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Analysis of noise stability of strip-transformation

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Abstract. In the paper, variations of the strip-method are investigated. Namely, we considered variants based on using different matrices: Hadamard, Haar, Frobenius, S-matrices, etc. These variants of the strip-method were implemented. The main purpose is to study the quality of restoration of one-dimensional signals (images are not considered) for various matrices in the case of impulse hindrances. Various types of matrices and signals have been tested. A theoretical estimation in terms of spectral coefficients of decomposition is proposed for the norm of error for the strip-transformation based on a Hadamard matrix in the case of an impulse hindrance.

Keywords: signal processing, orthogonal transformation, strip-method, Hadamard matrix, impulse hindrance, error estimation.

1. Introduction

An important problem of signal transmission through communication channels is the reduction of the level of hindrances and distortions and the accuracy increasing (or error decreasing). One of the methods of increasing the noise stability by information transfer and storage is the strip-method [1].

The advantage of this method is as follows: we have a linear combination of fragments of an initial signal or image at the starting point, so each fragment of the transferred message bears information about all fragments of the initial message without exceptions. It allows us, in case of loss or damage of one of fragments, to restore the whole signal or image without appreciable distortions.

At the terminal (receiving) point, a mixture of the signal and noise (received from the communication channel) is exposed to the inverse procedure. As a result, impulse hindrances are "stretched" lengthwise for the whole duration of the signal with simultaneous reduction of their amplitude. This leads to the reduction of a relative level of hindrances and, accordingly, to an increase in noise stability. The method reminds somewhat of the hologram, and it is widely used at transferring signals and images from satellites, since in ionosphere there can be a short-term loss of communication, i.e. the whole fragments of signals or images may completely vanish.

In the strip-method, the Hadamard matrices from the class of orthogonal matrices are usually used. Orthogonal matrices and the transformations constructed on their basis are widely applied to signal and image processing [2–5]. We can notice that although the strip-method is used for data transmission

from satellites, it is poorly studied even for the elementary impulse hindrances how the size of errors at signal restoration depends on the matrices.
In this paper, variations of the strip-method are investigated. Namely, different variants of matrices used in this method were considered: Hadamard, Haar, Frobenius, S-matrices, etc. The main purpose is to study the quality of restoration of one-dimensional signals (images are not considered) for various matrices in the case of impulse hindrances. Experiments are carried out and a theoretical estimation is presented.

2. STRIP-transformation of signals

According to [1], we give a definition of the notion "strip-transformation". At the first stage of this transformation, one-dimensional signals are split into blocks of identical length.

We assume that the vector $\vec{v} = (v_1, \ldots, v_n)$ is given and $n = k \cdot m$. Therefore it is possible to write $\vec{v} = \left(v_1, \cdots, v_m, v_{m+1}, \cdots, v_{2m}, \cdots, v_{(k-1)m+1}, \cdots, v_n\right)$. With this vector, the following matrix of the size $k \times m$ can be associated

$$A = \begin{pmatrix} v_1 & \cdots & \cdots & v_m \\ v_{m+1} & \cdots & \cdots & v_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ v_{(k-1)m+1} & \cdots & \cdots & v_n \end{pmatrix}.$$

In addition, we assume that the matrix of the size $k \times k$ is also given

$$H = \begin{pmatrix} h_{11} & \cdots & h_{1k} \\ & \cdots & \\ h_{k1} & \cdots & h_{kk} \end{pmatrix}.$$

Usually H is a Hadamard matrix. Thus, the direct strip-transformation consists in multiplication of these matrices , and we obtain

$$Q = H \cdot A = \begin{pmatrix} v_1^{'} & \cdots & v_m^{'} \\ & \cdots & \\ v_{(k-1)m+1}^{'} & & v_n^{'} \end{pmatrix}.$$

Here it is necessary to take into account that any i-th line of the result is a linear combination of all initial line-pieces of a signal with coefficients from the i-th line of the matrix H. The holographic property of the transformation follows from the fact that the i-th line-piece of the result comprises the information about all line-pieces of the initial signal.

