2) k_n^m\big) \cos m\varphi \sin\varphi \cos\theta + \\ & \big(\bar{l}_n^m - (n-1/2) l_n^m\big) \sin m\varphi \sin\varphi \cos\theta - \end{aligned}$$

$$
\begin{aligned}
&\big(\bar{q}_n^m - (n-1/2)q_n^m\big)\cos m\varphi\sin\theta - \\
&\big(\bar{p}_n^m - (n-1/2)p_n^m\big)\sin m\varphi\sin\theta\big] P_n^m(\cos\theta),
\end{aligned}
$$

$$
\begin{aligned}
H_r^{\mathrm{MT}} = {} & \sum_{n=1}^{\infty}\sum_{m=0}^{n} i_n^m\Big(\cos m\varphi\sin\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \\
& \sin m\varphi\cos\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) + \\
& j_n^m\Big(\sin m\varphi\sin\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \cos m\varphi\cos\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& k_n^m\Big(\cos m\varphi\cos\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} + \sin m\varphi\sin\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& l_n^m\Big(\sin m\varphi\cos\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \cos m\varphi\sin\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big) - \\
& (p_n^m\cos m\varphi - q_n^m\sin m\varphi) m P_n^m(\cos\theta),
\end{aligned}
$$

$$
\begin{aligned}
E_\theta^{\mathrm{MT}} = {} & -i\omega\mu_0 R_0\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big(i_n^m\cos m\varphi\cos\varphi\cos\theta + j_n^m\sin m\varphi\cos\varphi\cos\theta + \\
& k_n^m\cos m\varphi\sin\varphi\cos\theta + l_n^m\sin m\varphi\sin\varphi\cos\theta - \\
& q_n^m\cos m\varphi\sin\theta - p_n^m\sin m\varphi\sin\theta\Big) P_n^m(\cos\theta),
\end{aligned}
$$

$$
\begin{aligned}
E_\varphi^{\mathrm{MT}} = {} & i\omega\mu_0 R_0\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big(i_n^m\cos m\varphi\sin\varphi + j_n^m\sin m\varphi\sin\varphi - \\
& k_n^m\cos m\varphi\cos\varphi - l_n^m\sin m\varphi\cos\varphi\Big) P_n^m(\cos\theta),
\end{aligned}
$$

The total field of the ET-mode on the Earth's surface for $r = R_0$ has the form

$$
\begin{aligned}
H_\theta^{\mathrm{ET}} = {} & \sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[-i_n^m(m\sin m\varphi\cos\varphi + \cos m\varphi\sin\varphi) + \\
& j_n^m(m\cos m\varphi\cos\varphi - \sin m\varphi\sin\varphi) - \\
& k_n^m(m\sin m\varphi\sin\varphi - \cos m\varphi\cos\varphi) + \\
& l_n^m(m\cos m\varphi\sin\varphi + \sin m\varphi\cos\varphi) - \\
& (q_n^m\sin m\varphi - p_n^m\cos m\varphi) m\operatorname{ctg}\theta\Big] P_n^m(\cos\theta),
\end{aligned}
\tag{2.41}
$$

$$
\begin{aligned}
H_\varphi^{\mathrm{ET}} = {} & -\sum_{n=1}^{\infty}\sum_{m=0}^{n} i_n^m\cos m\varphi\cos\varphi \times \\
& \Big(\cos\theta P_n^m(\cos\theta) + \sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +
\end{aligned}
$$

$$j_n^m \sin m\varphi \cos\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$k_n^m \cos m\varphi \sin\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$l_n^m \sin m\varphi \sin\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$q_n^m \cos m\varphi\Big(\cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big) +$$
$$p_n^m \sin m\varphi\Big(\cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big),$$

$$E_\theta^{\mathrm{ET}} = \frac{1}{\sigma'_{\mathrm{E}} R_0} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \big(\bar{i}_n^m - (n-3/2) i_n^m\big) \cos m\varphi \cos\varphi \times$$
$$\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$\big(\bar{j}_n^m - (n-3/2) j_n^m\big) \sin m\varphi \cos\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$\big(\bar{k}_n^m - (n-3/2) k_n^m\big) \cos m\varphi \sin\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$\big(\bar{l}_n^m - (n-3/2) l_n^m\big) \sin m\varphi \sin\varphi\Big(\cos\theta P_n^m(\cos\theta) + \sin\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) +$$
$$\big(\bar{q}_n^m - (n-3/2) q_n^m\big) \cos m\varphi\Big(\cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big) +$$
$$\big(\bar{p}_n^m - (n-3/2) p_n^m\big) \sin m\varphi\Big(\cos\theta \frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big),$$

$$E_\varphi^{\mathrm{ET}} = \frac{1}{\sigma'_{\mathrm{E}} R_0} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \Big[-\big(\bar{i}_n^m - (n-3/2) i_n^m\big) \times$$
$$(m \sin m\varphi \cos\varphi + \cos m\varphi \sin\varphi) +$$
$$\big(\bar{j}_n^m - (n-3/2) j_n^m\big)(m \cos m\varphi \cos\varphi - \sin m\varphi \sin\varphi) -$$
$$\big(\bar{k}_n^m - (n-3/2) k_n^m\big)(m \sin m\varphi \sin\varphi - \cos m\varphi \cos\varphi) +$$
$$\big(\bar{l}_n^m - (n-3/2) l_n^m\big)(m \cos m\varphi \sin\varphi + \sin m\varphi \cos\varphi) -$$
$$\big(\bar{q}_n^m - (n-3/2) q_n^m\big) \sin m\varphi m \operatorname{ctg}\theta +$$
$$\big(\bar{p}_n^m - (n-3/2) p_n^m\big) \cos m\varphi m \operatorname{ctg}\theta\Big] P_n^m(\cos\theta),$$

$$E_r^{\mathrm{ET}} = \frac{1}{\sigma' R_0} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \Big[\big(\bar{i}_n^m + (n^2 - 9/4) i_n^m\big) \cos m\varphi \cos\varphi \sin\theta +$$
$$\big(\bar{j}_n^m + (n^2 - 9/4) j_n^m\big) \sin m\varphi \cos\varphi \sin\theta +$$
$$\big(\bar{k}_n^m + (n^2 - 9/4) k_n^m\big) \cos m\varphi \sin\varphi \sin\theta +$$
$$\big(\bar{l}_n^m + (n^2 - 9/4) l_n^m\big) \sin m\varphi \sin\varphi \sin\theta +$$

$$\big(\bar{q}_n^m + (n^2 - 9/4)q_n^m\big)\cos m\varphi \cos\theta +$$
$$\big(\bar{p}_n^m + (n^2 - 9/4)p_n^m\big)\sin m\varphi \cos\theta\Big]P_n^m(\cos\theta).$$

The total field of variations, observed on the Earth's surface, is obtained by superposition of the MT- and the ET-fields. When implementing such a superposition one needs to take into consideration two more important circumstances. The first is in that in practice (at stations) H_θ-component is observed with a reverse sign (a measuring device is generally oriented to the North), values of intensity in components observed being given in nanotesla (nT).

The second circumstance is in that the intensity of electric components is measured in volts per meter (V/m), therefore the above-obtained formulas should be multiplied by the transfer coefficient $10^{-2}/4\pi$. We must also keep in mind that all the formulas presented have been obtained obtained for one temporal harmonic. The observed field contains a great number of such harmonics, therefore before carrying out the spherical analysis, for which all the formulas were obtained, the Fourier harmonic analysis is made, whose description for the sake of simplicity is omitted.

Thus, components of the total observed field of global electromagnetic variations of the Earth's MGF have the following expansions

$$H_\theta = -\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[\bar{i}_n^m \cos m\varphi \sin\varphi + \bar{j}_n^m \sin m\varphi \sin\varphi - \tag{2.42}$$
$$\bar{k}_n^m \cos m\varphi \cos\varphi - \bar{l}_n^m \sin m\varphi \cos\varphi -$$
$$i_n^m\big(m \sin m\varphi \cos\varphi + (n+1/2)\cos m\varphi \sin\varphi\big) +$$
$$j_n^m\big(m \cos m\varphi \cos\varphi - (n+1/2)\sin m\varphi \sin\varphi\big) -$$
$$k_n^m\big(m \sin m\varphi \sin\varphi - (n+1/2)\cos m\varphi \cos\varphi\big) +$$
$$l_n^m\big(m \cos m\varphi \sin\varphi + (n+1/2)\sin m\varphi \cos\varphi\big) +$$
$$(p_n^m \cos m\varphi - q_n^m \sin m\varphi)m \operatorname{ctg}\theta\big]P_n^m(\cos\theta),$$

$$H_\varphi = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[\bar{i}_n^m \cos m\varphi \cos\varphi \cos\theta + \bar{j}_n^m \sin m\varphi \cos\varphi \cos\theta +$$
$$\bar{k}_n^m \cos m\varphi \sin\varphi \cos\theta + \bar{l}_n^m \sin m\varphi \sin\varphi \cos\theta -$$
$$(\bar{p}_n^m \sin m\varphi + \bar{q}_n^m \cos m\varphi)\sin\theta\big]P_n^m(\cos\theta) -$$
$$i_n^m\Big((n{+}1/2)\cos\theta P_n^m(\cos\theta) + \sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\cos m\varphi \cos\varphi -$$
$$j_n^m\Big((n{+}1/2)\cos\theta P_n^m(\cos\theta) + \sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\sin m\varphi \cos\varphi -$$
$$k_n^m\Big((n{+}1/2)\cos\theta P_n^m(\cos\theta) + \sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\cos m\varphi \sin\varphi -$$

$$l_n^m\Big((n{+}1/2)\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\sin m\varphi\sin\varphi -$$
$$p_n^m\Big(\cos\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}-(n{+}1/2)\sin\theta P_n^m(\cos\theta)\Big)\sin m\varphi -$$
$$q_n^m\Big(\cos\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}-(n+1/2)\sin\theta P_n^m(\cos\theta)\Big)\cos m\varphi,$$

