Bull. Nov. Comp. Center, Num. Anal., 9 (2000), 69–75

Finite element trace theorems

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The theorems on traces of functions from the Sobolev spaces play an important role in studying boundary value problems of mathematical physics. These theorems are commonly used for a priori estimates of the stability with respect to boundary conditions. The trace theorems play also very important role to construct and to investigate effective domain decomposition methods. The main focus of this talk is to study the case when norms of functions given in some domain dependent on parameters. Corresponding Sobolev spaces with the parameter dependent norms are generated, for instance, by elliptic problems with disproportional anisotropic coefficients. The main goal is to introduce the parameter dependent norms of traces of functions on the boundary such that the corresponding constants in the trace theorems are independent of the parameters. In the finite element case (finite element functions in the domain and finite element traces on the boundary), the corresponding constants should be independent of the mesh step too.

1. The finite element trace theorem for the Sobolev spaces $H^1_{p,q}$

In this section, we design equivalent norms in the trace space of finite element functions for solving by the domain decomposition methods [2–11, 13, 14] the system of grid equations approximating the following boundary value problem:

$$\begin{aligned} -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + a_0(x) u &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \quad\quad\quad x \in \Gamma. \end{aligned} \tag{1}$$

We assume that Ω is a bounded, polygonal region and Γ is its boundary. Let Ω^h be a regular triangulation of Ω which is characterized by a parameter h.

Let us introduce the Sobolev spaces $H^1_{p,q}(\Omega)$ with the norms

$$\begin{aligned} \|u\|^2_{H^1_{p,q}(\Omega)} &= p|u|^2_{H^1(\Omega)} + q\|u\|^2_{L_2(\Omega)}, \\ \|u\|^2_{L_2(\Omega)} &= \int_\Omega u^2(x)dx, \\ |u|^2_{H^1(\Omega)} &= \int_\Omega (|\nabla u(x)|)^2 \, dx. \end{aligned} \tag{2}$$

Here

$$p \equiv \text{const} \geq 0, \qquad q \equiv \text{const} \geq 0, \qquad p + q > 0.$$

Introduce the bilinear form

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}(x)\, \frac{\partial u}{\partial x_j}\, \frac{\partial v}{\partial x_i} + a_0(x)uv \right) dx.$$

We assume that the coefficients of problem (1) are such that $a(u,v)$ is a symmetric, coercive and continuous form in $H^1_{p,q}(\Omega)$, i.e.,

$$a(u,v) = a(v,u), \qquad \forall u,v \in H^1_{p,q}(\Omega),$$
$$\mu_0 \|u\|^2_{H^1_{p,q}(\Omega)} \le a(u,u) \le \mu_1 \|u\|^2_{H^1_{p,q}(\Omega)}, \quad \forall u \in H^1_{p,q}(\Omega),$$

Here μ_0, μ_1 are positive constants independent of p and q.

The main goal of this section is the study of a space of the traces of the finite element functions on the boundaries of Ω which is generated by the norm $H^1_{p,q}(\Omega)$.

Let $p > 0$, $q \ge 0$. Introduce the following norm in the Sobolev space $H^{1/2}_{p,q}(\Gamma)$:

$$\begin{aligned} \|\varphi\|^2_{L^2(\Gamma)} &= \int_\Gamma \varphi^2(x)\, dx, \\ |\varphi|^2_{H^{1/2}(\Gamma)} &= \int_\Gamma \int_\Gamma \frac{(\varphi(x) - \varphi(y))^2}{|x-y|^2}\, dx\, dy, \\ \|\varphi\|^2_{H^{1/2}_{p,q}(\Gamma)} &= \begin{cases} p|\varphi|^2_{H^{1/2}(\Gamma)} + (pq)^{1/2}\|\varphi\|^2_{L_2(\Gamma)} & \text{if } 0 < p/q \le 1, \\ p|\varphi|^2_{H^{1/2}(\Gamma)} + q\|\varphi\|^2_{L_2(\Gamma)} & \text{if } p/q \ge 1. \end{cases} \end{aligned} \tag{3}$$

The following theorem is valid.

Theorem 1. *There exist positive constants c_1, c_2, independent of p, q, such that*

$$\|\varphi\|_{H^{1/2}_{p,q}(\Gamma)} \le c_1 \|u\|_{H^1_{p,q}(\Omega)}$$

for any function $u \in H^1_{p,q}(\Omega)$, where $\varphi \in H^{1/2}_{p,q}(\Gamma)$ is the trace of u on the boundary Γ. Conversely, for any function $\varphi \in H^{1/2}_{p,q}(\Gamma)$ there exists $u \in H^1_{p,q}(\Omega)$ such that

$$u(x) = \varphi(x), \quad x \in \Gamma, \qquad \|u\|_{H^1_{p,q}(\Omega)} \le c_2 \|\varphi\|_{H^{1/2}_{p,q}(\Gamma)}.$$

Proof. In fact, the case $0 < p/q \le 1$ was considered in [1]. Using the trace theorem for the seminorm (see, for instance, [9]), the case $1 \le p/q$ can be easily proved. □

Denote by W a subspace of real-valued continuous functions linear on triangles and V the space of traces of functions from W at Γ.