Further, the matrix can be transformed to a vector and as a result we obtain $\vec{v}' = \left(v_1', \cdots, v_m', v_{m+1}', \cdots, v_{2m}', \cdots, v_{(k-1)m+1}', \cdots, v_n'\right)$.

The vector $\vec{v}'$ is called the result of the strip-transformation of the vector $\vec{v}$ with the help of the matrix H, and this fact is designated as $\vec{v}' = str_H(\vec{v})$.

For the Hadamard and other similar matrices, there is a property $H \cdot H^T = H^T \cdot H = k \cdot E$. Therefore, to execute the inverse transformation, it is necessary to use H^T. Actually, different estimations show that it is better to use orthogonal matrices. They give an error smaller in size than others. As a result, we obtain $A = \frac{1}{k} \cdot (H^T \cdot Q)$. The initial signal is restored from the matrix A by its extension into a line.

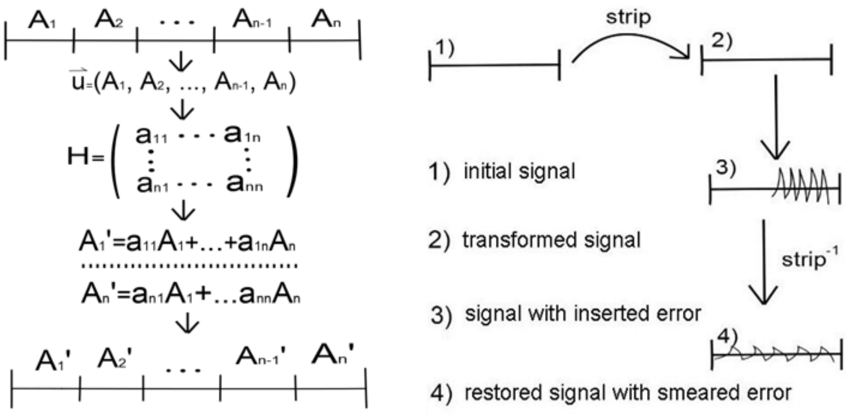

Figure 1. A general scheme of the strip-transformation

3. Some classes of matrices

Further, let us use the following designations. We suppose that the lines and columns of matrices are numbered by integer non-negative numbers, including zero. Next, if a_{ij} is an element of a matrix, then *i* is the number of a line, *j* is the number of a column. We assign an impulse hindrance setting one element of the matrix Q equal to zero. It is obvious that it is equivalent to setting one element of the vector $\vec{v}'$ to zero.

The matrix $A = (a_{ij}), (0 \le i, j \le n-1)$ is called orthogonal, if a scalar product of any two different lines is equal to zero, i.e. for any $i_1 \ne i_2$ the equality

$\sum_{j=0}^{n-1} a_{i_1 j} \cdot a_{i_2 j} = 0$ is valid. It is known that it is equivalent to orthogonality by columns. The orthogonal matrices whose elements are ± 1 are called the Hadamard matrices. We will use the various orthogonal matrices as the matrix H from the strip-method definition.

3.1 Hadamard matrices

In papers [6–8], the methods of construction of normalized Hadamard matrices, known as the Paley constructions, are described. Various examples of application of Hadamard transformation can be found in [9–11].

Definition 1. Let p be a prime number, $p \neq 2$, α be an arbitrary integer which is not divided by p. A Legendre symbol (α/p) is equal to 1 if the equation $x^2 \equiv \alpha (\mathrm{mod}\ p)$ has a decision and –1 otherwise.

It is well-known [12] that the following formula is true

$$(\alpha/p) = -1^M, \quad M = \sum_{x=1}^{p_1} \left[\tfrac{2\alpha x}{p}\right], \quad p_1 = \tfrac{1}{2}(p-1).$$

Here square brackets designate the integral part of a fraction. We notice that the resulted formula is very simple for calculation. Certainly, to calculate it on a computer, it is not necessary to raise –1 to the power. It is enough to supervise a property M of being even. Setting $R = M - 2\left[\frac{M}{2}\right]$, we obtain $-1^M = 1 - 2R$.