$$H_r=\sum_{n=1}^{\infty}\sum_{m=0}^{n} i_n^m\Big(\cos m\varphi\sin\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta}-$$
$$\sin m\varphi\cos\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big)+$$
$$j_n^m\Big(\sin m\varphi\sin\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta}+\cos m\varphi\cos\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big)-$$
$$k_n^m\Big(\cos m\varphi\cos\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta}+\sin m\varphi\sin\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big)-$$
$$l_n^m\Big(\sin m\varphi\cos\varphi\frac{\partial P_n^m(\cos\theta)}{\partial\theta}-\cos m\varphi\sin\varphi m\operatorname{ctg}\theta P_n^m(\cos\theta)\Big)-$$
$$(p_n^m\cos m\varphi-q_n^m\sin m\varphi)mP_n^m(\cos\theta),$$

$$E_\theta=\frac{10^{-2}}{4\pi\sigma'_{\mathrm{E}}R_0}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[-i_n^m\Big((\varkappa^2+n-3/2)\cos\theta P_n^m(\cos\theta)+$$
$$(n-3/2)\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)+$$
$$\bar{i}_n^m\Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\Big]\cos m\varphi\cos\varphi+$$
$$\Big[-j_n^m\Big((\varkappa^2{+}n{-}3/2)\cos\theta P_n^m(\cos\theta){+}(n{-}3/2)\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)+$$
$$\bar{j}_n^m\Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\Big]\sin m\varphi\cos\varphi+$$
$$\Big[-k_n^m\Big((\varkappa^2{+}n{-}3/2)\cos\theta P_n^m(\cos\theta){+}(n{-}3/2)\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)+$$
$$\bar{k}_n^m\Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\Big]\cos m\varphi\sin\varphi+$$
$$\Big[-l_n^m\Big((\varkappa^2{+}n{-}3/2)\cos\theta P_n^m(\cos\theta){+}(n{-}3/2)\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)+$$
$$\bar{l}_n^m\Big(\cos\theta P_n^m(\cos\theta)+\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)\Big]\sin m\varphi\sin\varphi+$$
$$\Big[q_n^m\Big((\varkappa^2{+}n{-}3/2)\sin\theta P_n^m(\cos\theta){-}(n{-}3/2)\cos\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big)+$$
$$\bar{q}_n^m\Big(\cos\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}-\sin\theta P_n^m(\cos\theta)\Big)\Big]\cos m\varphi+$$

$$\Big[p_n^m\Big((\varkappa^2{+}n{-}3/2)\sin\theta P_n^m(\cos\theta){-}(n{-}3/2)\cos\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\Big) -$$
$$\bar p_n^m\Big(\cos\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} - \sin\theta P_n^m(\cos\theta)\Big)\Big]\sin m\varphi,$$

$$E_\varphi = \frac{10^{-2}}{4\pi\sigma'_{\mathrm{E}}R_0}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[i_n^m\big((\varkappa^2+n-3/2)\cos m\varphi\sin\varphi +$$
$$(n-3/2)m\sin m\varphi\cos\varphi\big) -$$
$$\bar i_n^m(m\sin m\varphi\cos\varphi + \cos m\varphi\sin\varphi) +$$
$$\bar j_n^m(m\cos m\varphi\cos\varphi - \sin m\varphi\sin\varphi) +$$
$$j_n^m\big((\varkappa^2{+}n{-}3/2)\sin m\varphi\sin\varphi{-}(n{-}3/2)m\cos m\varphi\cos\varphi\big) -$$
$$\bar k_n^m(m\sin m\varphi\sin\varphi - \cos m\varphi\cos\varphi) -$$
$$k_n^m\big((\varkappa^2{+}n{-}3/2)\cos m\varphi\cos\varphi{-}(n{-}3/2)m\sin m\varphi\sin\varphi\big) +$$
$$\bar l_n^m(m\cos m\varphi\sin\varphi + \sin m\varphi\cos\varphi) -$$
$$l_n^m\big((\varkappa^2{+}n{-}3/2)\sin m\varphi\cos\varphi{+}(n{-}3/2)m\cos m\varphi\sin\varphi\big) -$$
$$\big(\bar q_n^m - (n-3/2)q_n^m\big)\sin m\varphi m\,\mathrm{ctg}\,\theta +$$
$$\big(\bar p_n^m - (n-3/2)p_n^m\big)\cos m\varphi m\,\mathrm{ctg}\,\theta\Big]P_n^m(\cos\theta),$$

$$E_r = \frac{10^{-2}}{4\pi\sigma' R_0}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[\big(\bar i_n^m + (n^2-9/4)i_n^m\big)\cos m\varphi\cos\varphi\sin\theta +$$
$$\big(\bar j_n^m + (n^2-9/4)j_n^m\big)\sin m\varphi\cos\varphi\sin\theta +$$
$$\big(\bar k_n^m + (n^2-9/4)k_n^m\big)\cos m\varphi\sin\varphi\sin\theta +$$
$$\big(\bar l_n^m + (n^2-9/4)l_n^m\big)\sin m\varphi\sin\varphi\sin\theta +$$
$$\big(\bar q_n^m + (n^2-9/4)q_n^m\big)\cos m\varphi\cos\theta +$$
$$\big(\bar p_n^m + (n^2-9/4)p_n^m\big)\sin m\varphi\cos\theta\Big]P_n^m(\cos\theta).$$

The analysis of these expansions has revealed that all the components of the observed EMF have the same set of unknown constants. Three magnetic components are generally measured at the world network of stations, a few stations measuring toroidal electric components. Therefore it is necessary to include into the spherical analysis all available data about components of the field. In this case, it is possible to essentially smooth the influence of random errors of measurements when converting the main matrix (see Section 1.5.5).

From (2.42) it follows that it is possible not to measure electric components of the field of global electromagnetic variations. They can be calculated with the results of the spherical analysis of magnetic components only. This is essential because measurements are aimed merely at magnetic components only.

According to (2.42), a vertical electric component of the field is potential on the Earth's surface. Its induction part is compensated by the potential part. Analysis of (2.39) and (2.42) makes possible to formulate the following theorem for a variable EMF.

Theorem 13. *Complete separation of observed electromagnetic fields into the MT- and the ET-modifications, as well as into external and internal fields in them, is uniquely solvable when the vertical component and one of the horizontal components of a magnetic field are known.*

The algorithm of solving the complete separation problem is in the following. With the magnetic components of the field of variations, measured at the world network of stations with the use of the spherical analysis, a certain number (up to a given upper limit of summation n in (2.42)) of coefficients with a stroke and without it. Then with the help of these coefficients in terms of (2.42) the total field is synthesized. If the synthesis of the field at each point slightly differs from the observed one (by a prescribed ε), then the number of such coefficients is considered to be sufficient for presenting the observed field on the sphere.

After that, equations (2.39) are solved, and with the coefficients obtained the MT- and ET-modes of the external and internal fields are synthesized. The MT- and the ET-modes can be calculated by formulas (2.40) and (2.41) in the total field with the same coefficients that are obtained by the spherical analysis method.

The Gauss–Shmidt method of the spherical analysis of observational data is approximate for calculation of coefficients. The degree of approximation depends on termination of infinite sums of expansions (2.42) on a certain prescribed number n. The value of number n that is potential for termination depends on the number of observational points available on the Earth's surface. The greater is their number, up to the greater n the field can be expanded. Whether the field of a selected number n is sufficient for interpolation is determined by comparing the observed field to its synthesis.

2.5. Magnetic and electric fields of the solar-daily variations in the MGG: 1957–1958

The best-understood and extensively studied in geophysics are solar-daily variations (S_q -variations), whose global propagation along the Earth's surface is now beyond question. The research into the solar-daily variations began as long ago as in the 30–40-s of the last century [7]. At that time, using the data of the International Geophysical Year (IGY-1933), Benkova examined magnetic components of the solar-daily variations by the method of spherical analysis with the Gauss–Shmidt formulas. Rather a large non-potential constituting part in magnetic components was fixed, which further was recalculated into currents. The densities of these currents appeared to be essentially exceeding those observed in the air, which caused criticizing the results obtained in [7] as viewed from the public opinion among geophysicists. Gradually "the baby was thrown out along with the bathwater". This appeared to be a widespread belief that there cannot be a non-potential part in the field S_q of variations in the atmosphere.

The author earlier mentioned that only one incorrect step was made in [7], namely, when the observed non-potential magnetic field was recalculated into electric currents. This should not been done, because $\nabla \times \boldsymbol{H}_T^{\mathrm{ET}} = \boldsymbol{H}_P^{\mathrm{MT}}$, i.e., vortices of a toroidal non-potential magnetic field do not generate electric currents in the Earth's atmosphere. Such vortices always excite only a poloidal magnetic field of the MT-mode. Therefore due to the existence of a toroidal part of the magnetic field of S_q-variations in the atmosphere an unusual situation arises: there is a non-potential magnetic field in the atmosphere, while there is no electric current through the atmosphere. As was noted above, the toroidal magnetic field in the atmosphere is generated by the spherical components j_0 and j_φ of the toroidal electric current, which are arranged in the ionosphere and in the Earth's spherical layers. The worthy cause for the occurrence of a toroidal part of the magnetic field, as noted above, is a spherical property of sources, in our case this is a spherical property of the sources of S_q-variations.