Unfortunately, the norm (3) does not work for the finite element case (with the constant c_2, independent of h). Indeed, let Ω be a unit square with the uniform grid. Let

$$p/q \ll h^2; \qquad \varphi^h(x) = 1, \quad x \in \Gamma.$$

Then for any $u^h(x) \in W$, such that

$$u^h(x) = \varphi^h(x), \quad x \in \Gamma,$$

we have

$$\|u^h\|^2_{H^1_{p,q}(\Omega)} \geq \frac{1}{6} qh, \qquad \|\varphi^h\|^2_{H^{1/2}_{p,q}(\Gamma)} = 4q(p/q)^{1/2}.$$

Let now $p \geq 0$, $q \geq 0$, and $p + q > 0$. Define the following norm in the finite element space V:

$$\|\varphi^h\|^2_{H^{1/2}_{p,q,h}(\Gamma)} = \begin{cases} hq\|\varphi^h\|^2_{L_2(\Gamma)} & \text{if } p/q \leq h^2, \\ p|\varphi^h|^2_{H^{1/2}(\Gamma)} + (pq)^{1/2}\|\varphi^h\|^2_{L_2(\Gamma)} & \text{if } h^2 \leq p/q \leq 1, \\ p|\varphi^h|^2_{H^{1/2}(\Gamma)} + q\|\varphi^h\|^2_{L_2(\Gamma)} & \text{if } p/q \geq 1. \end{cases}$$

The following theorem is valid.

Theorem 2. *There exist positive constants c_1, c_2, independent of p, q, h, such that*

$$\|\varphi^h\|_{H^{1/2}_{p,q,h}(\Gamma)} \leq c_1\|u^h\|_{H^1_{p,q}(\Omega)},$$

for any function $u^h \in W$, where $\varphi^h \in V$ is the trace of u^h on the boundary Γ. Conversely, for any function $\varphi^h \in V$ there exists $u^h \in W$, such that

$$u^h(x) = \varphi^h(x), \quad x \in \Gamma, \qquad \|u^h\|_{H^1_{p,q}(\Omega)} \leq c_2\|\varphi^h\|_{H^{1/2}_{p,q,h}(\Gamma)}.$$

Proof. For the case $h^2 \leq p/q$ the proof is based on the technique from [7, 9, 11]. For the case $p/q \leq h^2$ we use the trivial extension of φ^h by zero onto inside nodes. □

2. Analysis of Poincare–Steklov operators for anisotropic elliptic problems

In this section, we consider a model anisotropic problem which generates the bilinear form $a(u, v)$

$$a(u,v) = \int_\Omega \left(p_1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + p_2 \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) dx,$$

where

$$p_1 \equiv \text{const} > 0, \qquad p_2 \equiv \text{const} > 0.$$

Assume that $p_1 < p_2$. Let Ω be the unit square. The analysis of the Poincare–Steklov operators which corresponds to the bilinear form $a(u,v)$ is equivalent to the analysis of traces of functions on the boundary Γ of the domain Ω with respect to the norm

$$\|u\|^2 = a(u,u).$$

Using evident scaling of variables, we can reduce the analysis of traces with respect to the anisotropic norm $\|u\|$ to the analysis with respect to the isotropic norm but in the anisotropic domain $\widetilde{\Omega}$

$$\|u\| = (p_1/p_2)^{1/2} \|u\|_{H^1(\widetilde{\Omega})}.$$

Here

$$\widetilde{\Omega} = \{(x,y) \mid 0 < x < 1,\ 0 < y < H\},$$

where $H = (p_1/p_2)^{1/2}$. Denote by k an integer part of $1/H$ and set $H_1 = 1/k$,

$$\begin{aligned}
&S_i^- = \{(x,0) \mid (i-1)H_1 \le x < (i+1)H_1\}, \\
&S_i^+ = \{(x,H) \mid (i-1)H_1 \le x < (i+1)H_1\}, \quad i = 1,\dots,k-1, \\
&L = \{(0,y) \mid 0 \le y < H\}, \\
&R = \{(1,y) \mid 0 \le y < H\}, \\
&S_0^- = L \cup S_1^-, \quad S_0^+ = L \cup S_1^+, \quad S_k^- = R \cup S_{k-1}^-, \quad S_k^+ = R \cup S_{k-1}^+.
\end{aligned}$$