Definition 2. If $A = (\alpha_{ij})$ is $(n \times n)$ - matrix, $B = (b_{ks})$ is $(m \times m)$ - matrix, then Kronecker product $A \times B$ is called $(nm \times nm)$ - matrix.

$$A \times B = \begin{pmatrix} \alpha_{00} B & \alpha_{01} B & \dots & \alpha_{0n-1} B \\ \alpha_{10} B & \alpha_{11} B & \dots & \alpha_{1n-1} B \\ \dots & \dots & \dots & \dots \\ \alpha_{n-1,0} B & \alpha_{n-1,1} B & \dots & \alpha_{n-1,n-1} B \end{pmatrix}.$$

Let us give a short description of some classes of Hadamard matrices.

3.1.1. Matrices of order $n = 2^k$ are defined by induction.

For $\kappa = 1$, let

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

If H_k is already defined, then let

$H_{k+1} = H_1 \times H_k$ or, what is the same, $H_{k+1} = \begin{pmatrix} H_k & H_k \\ H_k & -H_k \end{pmatrix}$.

3.1.2. Matrices of order $n = p+1$, where $p \equiv 3(\mathrm{mod}\,4)$ is a prime number.

We define $\chi(k) = (k/p)$ and

$\alpha_{ij} = +1$, $(i = 0$ or $j = 0)$,

$\alpha_{ij} = \chi(j-i)$, $(1 \le i, j \le p, i \ne j)$,

$\alpha_{ii} = -1$, $(1 \le i \le p)$.

Proposition. The following equalities are valid:

1) $\alpha_{ij} = -\alpha_{ij}$, $(i, j \ge 1, i \ne j)$,

2) $\alpha_{ij} = \alpha_{i+k, j+k}$, $(i, j \ge 1, i \ne j)$,

3) $\alpha_{i, i+k} = \chi(k)$, $(i, k \ge 1)$,

4) $\alpha_{i, i+k} = -\alpha_{i, i+p-k}$, $(i, k \ge 1)$.

The proof directly follows from well-known properties [12] of the Legendre symbol.

The first equality from this proposition means that the matrix A is antisymmetric. The second equality shows that on any line parallel to the main diagonal all elements are equal. The third equality gives, in particular, a structure of the first line: it is $\langle 1, -1, \chi(1), \chi(2), \ldots \rangle$. From the fourth equality, it follows that in the first line, if we consider its piece $\langle \alpha_{12}, \alpha_{13}, \ldots, \alpha_{1,p-1} \rangle$ laying above the main diagonal, the elements equidistant from the ends have opposite signs. Thus the whole piece is restored by its half.

As an example, we can consider a matrix of the 12-th order, using for brevity ± instead ±1, respectively.

$$\begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + \\
+ & - & + & - & + & + & + & - & - & - & + & - \\
+ & - & - & + & - & + & + & + & - & - & - & + \\
+ & + & - & - & + & - & + & + & + & - & - & - \\
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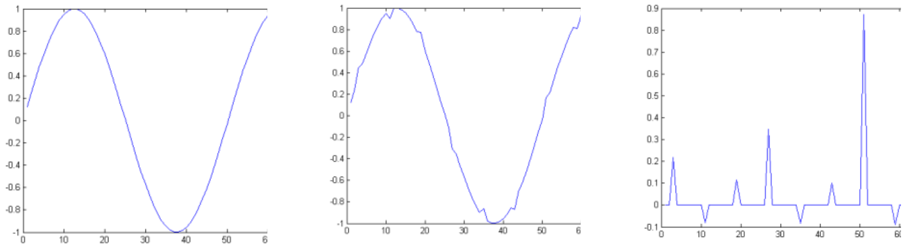
+ & + & - & + & + & + & - & - & - & + & - & -
\end{pmatrix}$$