The author's research was based on the important result that a toroidal variable magnetic field does not generate electric currents in the atmosphere. The author (monograph [6] and many other publications) presents his results employing expansions (2.42) of S_q-variations on evidence derived from the MGG-1958, processed by Rotanova and Borisova [16]. The data in question were published long ago and are well known, that is why here we only give a reference. Monograph [6] presents tables of coefficients calculated on evidence derived from [16]. These tables give real (small letters) and fictitious (capital letters) parts of unknown complex expansion coefficients (2.42). According to Theorem 1, the complexity of coefficients provides the completeness of expansions (2.42) for five ($k = 5$) temporal and five ($n = 5$) spatial harmonics. The considered coefficients made possible to synthesize the toroidal and poloidal parts of the MT- and the ET-modes of S_q-variations. As an example we present Table 2.1 of the daily variation of the total field of the MT- and the ET-modes at the station "Moskva" ($\theta = 34.52°$, $\varphi = 37.42°$) in the MGG-1957/1958. Table 2.1 gives the world

Table 2.1

t, h	H_θ	H_ϕ	H_r	H_θ^{MT}	H_ϕ^{MT}	H_r^{MT}	H_θ^{ET}	H_ϕ^{ET}
0	2	2	4	−10	17	4	12	−15
1	1	4	6	−14	20	6	15	−16
2	4	9	7	−24	25	7	28	−15
3	6	18	8	−26	30	8	32	−12
4	7	26	8	−26	33	8	33	−6
5	3	34	6	−30	36	6	32	−1
6	−7	39	2	−31	38	2	24	1
7	−18	37	0	−31	35	0	13	1
8	−29	28	−2	−24	26	−2	−5	1
9	−35	11	−6	−15	8	−6	−20	3
10	−35	−9	−1	−7	−14	−10	−28	5
11	−27	−27	−13	2	−33	−13	−29	5
12	−15	−40	−13	9	−44	−13	−24	4
13	−3	−42	−9	14	−44	−9	−17	2
14	7	−34	−3	12	−35	−3	−4	1
15	14	−21	3	6	−24	3	8	3
16	16	−9	6	9	−18	6	7	10
17	18	−2	4	22	−18	4	−4	16
18	20	−2	1	31	−18	1	−12	16
19	20	−5	0	35	−17	0	−15	12
20	19	−6	0	36	−12	0	−16	6
21	16	−5	−1	33	−5	−1	−17	0
22	11	−4	−1	26	3	−1	−15	−7
23	6	−2	2	4	11	2	2	−13

time (UT) in hours. The values of intensity of components are given in nanotesla. Table 2.1 displays the observed components as well as those synthesized by the spherical analysis: the poloidal part of the MT-mode and the toroidal part of the ET-mode.

The analysis of the table reveals that the poloidal and toroidal parts of the magnetic field of S_q variations at the station "Moskva" are equal in the value of intensity in maxima and in minima of the daily variation from 0 up to 23 hours of the world time. The spherical property of the source of S_q-variations so strongly influences the observed field that its toroidal part becomes not only appreciable but compatible to the poloidal one. This important fact is observed at all (without exception) stations of the world network, fixing S_q-variations of the magnetic field [6]. The list of the stations is given in [16]. Nevertheless, it is impossible to recalculate such an appreciable toroidal magnetic field into the atmospheric electric currents because $\nabla \times \boldsymbol{H}_T^{\mathrm{ET}} = \boldsymbol{H}_P^{\mathrm{MT}}$.

Thus, the two-modality of the magnetic field of S_q-variations is available. The toroidal part of the magnetic field is compatible in intensity to the poloidal field. This fact should not be ignored both in theory and practice of the research and applications of the magnetic field of S_q-variations as well as of any other variations of global propagation: there may arise serious errors if the toroidal magnetic field in the Earth's atmosphere is not taken into account.

In [1], diagrams of external and internal parts of the MT- and the ET-modes are presented. Since both the internal and the external electromagnetic fields of S_q-variations are two-modal, it is not of primary importance where the initial source is situated: either in the ionosphere or in the Earth, the main thing its being spherical. The spherical property is the main reason for the existence of the MT- and the ET-modes.

The structure of the electric field of global electromagnetic variations in the atmosphere is rather complicated. As a magnetic field, the electric field is two-modal. In addition, as elucidated above, the induction part of the radial (vertical) component of the electric field on the Earth's surface is compensated by the potential part, however, it can be reconstructed by calculation results of unknown coefficients with the method of the spherical or spatial analysis, as is done in [3] when analyzing S_q-variations.

The intensity of these components is different as well as their behavior from hour to hour local time (LT) (Figure 2.1). However the analysis of invisible radial component enables us to make interesting conclusions about sources of the solar-daily S_q-variations and to apply a radial component of the electric field to the two-modal sounding of the Earth.

Thus, the electric field of global S_q-variations has a radial component consisting of a visible potential and an invisible on the Earth's surface inductive parts. Their examples are displayed in Figure 2.1. At some points, the intensity of the radial potential part attains 1 kV/m, the intensity of the invisible part attaining millivolts per kilometer.

So, the debates about the presence or the absence in the field of variations of a radial (vertical or normal) component of the electric field [20] argues for its presence in the Earth's atmosphere. Moreover, this component in the atmosphere appeared to consist of two different parts: a potential (visible) part and an inductive (invisible) part that can be readily reconstructed with the help of results of the spherical analysis of independent magnetic components of the observed field. And these are not all "surprises" that are intrinsic of the electric field of global electromagnetic variations. In Section 2.3, when analyzing boundary conditions it was revealed that counting the intensity in their potential components should start with a different specific conductivity. These are, respectively, the Earth's conductivity σ'_{E} or the ionosphere conductivity σ'_{I} in the components, tangential to the Earth's surface and the ionosphere. In the vertical (radial) component, which is completely potential, the count begins with the atmosphere conductivity that is very low. Therefore, the potential part of E_r-component (formula (2.42)), which is counted from a very low atmosphere conductivity (of order $10^{-12} \div 10^{-14}(\mathrm{Ohm} \cdot \mathrm{m})^{-1}$) has the intensity of up to the first kV/m. How-

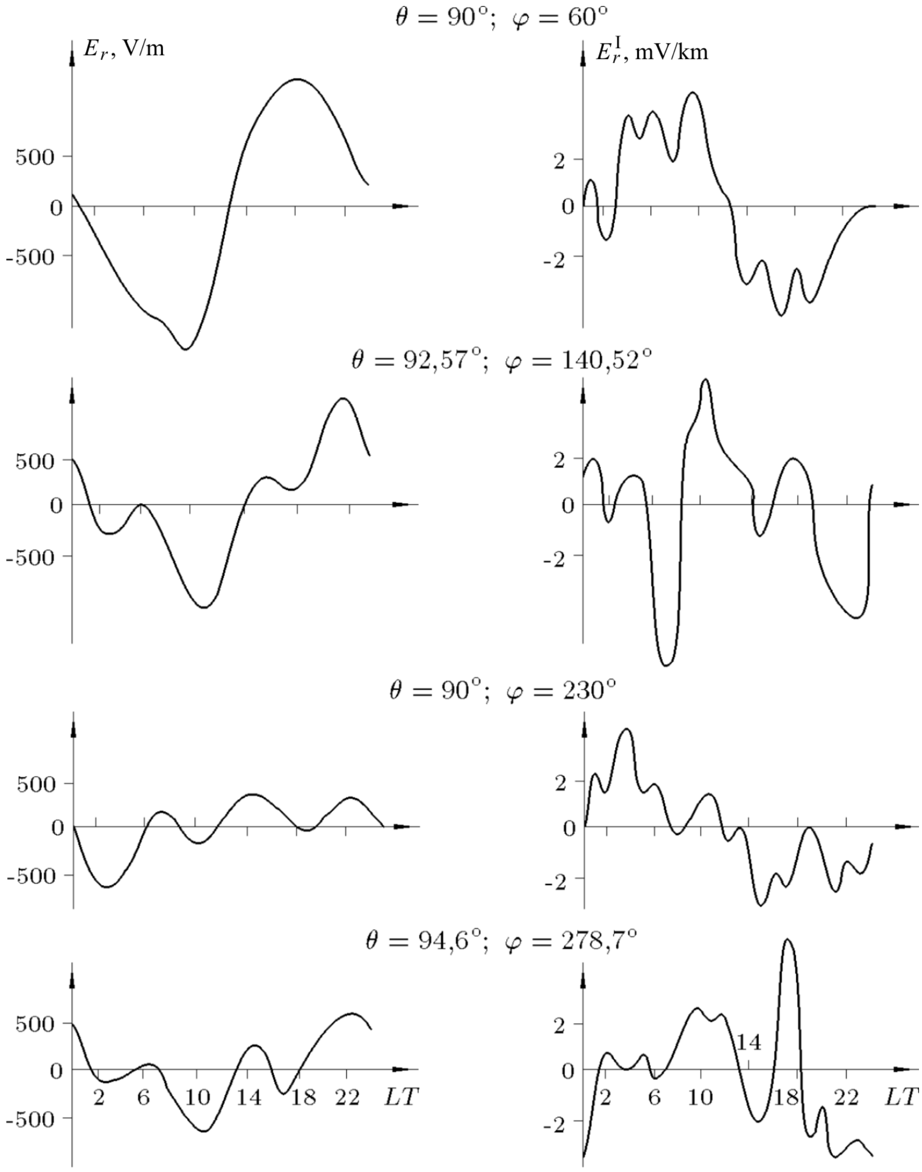


Figure 2.1. Electric field of external sources of variations at equatorial stations

ever, the intensity of the tangential components E_θ and E_φ of the potential electric field due to the high conductivity of the Earth's upper layer does not exceed the first microvolts per kilometer. Although such values of the field intensity are very low, they are measurable. Due to their smallness, they are not taken into consideration when interpreted.