Define

$$\begin{aligned}
\|\varphi\|^2_{H^{1/2}(\Gamma)} &= H\|\varphi\|^2_{L^2(\Gamma)} + |\varphi|^2_{H^{1/2}(\Gamma)}, \\
\|\varphi\|^2_{L^2(\Gamma)} &= \int\limits_\Gamma \varphi^2(x)\,dx, \\
|\varphi|^2_{H^{1/2}(\Gamma)} &= \sum_{i=0}^{k} \int\limits_{S_i^-}\int\limits_{S_i^-} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dx\,dy + \\
&\int\limits_{S_i^+}\int\limits_{S_i^+} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dx\,dy + \int\limits_{S_i^-}\int\limits_{S_i^+} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}\,dx\,dy.
\end{aligned}$$

The following lemma holds [10]:

Lemma 1. *There exists a positive constant c_1 independent of H, such that*

$$\|\varphi\|_{H^{1/2}(\Gamma)} \leq c_1\|u\|_{H^1(\Omega)},$$
$$|\varphi|_{H^{1/2}(\Gamma)} \leq c_1|u|_{H^1(\Omega)}$$

for any function $u \in H^1(\Omega)$, where $\varphi \in H^{1/2}(\Gamma)$ is the trace of u at the boundary Γ. Conversely, there exists a positive constant c_2, independent of H, such that for any function $\varphi \in H^{1/2}(\Gamma)$ there exists $u \in H^1(\Omega)$ such that

$$u(x) = \varphi(x), \quad x \in \Gamma,$$
$$\|u\|_{H^1)(\Omega)} \leq c_2\|\varphi\|_{H^{1/2}(\Gamma)}$$
$$|u|_{H^1(\Omega)} \leq c_2|\varphi|_{H^{\frac{1}{2}}(\Gamma)}.$$

Unfortunately, in the case of finite element spaces the above norm works only for isotropic grids in $\tilde{\Omega}$. To consider anisotropic grids, we need to define the grid dependent norms. Assume that there is a rectangular grid in Ω with the grid steps h_1 (in x direction) and h_2 (in y direction). Denote by $H_h(\Omega)$ the piecewise linear finite element space for this grid. The sides of Ω denote by

$$I_1 = \{(x,0) \mid 0 < x < 1\}, \qquad I_2 = \{(x,1) \mid 0 < x < 1\},$$
$$I_3 = \{(0,y) \mid 0 < y < 1\}, \qquad I_4 = \{(1,y) \mid 0 < y < 1\}.$$

For any finite element function $\varphi^h \in H_h(\Gamma)$ we put in correspondence the vector φ in the standard way.

The following lemmas hold.

Lemma 2. *Let $\varphi^h \in H_h(\Gamma)$, such that*

$$\varphi^h(x) = 0, \quad x \in I_2 \cup I_3 \cup I_4$$

Define the matrix S

$$(S\varphi, \varphi) = \inf |u^h|^2_{H^1(\Omega)}$$

for any $u^h \in H_h(\Omega)$, such that $u^h(x) = \varphi^h(x)$, $x \in \Gamma$.

Then there exist constants c_1, c_2, independent of h_1 and h_2, such that

$$c_1(S\varphi, \varphi) \leq \|\varphi^h\|^2_{H^{1/2}(\Gamma)} + h_2|\varphi|^2_{H^1(I_1)} \leq c_2(S\varphi, \varphi).$$

Lemma 3. *Let $\varphi^h \in H_h(I_1)$. Define the matrix S*

$$(S\varphi, \varphi) = \inf \|u^h\|^2_{H^1(\Omega)}$$

for any $u^h \in H_h(\Omega)$, such that $u^h(x) = \varphi^h(x)$, $x \in \Gamma$.

Then there exist constants c_1, c_2, independent of h_1 and h_2, such that

$$c_1(S\varphi, \varphi) \leq \|\varphi^h\|^2_{H^{1/2}(I_1)} + h_2|\varphi|^2_{H^1(I_1)} \leq c_2(S\varphi, \varphi).$$

Finally, we have the following theorem.

Theorem 3. *Let $\varphi^h \in H_h(\Gamma)$. Define the matrix S*

$$(S\varphi, \varphi) = \inf \|u^h\|^2_{H^1(\Omega)}$$

for any $u^h \in H_h(\Omega)$, such that $u^h(x) = \varphi^h(x)$, $x \in \Gamma$.

Then there exist constants c_1, c_2, independent of h_1 and h_2, such that

$$c_1(S\varphi, \varphi) \leq \|\varphi^h\|^2_{H^{1/2}(\Gamma)} + h_2\Big(|\varphi|^2_{H^1(I_1)} + |\varphi|^2_{H^1(I_2)}\Big) + h_1\Big(|\varphi|^2_{H^1(I_3)} + |\varphi|^2_{H^1(I_4)}\Big) \\ \leq c_2(S\varphi, \varphi).$$

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