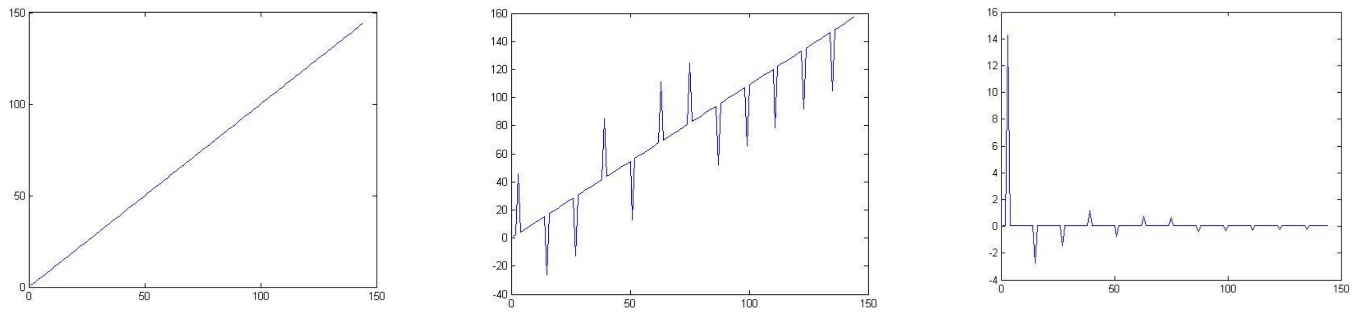
a) initial signal b) restored signal c) error

Figure 2. Strip-transformation of the function $\sin(x/8)$; a Hadamard matrix is used; $n = 2^k$ $n=16$

3.1.3. Another class of Hadamard matrices has been considered. These matrices can be represented in a form $H_n = X_1 \times P_1 + X_2 \times P_2$, where $\times$ is the Kronecker product. Thereby, P_1 and P_2 are the Hadamard matrices of the second order satisfying the condition $P_1^T P_2 + P_2^T P_1 = 0$; X_1 and X_2 are rarefied matrices which satisfy the condition $X_1 X_2^T - X_2 X_1^T = 0$.

An elementary example: $X_1 = P_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $X_2 = P_2 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$.

Computational experiments for matrices mentioned above are represented below.



a) initial signal b) restored signal c) error

Figure 3. Strip-transformation of the linear function $Q(2,3) = 0$

3.2. Conference matrices

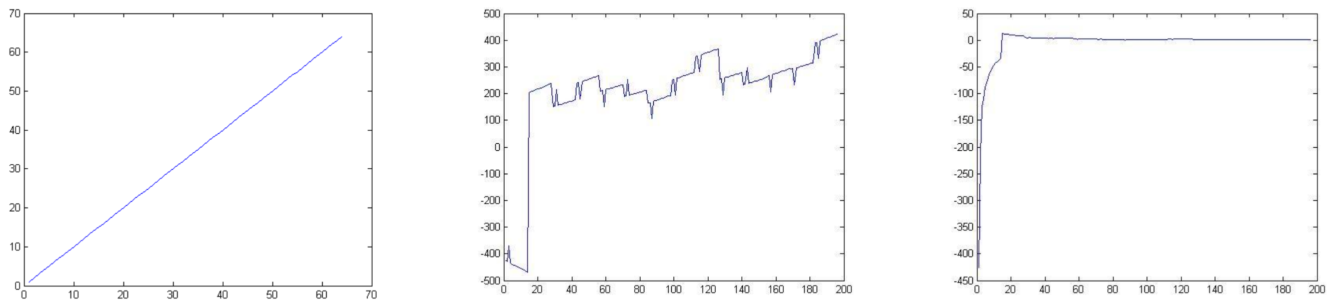
A conference matrix C_n of order n is $n \times n$ matrix [14] which has zeros on the main diagonal and other its elements are equal to 1 or –1, and it satisfies the equality $C_n C_n^T = (n-1)E_n$. The name "conference" is the result of using such matrices when constructing the networks having one and the same attenuation between any pair of terminals. Let $n = 4t + 1 = p + 1$, where p is an odd prime number.