An electric field of the MT-mode, as mentioned above, is toroidal, its intensity being the first tens mV/km in S_q-variations. According to the boundary conditions (Section 2.3), a toroidal electric field of the MT-mode coincides with the field of terrestrial currents in the Earth and is measured at some stations of the world network. In [1], the author analyzes the data on the toroidal electric components observed at the station Pleschenitsy, the Minsk region ($\theta = 45.5°$, $\varphi = 27.9°$). The theoretically obtained results and the observations are in good agreement, although the observations and calculation were made with the data referring to different epochs. However, due to the repeatability of the solar-daily variations, the results of the experiment and the calculation essentially coincided. Table 2.2 presents the observed daily variations (the first two columns) and the calculated (the next two columns) toroidal electric components of the MT-mode at the station Pleschenitsy. The field of variations contain the toroidal electric components of the MT-mode, their intensity at certain points of the Earth's surface exceeding the first ten mV/km (see Table 2.2). Such fields are convincingly observed and can be used for the calculation of the Earth's and ionosphere conductivity.

Thus, the global field the solar-daily variations consists not only of the toroidal and the poloidal parts of the magnetic field, but also contains in the

Table 2.2

t, h	Station Pleschenitsy				Station Moskva	
	E_θ^{MT}	E_ϕ^{MT}	E_θ^{MT}	E_ϕ^{MT}	E_θ^{MT}	E_ϕ^{MT}
0	−2	−5	−1	−1	1	7
1	−1	−6	−1	−2	1	4
2	0	−6	2	−2	2	3
3	2	−1	2	−3	1	−1
4	5	−5	4	−2	2	1
5	6	4	6	0	4	−4
6	3	5	2	3	2	−7
7	2	6	−1	4	−1	−6
8	0	9	1	5	−1	−8
9	−1	8	−1	4	−3	−8
10	1	8	−2	2	−4	−8
11	−1	1	−3	1	−5	0
12	−3	−4	−7	−2	−8	7
13	−4	−2	−5	−2	−4	4
14	−7	−7	0	−3	1	9
15	−4	1	2	−3	2	5
16	0	1	4	−1	3	−5
17	2	1	3	−1	1	−2
18	5	1	1	−1	0	0
19	2	3	−1	−1	2	−1
20	1	8	−1	0	1	−1
21	−1	−2	−2	1	0	−5
22	−1	−8	−2	1	1	1
23	−2	−4	−1	−1	2	13

atmosphere both non-potential toroidal components of the electric field that are horizontal to the Earth's surface and horizontal poloidal potential electric components and a potential radial (vertical) component of the electric field.

In addition, the algorithms developed can aid in the reconstruction of the compensated inductive part of the radial component of the electric field and use it for a detailed analysis of sources of the solar-daily variations and for the Earth's sounding. The formula for the induction invisible part of the electric field has the form

$$
\begin{aligned}
E_r^{\mathrm{I}} = -\frac{i\omega\mu_0 R_0 \cdot 10^{-2}}{4\pi} \sum_{k=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} k \times \\
\big[(i_n^m \cos m\varphi + j_n^m \sin m\varphi) \sin\theta \cos\varphi + \\
(k_n^m \cos m\varphi + l_n^m \sin m\varphi) \sin\theta \sin\varphi + \\
(q_n^m \cos m\varphi + p_n^m \sin m\varphi) \cos\theta\big]_k \; P_n^m(\cos\theta).
\end{aligned}
\tag{2.43}
$$

2.6. A source of the solar-daily variations

The simulation of sources of geoelectromagnetic variations is a necessary element of interpretation and aids in the solution of a variety of problems of studying morphological peculiarities of variations, their generation, as well as a general geophysical nature of the event. In addition, a model of a source of one or another variation is of interest not only because it reflects a geophysical event as it is, but it makes possible, for example with the use of variations, to study the system of winds at the height of a source, as in the first approximation the direction of the electric current reflects the direction of wind in the ionosphere. The research into the morphology of the current systems of different variations aids, in addition, in the solution of a more complicated global problem of the solar-terrestrial connections. Therefore, the adequacy of a source to the field, measured on the Earth's surface, is of primary importance in the variations interpretation.

The sources of geoelectromagnetic variations are simulated with the use of results of the spherical analysis of their field, observed on the Earth's spherical surface (the Earth in the first approximation is considered to be spherical). The complete separation of variations fields in this case contributes to solving the problem of simulation by providing the separation of the field of external origin in both modifications. The principle of modeling equivalent sources, based on the transfer from the magnetic field of external sources to the double layer of charges, whose power was identified with the force of an equivalent current, was introduced into the theory of the current systems comparatively long ago. Contours of the equal force current form a current system that is similar to that existing in nature. Numerous scientific studies are devoted to the investigation of equivalent current systems.

However, due to the existence of an ET- field, it is necessary to solve a number of new problems. The first is in developing the physical basis of the way of constructing a current system, adequate to the observed field. The key to solving this problem is in separation of two modifications of the field and in development of a physically justified way of the transfer from the observed field to its source.

The second problem is in the derivation of all necessary formulas suitable for the numerical calculation.

The third problem is to apply theory to a concrete material of observations and to reveal new peculiarities of sources, whose existence is predetermined by the existence of the ET-field that was not earlier taken into account in the theory.

The simulation of sources of the MT-field should be based on a few assumptions. First, the Earth is considered to be spherical, the electric currents, accompanied by the MT-field, to be concentrated in the ionosphere thin layer E at the height $h = 120\,\mathrm{km}$ from the Earth's surface. Second, from the physical stand point and the fact that the ET-type field is excluded beforehand follows that a thin layer, which is a carrier of the current system, is spherically symmetric to the Earth. In the opposite case, an ET- field that was excluded beforehand, would arise. In this case it is natural to assume that the interaction of currents has occurred before the separation of fields, so the MT-field would reflect the final picture in distributing the current in a source.

The next important assumption is that the surface—the current carrier—is supposed to be infinitely thin: above and below it, all the space is filled with the air with the conductivity σ' or is empty $\sigma' = 0$.

If the current is concentrated on a thin surface, the tangential to it components of the magnetic field satisfy on this surface the boundary condition of the form

$$j(p) = \big[n, \big(H^{+}(p) - H^{-}(p)\big)\big]. \tag{2.44}$$

Here $H^{+}(p)$ is a magnetic field at the point p above the layer with current, $H^{-}(p)$ is a magnetic field at the point p under the layer with current, $j(p)$ is the surface current, n is an external normal. In the above-discussed statement, properties of the medium above and below the layer with current are considered to be identical, that is why the magnetic field components will differ only by a sign. The expansion coefficients of the field under the layer and below it coincide on the layer. In order to make sure that it is so, let us consider a simple example. Let the conductivity above and below the layer be equal to zero $\sigma' = 0$, then it will be possible to apply the reduced expansions for the MT field. The coefficients of these reduced expansions are expressed via integrals. Let us take, for example, one coefficient from the external side of the surface:

$$\varkappa_n^m = \frac{\bar{c}_n^m}{4\pi \tilde{R}^{n+2}} \int\limits_W r'^n j_{x'}^{\mathrm{CT}}(x', y', z') \cos m\phi' P_n^m(\cos\theta')\, dw', \tag{2.45}$$

where W is a layer above the Earth, $\tilde{R}$ is a distance from the Earth's center up to the layer W. From the inner side of the layer the corresponding coefficient will have the form

$$a_n^m = \frac{\bar{c}_n^m \tilde{R}^{n-1}}{4\pi} \int\limits_W \frac{1}{r'^{n+1}} j_{x'}^{\mathrm{CT}}(x', y', z') \cos m\phi' P_n^m(\cos\theta')\, dw'. \tag{2.46}$$

Here r' is a current radius. If the layer thickness tends to zero, the radii are transformed as follows:

$$\tilde{R} = R_0 + h, \qquad r' = R_0 + h, \tag{2.47}$$

where R_0 is the Earth's radius, h is a height from the Earth's surface to the surface with current. In this case in integrals (2.45) and (2.46), the steady volume current should be changed for the surface current. As a result we come to

$$\begin{aligned} \varkappa_n^m &= \frac{\bar{c}_n^m}{4\pi (R_0 + h)^2} \int\limits_\Sigma j_{x'}^{\mathrm{CT}}(x', y', z') \cos m\phi' P_n^m(\cos\theta')\, ds', \\ a_n^m &= \frac{\bar{c}_n^m}{4\pi (R_0 + h)^2} \int\limits_\Sigma j_{x'}^{\mathrm{CT}}(x', y', z') \cos m\phi' P_n^m(\cos\theta')\, ds', \end{aligned} \tag{2.48}$$

where Σ is the surface with current, ds' is an element of the surface.

There is no difficulty to see that in (2.48) the coefficients on different sides of an infinitely thin surface with current are described by the same formulas. Similar operations can be carried out with the rest coefficients and to make sure that they coincide, i.e., really, when transferring a thin layer with electric current the tangential components change only a sign, the coefficients on both sides coinciding. Thus, when constructing formulas for the surface current density it is possible to employ the expansions of the field below the surface with current.

From the external side of the surface with current, the tangential components have the form

$$
\begin{aligned}
H_\theta^{+\mathrm{MT}} = &-\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m\cos m\phi + b_n^m\sin m\phi)\sin\phi - \\
&(c_n^m\cos m\phi + d_n^m\sin m\phi)\cos\phi\big]P_n^m(\cos\theta)\times \\
&\Big(-\frac{n-1/2}{R_0+h}I_{n+1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big) + \frac{\varkappa}{R_0}I_{n-1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big)\Big), \\
H_\phi^{+\mathrm{MT}} = &-\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m\cos m\phi + b_n^m\sin m\phi)\cos\theta\cos\phi + \\
&(c_n^m\cos m\phi + d_n^m\sin m\phi)\cos\theta\sin\phi - \\
&(e_n^m\cos m\phi + f_n^m\sin m\phi)\sin\theta\big]P_n^m(\cos\theta)\times \\
&\Big(-\frac{n-1/2}{R_0+h}I_{n+1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big) + \frac{\varkappa}{R_0}I_{n-1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big)\Big).
\end{aligned} \tag{2.49}
$$

From the inner side of the surface with electric current, the formulas for tangential components of the magnetic field are written down as

$$
\begin{aligned}
H_\theta^{-\mathrm{MT}} = &\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m\cos m\phi + b_n^m\sin m\phi)\sin\phi - \\
&(c_n^m\cos m\phi + d_n^m\sin m\phi)\cos\phi\big]P_n^m(\cos\theta)\times \\
&\Big(-\frac{n-1/2}{R_0+h}I_{n+1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big) + \frac{\varkappa}{R_0}I_{n-1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big)\Big), \\
H_\phi^{-MT} = &\sum_{n=1}^{\infty}\sum_{m=0}^{n}\big[(a_n^m\cos m\phi + b_n^m\sin m\phi)\cos\theta\cos\phi + \\
&(c_n^m\cos m\phi + d_n^m\sin m\phi)\cos\theta\sin\phi - \\
&(e_n^m\cos m\phi + f_n^m\sin m\phi)\sin\theta\big]P_n^m(\cos\theta)\times \\
&\Big(-\frac{n-1/2}{R_0+h}I_{n+1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big) + \frac{\varkappa}{R_0}I_{n-1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big)\Big).
\end{aligned} \tag{2.50}
$$

Now formulas (2.49) and (2.50) should be substituted into (2.44). In so doing, it is convenient to present it as components:

$$
j_\theta^{\Pi} = H_\phi^{+\mathrm{MT}} - H_\phi^{-\mathrm{MT}}, \qquad j_\phi^{\Pi} = H_\theta^{+\mathrm{MT}} - H_\theta^{-\mathrm{MT}}. \tag{2.51}
$$

For the radial function, we may introduce the notation

$$
\psi_n(\varkappa, h) = \Big(-\frac{n-1/2}{R_0+h}I_{n+1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big) + \frac{\varkappa}{R_0}I_{n-1/2}\Big(\varkappa\frac{R_0+h}{R_0}\Big)\Big). \tag{2.52}
$$

The factor $10^{-2}/4\pi$ should be introduced into the final calculation formulas, which transfers the values of coefficients obtained in nT to A/m.

Since it is required to study the behavior of the current systems in terms of time, it is convenient to reconstruct the earlier omitted dependence of the current density on time. With allowance for the above-said, the electric current components on an infinitely thin surface at a height h from the Earth, will have the form

$$\begin{aligned} j_\theta^\Pi = \frac{-10^{-2}}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \big[& (a_n^m \cos m\phi + b_n^m \sin m\phi) \cos\theta \cos\phi + \\ & (c_n^m \cos m\phi + d_n^m \sin m\phi) \cos\theta \sin\phi - \\ & (e_n^m \cos m\phi + f_n^m \sin m\phi) \sin\theta \big]_k \psi_n(\varkappa, h) P_n^m(\cos\theta) e^{i\omega kt}, \qquad (2.53) \\ j_\phi^\Pi = \frac{10^{-2}}{2\pi} \sum_{k=-\infty}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{n} \big[& (a_n^m \cos m\phi + b_n^m \sin m\phi) \sin\phi - \\ & (c_n^m \cos m\phi + d_n^m \sin m\phi) \cos\phi \big]_k P_n^m(\cos\theta) \psi_n(\varkappa, h) e^{ik\omega t}. \end{aligned}$$

In (2.53), components of the surface current density have the dimension in A/m and allow constructing the distribution of the current density vectors on a given surface. A set of such vectors, constructed at a given instant of the world time, represents a current system, generating a MT-field. The current system, constructed by the MT-field, is known to be solenoidal because an ET-field was excluded beforehand. The choice of the world time for constructing the current systems is explained by a synchronous change in variations in the unified world time. In this case, the adequacy of a source to the observed field is ensured by formula (2.51), as the electric current is directly calculated by the field components without any transformations.

The software package developed calculates the current systems by a MT-field. Figure 2.2 presents the current system of S_q-variations in 1958 [2]. This system has been obtained by 6 o'clock of the world time. The following details are characteristic. On the day side, known contours of the current present: the southern with clockwise rotation and the northern with counterclockwise rotation. The contours are divided by the electric current of the east direction that flows in the vicinity of the equator and corresponds to a known electrojet. The line connecting the centers of vortices on the day side tends to the direction of the force lines of the MGF. Earlier this fact was not pointed out. The night side has similar day northern and southern vortices, but of the opposite rotation direction and separated by the west direction current. The line connecting the centers of the night vortices is also close to the direction of the force lines of the MGF. An important result is that during the world day, the night vortices can shift closer to the morning or to the evening side. Such a symmetric picture of the current vortices of a MT-field was not noted earlier. In our opinion, it most of all corresponds to the physics of sources and is adequate to the observed field.

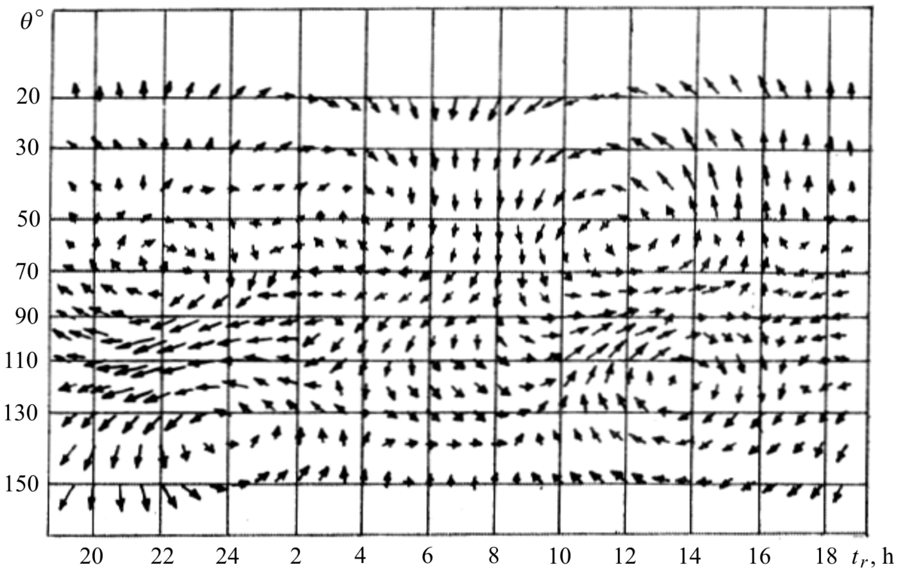


Figure 2.2. A source of the MT-field of S_q-variations at $UT = 6$ hours, each vector corresponds to the direction and value of the surface current

Thus, the proposed technique of constructing a model of a source of a MT-field enabled us to reveal a sufficiently trivial morphological structure of a source of the solar-daily variations. The current system of S_q-variations represents a set of the two symmetric variations on the day and night sides (or on the morning and evening sides) and oppositely rotating systems of currents, separated by electrojets of the east and the west directions, respectively. The source of S_q-variations has no other contours in addition to the above-mentioned. The symmetric picture of situation of the current vortices corresponds to the morphology of the daily S_q-variations, observed on the Earth's surface and is in agreement with the electrodynamics of the event.

Then by analyzing an invisible part of the vertical electric field in the equatorial zone, electrojets with vertical inflows on the morning and the evening sides of the day were revealed. Clearly, these vertical currents can exist only within the ionosphere, where the horizontal and vertical conductivities are present. Nevertheless, the presence of inflows is readily traced from abnormal behavior of the inductive part of the vertical electric field (Figure 2.1) in the morning and evening hours. This field has been constructed from the results of spherical analysis of magnetic components of the solar-daily variations. The combined analysis of all new facts taking place after separating the MT- and the ET-modes in S_q-variations and then after the separation in each of the modes of external and internal sources enabled the author to propose his model of the source of S_q-variations which, in our opinion, essentially contributes to a well-known classical model of currents of S_q-variations in the ionosphere (Figure 2.3).

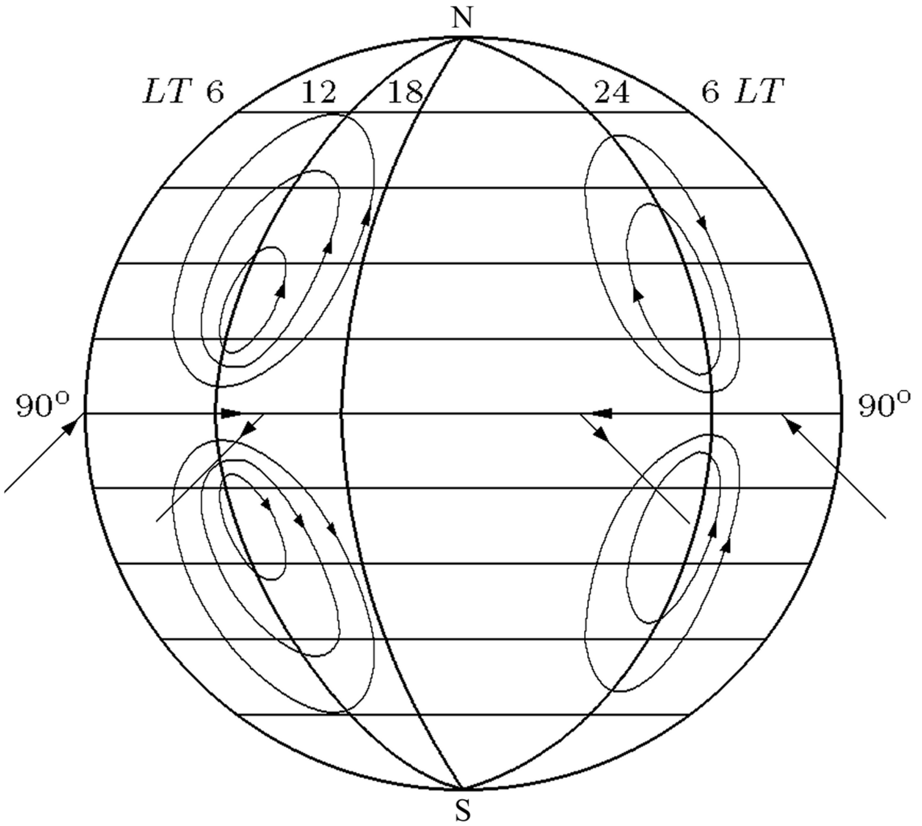


Figure 2.3. An external source of S_q-variations, simulated by the MT- and ET-fields, MGG 1957/1958 (LT is local time in hours)

The generation of such a current system of S_q-variations, displayed in Figure 2.3, must apparently depend on winds at the height E of the ionosphere layer, the plasma drifting in it on the equator to the west in the day time and to the east — at night (the current flowing in the opposite direction), as well as on stationary convection in the lower magnetosphere. The size of a contribution of one or another mechanism to generating regular S_q-currents is to be revealed. The mechanism of forming a vertical current on the equator on the morning and evening sides of the day, which is of a strong interest as it is, seems to be absolutely unclear. Nevertheless, the fact of appearance of vertical inflows on the equator is beyond question.