Further we define $C_n = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & B_{n-1} & \\ 1 & & & \end{pmatrix}$, and $B_{n-1} = (b_{ij})$ is a symmetric matrix.

Thereby we suppose that $b_{ij} = 0$ for $i = j$, $b_{ij} = 1$ when $j - i$ is a square modulo p, $b_{ij} = -1$ when $j - i$ is not a square modulo p. A cyclic matrix C_n will be a conference matrix. A simple example of such matrix is

$$C_n = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

Computational experiments for the matrix mentioned above are represented below.



a) initial signal b) restored signal c) error

Figure 4. Strip-transformation of a linear function with the help of a conference matrix

3.3. S-matrices

The S-matrix is defined [15] by induction. We enter a designation S_n for any matrix of order n. In fact, we construct a matrix of order $n-1$. The matrix S_{n-1} is derived from a Hadamard matrix of order $n = 2^k$ by deleting its first line and column. Afterwards, the elements in the Hadamard matrix are replaced by the following rule. The elements equal to 1 are replaced with zeros, $(1 \to 0)$, and elements equal to – 1 are replaced with units, $(-1 \to 1)$. Here matrices of the first and third order are represented

$$S_1 = (1),\quad S_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

These matrices are constructed on the basis of the following Hadamard matrices

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

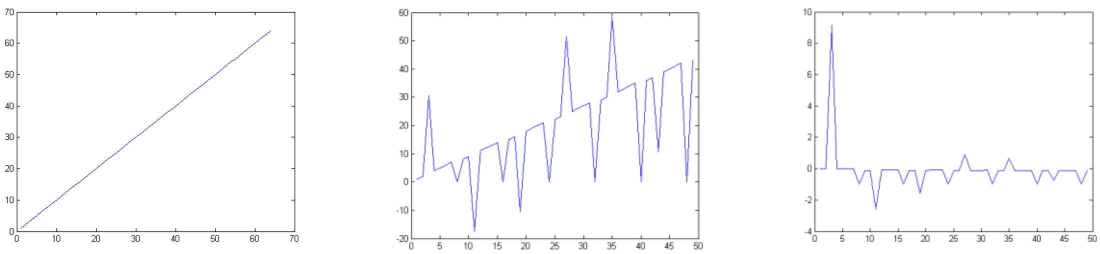
The next properties of S-matrices are known:

1) $S_n S_n^T = \frac{1}{4}(n+1)(I_n + J_n)$,

2) $S_n J_n = J_n S_n = \frac{1}{2}(n+1)J_n$,

3) $S_n^{-1} = \frac{2}{n+1}(2S_n^T - J_n)$,

where I_n is an identity matrix and J_n is a matrix all elements of which are units.



a) initial signal b) restored signal c) error

Figure 5. Strip-transformation of a linear function on the basis of S-matrix

3.4. Haar matrices

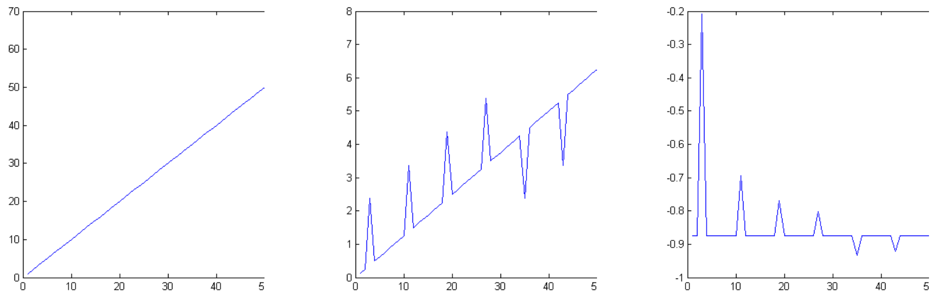
A Haar matrix [13] is an orthogonal matrix, i.e. $AA^T = A^T A = E$. Haar matrices are constructed for dimensions $n = 2^k$. It is well known that Haar transformation is one of the simplest wavelet-transformations [16–17].