Thus, investigation of a source only of S_q-variations made possible to obtain a number of the new interesting facts. Also, interesting facts can be revealed in sources of all other types of variations if for their investigation one applies expansions (2.42) with allowance for all currently known electrodynamic data on the global electromagnetic field of variations.

2.7. Generalized electrodynamic equations for the Earth's constant and alternating electromagnetic fields

Electromagnetic fields, observed on the Earth by the world network of magnetic observatories as well as by self-contained devices in the regional prospecting of mineral resources, are conventionally considered to be force and one-modal fields [15]. For the physico-mathematical simulation of the EMFs, observed on the Earth, a force component of Maxwell's equations is often used. This is done because of their low oscillation frequency. But for the mathematical modeling of the MGF, the potentiality of a force modification (mode) of the magnetic field in the air is used [15].

A somewhat different situation is observed in the theory of the MGF generation, where along with a force mode of an electric field its non-force part, called a toroidal magnetic and a poloidal electric field is used. This generation theory as compared to others, in due time was called the theory of dynamo-excitation of the MGF. Its main peculiarity is the conviction that the force mode is not to appear in the Earth's atmosphere due to enormous intensity (200–500 Hz) inside the Earth, which is necessary for the MGF generation [15, p. 166]. The author proves that this is not so [6]. A non-force EMF (a toroidal magnetic and a poloidal electric fields) is measured in the Earth's atmosphere along with a force component when the EMF intensity is directly fixed by a device [6] and appears to be an unavoidable obstacle when interpreting observations by a one-modal scheme.

It is natural that substitution of a one-modal for a two-modal interpretation requires the development of the original equations for a complete description of the EMFs observed on the Earth (in the atmosphere) in terms of a force and a non-force modifications. In this chapter, a force and a non-force parts of the observed EMF are expressed in terms of the concepts about toroidal and poloidal EMFs. In this connection Maxwell's force equations in a low-frequency zone, according to definitions from (2.10) and (2.20), can be written down as follows:

$$\begin{gathered} \nabla \times \boldsymbol{H}_P^{\mathrm{MT}} = \sigma \boldsymbol{E}_T^{\mathrm{MT}}, \quad \nabla \times \boldsymbol{E}_T^{\mathrm{MT}} = -\mu \frac{\partial \boldsymbol{H}_P^{\mathrm{MT}}}{\partial t}, \\ \nabla \cdot (\boldsymbol{H}_P^{\mathrm{MT}}, \boldsymbol{E}_T^{\mathrm{MT}}) = 0. \end{gathered} \tag{2.54}$$

A non-force EMF is given by formula (2.20). Similar to relation (2.54) for a non-force mode in a low-frequency area, we can write down equations

$$\begin{gathered} \nabla \times \boldsymbol{H}_T^{\mathrm{ET}} = \boldsymbol{H}_P^{\mathrm{MT}}, \quad \nabla \times \boldsymbol{E}_P^{\mathrm{ET}} = 0, \\ \nabla \cdot (\boldsymbol{H}_T^{\mathrm{ET}}, \boldsymbol{H}_P^{\mathrm{MT}}, \boldsymbol{E}_P^{\mathrm{ET}}) = 0. \end{gathered} \tag{2.55}$$

Formulas (2.55) include an important property of a toroidal magnetic field, namely, its vortices do not excite the electric current in any medium, including the Earth's atmosphere, but according to Theorem 3, generate a force magnetic field. In this connection, standard boundary conditions for a complete magnetic field, including those on the Earth's surface are written down by formulas (1.23) and (2.55).

Now it is possible to combine the force and the non-force EMFs in the generalized Maxwell's equations. The complete force modification of equations will look like

$$\nabla \times \boldsymbol{H}_P^{\mathrm{MT}} = \sigma \boldsymbol{E}_T^{\mathrm{MT}} + \chi \boldsymbol{H}_T^{\mathrm{ET}}, \quad \nabla \times \boldsymbol{E}_T^{\mathrm{MT}} = -\mu \frac{\partial \boldsymbol{H}_P^{\mathrm{MT}}}{\partial t},$$
$$\nabla \cdot (\boldsymbol{H}_P^{\mathrm{MT}}, \boldsymbol{H}_T^{\mathrm{ET}}, \boldsymbol{E}_T^{\mathrm{MT}}) = 0,$$

where

$$\chi = \begin{cases} \dfrac{\gamma}{\eta}, & t = 0, \\ \bar{\varkappa}^{1/2}, & t > 0. \end{cases} \tag{2.56}$$

In this case, a supplementary electric current $\chi \boldsymbol{H}_T^{\mathrm{ET}} = \sigma(\mu\gamma \boldsymbol{H}_T^{\mathrm{ET}}) = \sigma \boldsymbol{E}_T'^{\mathrm{MT}}$ arising due to a spherical property of a source is just the source of a non-force toroidal magnetic field, penetrating into the Earth's atmosphere with boundary conditions (1.23) and (2.25). The non-force mode can also be characterized by its equations:

$$\begin{gathered} \nabla \times \boldsymbol{H}_P^{\mathrm{ET}} = \boldsymbol{H}_P^{\mathrm{MT}}, \quad \nabla \times \boldsymbol{E}_P^{\mathrm{ET}} = 0, \quad \frac{\partial \boldsymbol{D}_P^{\mathrm{ET}}}{\partial t} = -\chi \boldsymbol{H}_T'^{\mathrm{ET}}, \\ \nabla \cdot (\boldsymbol{H}_T^{\mathrm{ET}}, \boldsymbol{H}_P^{\mathrm{MT}}, \boldsymbol{E}_P^{\mathrm{ET}}) = 0, \quad \nabla \cdot \boldsymbol{D}_P^{\mathrm{ET}} = \rho, \quad \boldsymbol{D}_P^{\mathrm{ET}} = \varepsilon \boldsymbol{E}_P^{\mathrm{ET}}, \end{gathered} \tag{2.57}$$

where ρ is the electric charges density, ε is dielectric permeability. A specific feature of equations (2.57) is the fact that they reflect a non-force toroidal magnetic field $\boldsymbol{H}_T'$ excited by the electric induction that is rapidly varying in time. In this case, the equation for a temporal derivative of the electric induction is a symmetric reflection of the second equation from (2.56). Really, if we assume that $\nabla\times \sim \frac{1}{L}$, where L is a characteristic size of a local domain with a magnetic field, then $\mu\frac{\partial \boldsymbol{H}_P^{\mathrm{MT}}}{\partial t} \sim -\frac{1}{L}E_T^{\mathrm{MT}}$. The coefficient χ from (2.57) has also a dimension of the inverse characteristic length $\chi \sim \frac{1}{L}$, i.e., $\frac{\partial \boldsymbol{D}_P^{\mathrm{ET}}}{\partial t} \sim -\frac{1}{L}\boldsymbol{H}_T'^{\mathrm{ET}}$.

Thus, in the temporal domain, the force and non-force EMFs are described by the equations with a certain symmetry.

Combining the force and the non-force EMFs in the united system of equations, we can write down their complete modification, called the generalized Maxwell's equations for the force and non-force EMFs, observed on the Earth:

$$
\begin{aligned}
&\nabla\times\boldsymbol{H}_P^{\mathrm{MT}}=\sigma\boldsymbol{E}_T^{\mathrm{MT}}+\frac{\partial\boldsymbol{D}_P^{\mathrm{ET}}}{\partial t}+\chi\boldsymbol{H}_T^{\mathrm{ET}}+\boldsymbol{J}^{\mathrm{ST}},\\
&\nabla\times\boldsymbol{H}_T^{\mathrm{ET}}=\boldsymbol{H}_P^{\mathrm{MT}},\quad \nabla\times\boldsymbol{E}_T^{\mathrm{MT}}=-\mu\frac{\partial\boldsymbol{H}_P^{\mathrm{MT}}}{\partial t},\quad \nabla\times\boldsymbol{E}_P^{\mathrm{ET}}=0,\\
&\frac{\partial\boldsymbol{D}_P^{\mathrm{ET}}}{\partial t}=-\chi\boldsymbol{H}_T^{\prime\mathrm{ET}},\quad \nabla\cdot(\boldsymbol{H}_P^{\mathrm{MT}},\boldsymbol{H}_T^{\mathrm{ET}},\boldsymbol{E}_T^{\mathrm{MT}})=0,\\
&\nabla\cdot\boldsymbol{D}_P^{\mathrm{ET}}=\rho',\quad \nabla\cdot\boldsymbol{D}_T^{\mathrm{MT}}=\rho,\\
&\boldsymbol{B}_P^{\mathrm{MT}}=\mu\boldsymbol{H}_P^{\mathrm{MT}},\quad \boldsymbol{D}_P^{\mathrm{ET}}=\varepsilon\boldsymbol{E}_P^{\mathrm{ET}},\ \boldsymbol{B}_T^{\mathrm{ET}}=\mu\boldsymbol{H}_T^{\mathrm{ET}},\ \boldsymbol{D}_T^{\mathrm{MT}}=\varepsilon\boldsymbol{E}_T^{\mathrm{MT}}.
\end{aligned}
\tag{2.58}
$$

Here $\boldsymbol{J}^{ST}$ is an extraneous electric current. Naturally, equations (2.58) are valid only in spherical domains and for spherical sources. In laboratory conditions, when $\boldsymbol{H}_T^{\mathrm{ET}}=0$, $\boldsymbol{E}_P^{\mathrm{ET}}=0$, they automatically transfer to standard Maxwell's equations.