Examples of matrices of the 4-th and 8-th order are represented below

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix},$$

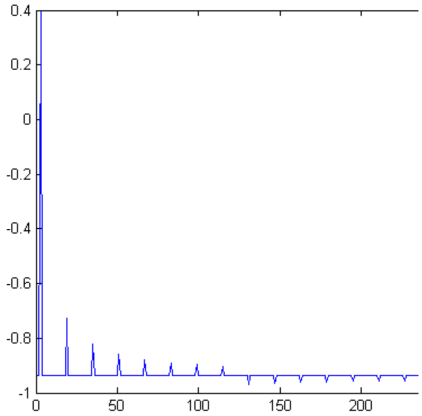
$$H_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}.$$

It is rather simply to formulate a general algorithm for constructing the Haar matrices. When the dimension increases, the subsequent "waves" are twice shorter than previous and their amplitudes are multiplied by $\sqrt{2}$.

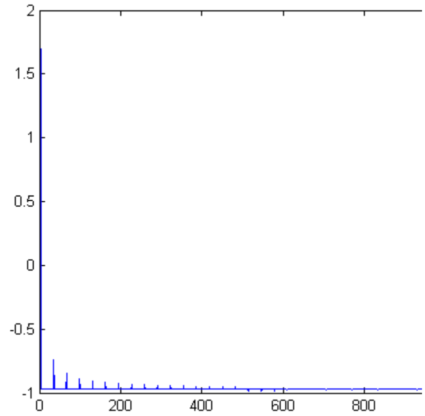


a) initial signal b) restored signal c) error

Figure 6. Strip-transformation of a linear function by means of a Haar matrix, *n*=64



a) error for n = 256



b) error for n = 1024

Figure 7. A Haar matrix is used, errors are given for high dimensions

3.5. Frobenius matrices

A Frobenius matrix A_n of order n, i.e. its dimension is $n \times n$, looks like [18]

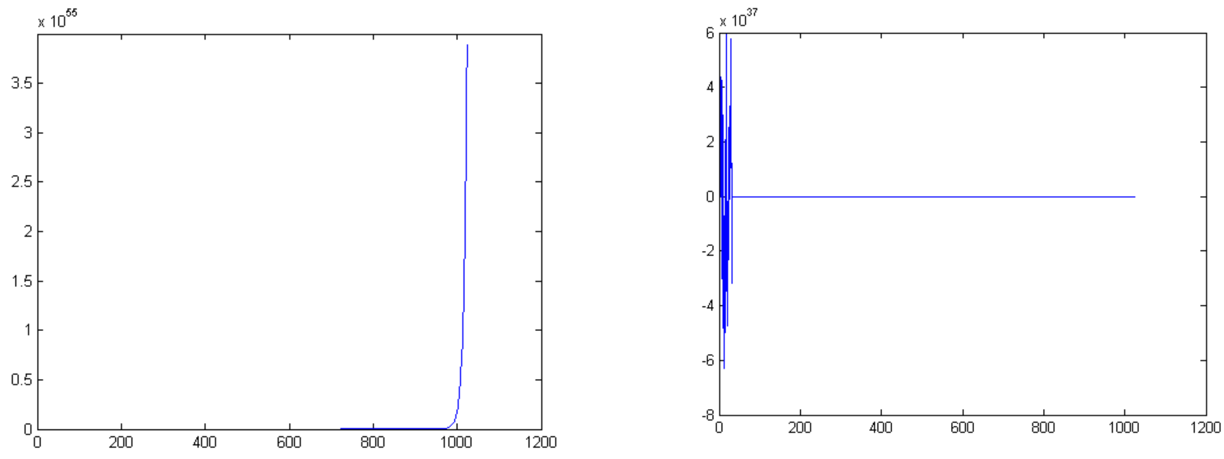
$$A = \begin{pmatrix} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix},$$

where the corresponding elements are taken from a polynomial $P(x) = x^n - a_{n-1}x^{n-1} - \ldots - a_0$.

Such matrix is called an accompanying matrix. If we apply induction by n and take into account decomposition by the first line, then we notice that

$\det(A - \lambda I) = (-1)^n(\lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \ldots - a_0) = (-1)^n P(\lambda)$, and

if $a_0 \neq 0$, then $$A^{-1} = \begin{pmatrix} -\dfrac{a_1}{a_0} & -\dfrac{a_2}{a_0} & \cdots & -\dfrac{a_{n-1}}{a_0} & \dfrac{1}{a_0} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$



a) restored exponential function b) error

Figure 8. Strip-transformation of exponential by a Frobenius matrix for n = 1024

It follows from experiments that this matrix works well, but in the initial part we have a "wild" error. As a decision of this problem, we suggest the following approach. We can start with any unnecessary signal, which allows us to omit the initial part and restore the next informative part of a signal without a strong error.

4. The restored signal error estimation for impulse hindrance in the case of STRIP-transformation based on Hadamard matrix

Let us consider the strip-transformation on the basis of a Hadamard matrix. We will try to estimate how the distortion received in a restored signal depends on the hindrance brought in an image of the strip-transformation.

It is well known that the lines of a Hadamard matrix are the Walsch functions [13], and further we use the corresponding terminology. Let us designate by $Wal(k)$ the function corresponding to the k-th line of a matrix.

Now let a signal be represented in the form $Y = \sum_{k=o}^{N-1} a(k)Wal(k)$. Further we apply the strip-transformation on the basis of a Hadamard matrix of order N. We assume that there is an impulse hindrance which nulls one component of the received vector. Let now Y' be the result of an inverse transformation, i.e. the restored signal.

Proposal. The following error estimation is valid

$$\| Y' - Y \|_\infty \le \frac{1}{N} \max |a(k)|.$$

Proof. If we use designations from Section 1, then we have $Q = HA,\ A = \frac{1}{N} H^{-1} Q$.

Further we have a matrix of hindrances Δ of the same dimension as the matrix A, in which only one component is not zero, i.e. Δ looks like as

$$\Delta = \begin{array}{cc} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \\ i \\ \\ \end{array} \\ \begin{array}{ccc} & j & \end{array} & \end{array} \cdot$$

Analogously, taking into account a hindrance, we obtain $A' = \frac{1}{N} H^{-1}(Q + \Delta) = A + \frac{1}{N} H^{-1} \Delta$.

So, it follows that $\| Y' - Y \|_\infty = \| A' - A \|_\infty = \frac{1}{N} |c| \le \frac{1}{N} \max |a(k)|$.

5. Conclusion

In this paper, different variants of matrices used in the strip-method are investigated: the Hadamard, the Haar, conference matrices, etc. The methods of construction of such matrices are considered. The main question is the quality of signals restored by using different matrices in the case of impulse hindrances.

A brief review of the results obtained is as follows. As for a linear function, we can say that the maximum error appears at the beginning of the restored signal for all matrices here considered. With the least error, such signal has been restored with the help of a Haar matrix. The same result is true for exponentials, but we do not give the details here. In this case, we have several peaks, one of which is essentially greater than the others. Thus, for exponentials, the Haar matrix works better. As for a sinusoid, the situation is more difficult. For example, a Hadamard matrix of order $n=p+1$, $p=3$ ($mod 4$), p is a prime number, is more suitable for processing a sinusoid signal. Probably, such matrices can be efficiently used for processing the limited periodic functions.

The estimation of the norm of error for the strip-transformation based on Hadamard matrix in the case of impulse hindrance in terms of spectral coefficients of decomposition is obtained. The structure of the matrix $H^{-1}\Delta$ considered in the last section of this paper is obvious. There is only one column with non-zero elements and they are equal to $\pm c$, since all elements of a Hadamard matrix are equal to ± 1. Also, we can see that the error in the restored signal is N times less than a hindrance brought in the image of the strip-transformation, and this is an important circumstance.

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