In a static EMF, similar equations are reduced as follows:

$$
\begin{aligned}
&\nabla\times\boldsymbol{H}_P^{\mathrm{MT}}=\sigma\boldsymbol{E}_T^{\mathrm{MT}}+\frac{\gamma}{\eta}\boldsymbol{H}_T^{\mathrm{ET}}+\boldsymbol{J}^{\mathrm{ST}},\quad \nabla\times\boldsymbol{H}_T^{\mathrm{ET}}=\boldsymbol{H}_P^{\mathrm{MT}},\\
&\nabla\times\boldsymbol{E}_{P,T}^{\mathrm{MT,ET}}=0,\quad \nabla\cdot(\boldsymbol{H}_P^{\mathrm{MT}},\boldsymbol{H}_T^{\mathrm{ET}},\boldsymbol{E}_T^{\mathrm{MT}})=0,\\
&\nabla\cdot\boldsymbol{D}_P^{\mathrm{ET}}=\rho',\quad \boldsymbol{B}_{P,T}^{\mathrm{MT,ET}}=\mu\boldsymbol{H}_{P,T}^{\mathrm{MT,ET}},\quad \boldsymbol{D}_{P,T}^{\mathrm{MT,ET}}=\varepsilon\boldsymbol{E}_{P,T}^{\mathrm{MT,ET}}.
\end{aligned}
\tag{2.59}
$$

Active algorithms of observations interpretation at the world network of magnetic observatories that were designed employing equations (2.58) and (2.59), i.e., with allowance for a non-force modification of the Earth's field.

The source of a non-force EMF, observed in the Earth's atmosphere are spherical (toroidal) electric conductivity currents, circulating in the Earth's spherical layers as well as electric currents in the ionosphere and in the vortex current, which also excite the MGF variations.

Clearly, a group of the Lorentz transforms for systems (2.58) and (2.59) remains the same as for the force part of Maxwell's equations.

Here it is necessary to investigate the inverse problem, namely: how to extract from standard Maxwell's equations in spherical coordinates supplementary electric currents that excite a non-force part of the magnetic field, being situated in spherical layers and on spherical surfaces that excite a non-force part of the magnetic field. To this end, let us present the following formulas for the total EMF:

$$
\begin{aligned}
&H_\theta=-\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)+\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi},\qquad E_\theta=-i\omega\mu_0A_\theta+\frac{1}{\sigma'}\nabla_\theta\nabla\cdot\boldsymbol{A},\\
&H_\phi=\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta},\qquad E_\phi=-i\omega\mu_0A_\phi+\frac{1}{\sigma'}\nabla_\phi\nabla\cdot\boldsymbol{A},\\
&H_r=\frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)-\frac{\partial}{\partial\phi}A_\theta\Big],\ E_r=-i\omega\mu_0A_r+\frac{1}{\sigma'}\nabla_r\nabla\cdot\boldsymbol{A}.
\end{aligned}
\tag{2.60}
$$

It will be recalled how to calculate the magnetic field rotor through a vector potential and the Laplace vector operator, based on the formula $\Delta = \nabla\nabla\cdot -\nabla\times\nabla\times$:

$$\begin{aligned}\nabla\times\boldsymbol{H} &= \nabla\nabla\cdot\boldsymbol{A} - \nabla\nabla\cdot\boldsymbol{A} + \nabla\times H = \nabla\nabla\cdot\boldsymbol{A} - (\nabla\nabla\cdot\boldsymbol{A} - \nabla\times H)\\ &= \nabla\nabla\cdot\boldsymbol{A} - (\nabla\nabla\cdot\boldsymbol{A} - \nabla\times\nabla\times\boldsymbol{A}) = \nabla\nabla\cdot\boldsymbol{A} - \Delta\boldsymbol{A}. \qquad (2.61)\end{aligned}$$

In (2.61), the term $\nabla\nabla\cdot\boldsymbol{A}$ is added and subtracted. Now, calculate θ-component of the magnetic field rotor:

$$\nabla_\theta\times\boldsymbol{H} = \Big(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}H_r - \frac{1}{r}\frac{\partial}{\partial r}H_\phi\Big)e_\theta. \qquad (2.62)$$

Here e_θ, e_ϕ, e_r are orths of a spherical coordinates system. In terms of potentials, this can be written down as

$$\begin{aligned}\nabla_\theta\times\boldsymbol{H} = &\frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}A_\phi + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}A_\phi -\\ &\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}A_\theta + \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}A_r - \frac{2}{r}\frac{\partial}{\partial r}A_\theta - \frac{\partial^2}{\partial r^2}A_\theta. \qquad (2.63)\end{aligned}$$

Now, add and subtract the expression $\nabla_\theta\nabla\cdot\boldsymbol{A}$:

$$\begin{aligned}\nabla_\theta\nabla\cdot\boldsymbol{A} = &\frac{2}{r^2}\frac{\partial}{\partial\theta}A_r + \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}A_r - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}A_\phi + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}A_\phi -\\ &\frac{\cos\theta}{r^2\sin\theta}\frac{\partial A_\theta}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}A_\theta - \frac{A_\theta}{r^2\sin^2\theta}. \qquad (2.64)\end{aligned}$$

It is not difficult to check that a difference between (2.63) and (2.64) is just a projection of the vector $\Delta\boldsymbol{A}$ onto a tangential plane to the spherical coordinate θ:

$$\begin{aligned}\Delta_\theta\boldsymbol{A} = &\frac{\partial^2}{\partial r^2}A_\theta + \frac{2}{r}\frac{\partial}{\partial r}A_\theta + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}A_\theta + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}A_\theta -\\ &\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}A_\theta + \frac{2}{r^2}\frac{\partial}{\partial\theta}A_r - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}A_\phi. \qquad (2.65)\end{aligned}$$

Thus,

$$\nabla_\theta\times\boldsymbol{H} = \nabla_\theta\nabla\cdot\boldsymbol{A} - \Delta_\theta\boldsymbol{A}. \qquad (2.66)$$

If in the third line from (2.60) we multiply E_θ by σ' and equate the result to (2.66), we will obtain

$$-i\omega\mu_0\sigma' A_\theta + \nabla_\theta\nabla\cdot\boldsymbol{A} = \nabla_\theta\nabla\cdot\boldsymbol{A} - \Delta_\theta\boldsymbol{A} \qquad (2.67)$$

or

$$\Delta_\theta\boldsymbol{A} = i\omega\mu_0\sigma' A_\theta. \qquad (2.68)$$

The latter equation already strictly corresponds to equation (2.6) in the domains, where $\boldsymbol{J}^{\mathrm{ST}}$=0. Thus, it appears possible to verify all the components of the rotor of the field, where it is also required to distinguish the needed electric currents components by the technique proposed. In order to show the feasibility of the second Maxwell's equation, it is sufficient to calculate the rotor from the latter components of (2.60). From the right, there will be strictly a magnetic field, because the gradient rotor identically becomes zero. The magnetic field divergence also becomes zero, as $\nabla \cdot \nabla \times \boldsymbol{A} \equiv 0$. As for the total current divergence, it is necessary to consider the following. Calculate the divergence from

$$\sigma' \boldsymbol{E} = -i\omega\mu_0\sigma'\boldsymbol{A} + \nabla\nabla \cdot \boldsymbol{A}. \tag{2.69}$$

Then we come to

$$\begin{aligned}
\sigma'\nabla \cdot \boldsymbol{E} &= -i\omega\mu_0\sigma'\nabla \cdot \boldsymbol{A} + \nabla \cdot \nabla\nabla \cdot \boldsymbol{A} \\
&= (-i\omega\mu_0\sigma'\phi + \Delta\phi) = \frac{1}{\sigma'}\nabla \cdot (\Delta\boldsymbol{A} - i\omega\mu_0\sigma'\boldsymbol{A}) = 0, \qquad (2.70)\\
\nabla \cdot \nabla \times \boldsymbol{H} &= \nabla \cdot (\Delta\boldsymbol{A} - i\omega\mu_0\sigma'\boldsymbol{A}) = \sigma'(\Delta\phi - i\omega\mu_0\sigma'\phi) = 0.
\end{aligned}$$

Comparing these two formulas, which were obtained from the magnetic and electric fields from (2.60), we see that the total current divergence is equal to zero if the potential satisfies equation (2.6) for $\boldsymbol{J}^{\mathrm{ST}} = 0$. Taking the above-said into account, let us divide (2.60) into the fields $\boldsymbol{H}_P^{\mathrm{MT}}$ and $\boldsymbol{H}_T^{\mathrm{ET}}$, and write down Maxwell's equations for each of them. As follows from the above, the fields $\boldsymbol{H}_P^{\mathrm{MT}}$ and $\boldsymbol{H}_T^{\mathrm{ET}}$ are formed by the same components of the total current, including $\boldsymbol{J}^{ST}$, which by definition is solenoidal:

$$\begin{aligned}
\nabla \cdot \sigma'\boldsymbol{E}_T &= -i\omega\mu_0\sigma'\nabla \cdot \boldsymbol{A} \\
&= -i\omega\mu_0\sigma'\Big(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_\phi + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta A_\theta\Big) = 0, \qquad (2.71)\\
\nabla \cdot \sigma'\boldsymbol{E}_P &= \Delta\phi - i\omega\mu_0\sigma'\phi = 0.
\end{aligned}$$

In formulas (2.71), the relation $\phi = \frac{1}{r^2}\frac{\partial}{\partial r}r^2 A_r$ is used. Hence, with allowance for the solenoidal property of the total current in the electric field, the division should be carried out as follows:

In the toroidal electric field

$$\begin{aligned}
E_{T\theta}^{\mathrm{MT}} &= -i\omega\mu_0 A_\theta, \qquad E_{T\phi}^{\mathrm{MT}} = -i\omega\mu_0 A_\phi,\\
E_{Tr}^{\mathrm{MT}} &= 0, \qquad \nabla \cdot \boldsymbol{E}_T^{\mathrm{MT}} = 0.
\end{aligned} \tag{2.72}$$

In the poloidal electric field

$$E_{P\theta}^{\mathrm{ET}} = \frac{1}{\sigma'}\nabla_\theta \nabla \cdot \boldsymbol{A}, \qquad E_{P\phi}^{\mathrm{ET}} = \frac{1}{\sigma'}\nabla_\phi \nabla \cdot \boldsymbol{A},$$
$$E_{Pr}^{\mathrm{ET}} = -i\omega\mu_0 A_r + \frac{1}{\sigma'}\nabla_r \nabla \cdot \boldsymbol{A}, \qquad \nabla \cdot \boldsymbol{E}_P^{\mathrm{ET}} = 0. \tag{2.73}$$

The magnetic field must also be divided with allowance for the solenoidal property of the currents generating it. The magnetic field of the poloidal type is formed of solenoidal components of the potentials A_θ and A_ϕ:

$$H_{P\theta}^{\mathrm{MT}} = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi), \qquad H_{P\phi}^{\mathrm{MT}} = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta),$$
$$H_{Pr}^{\mathrm{MT}} = \frac{1}{r\sin\theta}\Big[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial}{\partial\phi}A_\theta\Big]. \tag{2.74}$$

The magnetic field of the toroidal type will be components

$$H_{T\theta}^{\mathrm{ET}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r, \qquad H_{T\phi}^{\mathrm{ET}} = -\frac{1}{r}\frac{\partial}{\partial\theta}A_r. \tag{2.75}$$

It is not difficult to see that the poloidal and toroidal EMFs, obtained on the basis of other considerations, coincide by definition with those earlier obtained in (2.12) and (2.13). Moreover, equations for the rotor of these fields should be written down with allowance for (2.61)–(2.66), which reveal that in spherical coordinates one should distinguish the part, corresponding to the Laplacian. As the coordinates are spherical, this appears to be non-trivial. Let us carry out the following:

$$\nabla_\theta \boldsymbol{H}_P^{\mathrm{MT}} = \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}A_\phi + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}A_\phi -$$
$$= \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}A_\theta - \frac{2}{r}\frac{\partial}{\partial r}A_\theta - \frac{\partial^2}{\partial r^2}A_\theta, \tag{2.76}$$
$$\nabla_\theta \times \boldsymbol{H}_T^{\mathrm{ET}} = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}A_r.$$

To the poloidal field rotor in (2.76) we should add $\nabla_\theta\nabla\cdot A$ and then subtract it. In this case, for attaining a complete Laplacian it will lack a term of the form $\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}A_r$, to be separated from the potential divergence gradient, i.e.

$$\nabla_\theta \times \boldsymbol{H}_P^{\mathrm{MT}} + \nabla_\theta\nabla\cdot\boldsymbol{A} - \nabla_\theta\nabla\cdot\boldsymbol{A} = \Delta_\theta\boldsymbol{A} + \nabla_\theta\nabla\cdot\boldsymbol{A} + \frac{2}{r^2}\frac{\partial}{\partial\theta}A_r$$
$$= \Delta_\theta\boldsymbol{A} + \frac{2}{r^2}\frac{\partial}{\partial\theta}A_r = \sigma' E_{T\theta} + \frac{2}{r^2}\frac{\partial}{\partial\theta}A_r. \tag{2.77}$$

Hence follows that the rotor of θ-component of the poloidal type magnetic field can be presented as

$$\nabla_\theta \times \boldsymbol{H}_P^{\mathrm{MT}} = \sigma' E_{T\theta}^{\mathrm{MT}} + \frac{2}{r^2}\frac{\partial}{\partial\theta} A_r. \tag{2.78}$$

By carrying out similar calculations with all other rotor components, according to (1.117), we obtain

$$\nabla \times \boldsymbol{H}_P^{\mathrm{MT}} = \sigma' \boldsymbol{E}_T^{\mathrm{MT}} + \frac{2}{r}\,\mathrm{Grad}\, A_r = \sigma' \boldsymbol{E}_T^{\mathrm{MT}} + \frac{2}{r}\boldsymbol{H}_T^{\mathrm{ET}}. \tag{2.79}$$

Here Grad is a 2D gradient. At the same time, according to the second equation from (2.76), similar calculations for a toroidal field result in the formula

$$\nabla \times \boldsymbol{H}_T^{\mathrm{ET}} = \boldsymbol{H}_P^{\mathrm{MT}}. \tag{2.80}$$

In (1.79), a supplementary current $\frac{2}{r}\boldsymbol{H}_T^{\mathrm{ET}}$ is just the one that yields in spherical sources the additional toroidal field, defined in (1.29) and (1.31). Based on the calculations carried out, a formal mathematical solution to the inverse problem brings about Maxwell's equations separately for a force and a non-force parts of the magnetic field observed on the Earth. The force part of the EMF is

$$\begin{gathered} \nabla \times \boldsymbol{H}_P^{\mathrm{MT}} = \sigma' \boldsymbol{E}_T^{\mathrm{MT}} + \frac{2}{r}\boldsymbol{H}_T^{\mathrm{ET}}, \quad \nabla \times \boldsymbol{E}_T^{\mathrm{MT}} = -\mu_0 \frac{\partial}{\partial t}\boldsymbol{H}_P^{\mathrm{MT}}, \\ \nabla \cdot \boldsymbol{E}_T^{\mathrm{MT}} = 0, \quad \nabla \cdot \boldsymbol{H}_P^{\mathrm{MT}} = 0, \end{gathered} \tag{2.81}$$

its non-force part being

$$\begin{gathered} \nabla \times \boldsymbol{H}_T^{\mathrm{ET}} = \boldsymbol{H}_P^{\mathrm{MT}}, \quad \nabla \times \boldsymbol{E}_P^{\mathrm{ET}} = 0, \\ \nabla \cdot \boldsymbol{E}_P^{\mathrm{ET}} = 0, \quad \nabla \cdot \boldsymbol{H}_T^{\mathrm{ET}} = 0. \end{gathered} \tag{2.82}$$

Summing (2.81) and (2.82) and taking into account $\frac{\gamma}{\eta} \sim \frac{1}{r}$, and $(i\omega\mu_0\sigma')^{1/2} \sim \frac{1}{r}$, we obtain equations similar to (2.58) and (2.59). The main result of solving the inverse problem is that it appeared possible to reveal the mechanism of generation of a supplementary current causing the appearance in spherical sources of an additional non-force magnetic field $\boldsymbol{H}_T^{\mathrm{ET}}$. It should be noted that the solution to the direct problem in the previous sections yields a more adequate to the experiment result in terms of physics in the form of equations (2.58) and (2.59).

Conclusion

The proposed development of the electrodynamics of the MGF and its long-period variations, observed at the world network of geomagnetic stations, in the author's opinion, essentially extends the possibilities of studying both the MGF and its variations of any period. The form of presenting the material as formulated theorems of the principal problems is the most reasonable for the two main reasons: on the one hand, the development of problems of the MGF and its variations is associated with generalization of well-known classical theorems, such as the Helmholtz and the Gauss–Shmidt theorems, Maxwell's theorem of equations for geomagnetism, etc. On the other hand, although the above-mentioned generalized theorems have a concrete mathematical proof, their understanding being associated, basically, with the physics of events, is related direct ly to the MGF, and with the field of its variations at different periods.

The physics of events in geomagnetism is an intricate question, mainly, due to the absence of access to the MGF source. Nevertheless, the observations of the MGF, as well as its application to describing more general theorems, allowed us to make fundamental conclusions about the nature of the MGF source. As soon as was proved that a toroidal magnetic field is measured by magnetometers of the world network of stations in the Earth's atmosphere [3], there immediately arised a possibility to calculate its intensity not only in the atmosphere, but also in the vicinity of the liquid core. This has allowed rejecting the hypothesis of dynamo-excitation of the MGF, and following Elsasser and Frenkel, to turn to the development of the induction hypothesis.

The detected distance to the MGF source, its geometrical size, the current intensity in it, its stability as related to internal and external effects on the magnetic field, have allowed the verified conclusion about the MGF source as a toroidal electric current, supported by the Earth's stable rotation.

As for the long-period quiet solar-daily variations, their electromagnetic field is two-modal, its two-modality being analytically and numerically proved. In addition, the electromagnetic field of the electric type has been proved to be non-force in the sense that the Lorentz force in its magnetic field is absent and the electromotive force in the electric field is equal to zero. In this connection, the magnetic field of the electric type is not subject to skin-effect because it does not excite the electric current. It penetrates into the Earth three times as deep as the force magnetic field.

The properties of the MGF and its variations, investigated in this paper, in our opinion, do not exhaust all the problems known in the MGF and its variations. The author would like to expect that the approach proposed to studying the MGF and its variations could be useful in revealing still unknown features of the MGF and its variations.

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