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Numerical solution to first boundary-value problem for axially symmetric Poisson equations using finite element method

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Abstract. Unlike common approaches this paper does not rely on results for 3D problems. Degeneration of the operator on a part of the border requires the use of the corresponding weight functional spaces to analyze a variational statement to build a difference scheme and to prove the convergence. A result of convergence is proven using piecewise linear finite elements. The estimate of the convergence rate of the approximate solution to the exact one is not worse than in the case of non-degenerate equation. A numerical example that confirms the estimate is given.

1. Introduction

This paper deals with the first boundary value problem for the following degenerating elliptic equation:

$$\mathcal{A}(u) \equiv -\frac{\partial^2 u}{\partial x^2} - \frac{1}{y}\frac{\partial}{\partial y}\Big(y\frac{\partial u}{\partial y}\Big) = f(x,y), \qquad (1)$$

in the domain D, which lies in the top half-plane $y > 0$. The boundary Γ of the domain D consists of a section $\Gamma_0 = [a,b]$ of the axis $y = 0$ and a smooth curve Γ_1 with its ends at the points a and b. To exclude possible singularities in a junction of Γ_1 and Γ_0 and to simplify a numerical analysis, we would assume that there exists $d > 0$ such that a subregion $D \cap (y < d)$ is a rectangle.

Equation (1) is a common axially symmetric Poisson equation in the cylindrical coordinates. Various approaches to solution of this problem were considered in a numerous publications [1–3].

The main feature of the problem in question is its degeneration on part of the border Γ_0. This factor must be taken into account while formulating a variational statement, building a variation-difference scheme using a finite element method and when analyzing convergence. To analyze degenerating elliptical equations it is convenient to use appropriate weighted Sobolev's spaces [4].

First, let us rewrite equation (1) in the divergent form to apply a variational method

$$-\frac{\partial}{\partial x}\Big(y\frac{\partial u}{\partial x}\Big)-\frac{\partial}{\partial y}\Big(y\frac{\partial u}{\partial y}\Big)=yf(x,y). \tag{2}$$

The equation (2) is elliptic in the domain $D_\delta = D\cap(y>\delta)$, $\delta>0$, and it degenerates on the axis $y=0$. Note that depending on exponential order of degeneration, the boundary values may be required either on the whole border Γ or on Γ_1 only, while Γ_0 would be free of the boundary conditions. The case considered has the so-called "strong degeneration" and homogenous boundary conditions are required on Γ_1, only.

To state and to analyze the first boundary value problem for equation (2), we use weighted spaces $H^1_y(D)$, $\tilde H^2_y(D)$ and their subspaces.

Let $H^1_y(D)$ denote the space of measurable in D functions having all the distributional derivatives of the first order in D for which the following value is finite:

$$\|u\|_{H^1_y(D)}=\Big(\int_D y(u_x^2+u_y^2)\,dD+\int_D yu^2\,dD\Big)^{1/2}.$$

The former value is a norm in $H^1_y(D)$. Similarly, the space $\tilde H^2_y(D)$ is defined by finiteness of the value

$$\|u\|_{\tilde H^2_y(D)}=\Big(\int_D y(u_{xx}^2+2u_{xy}^2+u_{yy}^2)\,dD+\int_D y^{-1}u_y^2\,dD+\|u\|^2_{H^1_y(D)}\Big)^{1/2}.$$

We will use $\|\cdot\|$ to denote the norm in $L_2(D)$.

Next, $C_0^\infty(D)$ denotes a class of finite infinitely differentiable functions in D, $\overset{\circ}{C}{}^\infty_{\Gamma_1}(\overline{D})$ denotes a linear manifold of infinitely differentiable functions in D, which vanish in a strip adjacent to the boundary Γ_1 (a strip is unique for each function), and all the functions whose derivatives (including the function itself) can be continuously extended in $\overline{D}$. Finally, $E(D)$ denotes linear space of functions in $\overset{\circ}{C}{}^\infty_{\Gamma_1}(\overline{D})$ such that $u_y=0$ on Γ_0.

Now we are ready to define the working spaces. Let $\overset{\circ}{H}{}^1_y(D)$ be the closure of the lineal $C_0^\infty(D)$ in the norm of $H^1_y(D)$ and $\overset{\circ}{\tilde H}{}^2_y(D)$ be the closure of the linear space $E(D)$ in the norm of $\tilde H^2_y(D)$.

Lemma 1. *The function from $H^1_y(D)$ belongs to the space $\overset{\circ}{H}{}^1_y(D)$ if and only if its trace on the border Γ_1 of the domain D is zero.*

Proof. For any small $\delta>0$, a space $\overset{\circ}{H}{}^1_y(D_\delta)$ of restricted to $D_\delta=D\cap(y>\delta)$ functions from $\overset{\circ}{H}{}^1_y(D)$ is equivalent to the common Sobolev space $\overset{\circ}{H}{}^1_{\Gamma_1,\delta}(D_\delta)$ which is defined as the closure of the lineal $\overset{\circ}{C}{}^\infty_{\Gamma_1,\delta}(D_\delta)$ in the norm $H^1(D_\delta)$. The elements of the former space have zero trace on $\Gamma_{1,\delta}$ [5, p. 152]. This follows from the equality $u|_{\Gamma_{1,\delta}}=0$, where $\Gamma_{1,\delta}=\Gamma_1\cap(y\ge\delta)$.

On the line of degeneration Γ_0, a function $\overset{\circ}{H}{}^1_y(D)$ can have any finite value. Also it is possible that functions from $\overset{\circ}{H}{}^1_y(D)$ tend to infinity while $y \to 0$ (see [6]). $\square$

Lemma 2. *Let* $u \in \overset{\circ}{\tilde{H}}{}^2_y(D)$. *Then* $u_y\big|_{\Gamma_0} \in L_2(\Gamma_0)$ *and* $u_y\big|_{\Gamma_0} = 0$.

The proof of this lemma can be found in [7, 8].

Lemma 3. *The following inequality holds for* $u \in C_0^\infty(D)$

$$\int_D \sigma_\alpha(y) u^2(x,y)\, dD \le C(\alpha) \int_D y^{2\alpha} u_y^2\, dD, \tag{3}$$

where

$$\sigma_\alpha(y) = \begin{cases} y^{2\alpha-2}, & \alpha \ne 1/2, \\ y^{-1}|\ln y|^{-2-\varepsilon}, & \alpha = 1/2,\ \varepsilon > 0, \end{cases}$$

$$C(\alpha) = \begin{cases} 4/(2\alpha-1)^2, & \alpha \ne 1/2, \\ C > 0, & \alpha = 1/2. \end{cases}$$

The proof of this lemma can be found in [9, p. 513–568].

Lemma 4. *Any function* $u \in \overset{\circ}{H}{}^1_y(D)$ *has zero trace on* Γ_1.

Proof. For any small $\delta > 0$, the space $\overset{\circ}{H}{}^1_y(D_\delta)$ of functions from $\overset{\circ}{H}{}^1_y(D)$ restricted to $D_\delta = D \cap (y > \delta)$ is equivalent to the Sobolev space $\overset{\circ}{H}{}^1_{\Gamma_1,\delta}(D_\delta)$ which is defined as closure of the lineal $\overset{\circ}{C}{}^\infty_{\Gamma_1,\delta}(D_\delta)$ in the manifold norm $H^1(D_\delta)$ whose elements have zero trace on $\Gamma_{1,\delta}$, see [5, p. 148]. This leads to the equality $u|_{\Gamma_{1,\delta}} = 0$ where $\Gamma_{1,\delta} = \Gamma_1 \cap (y \ge \delta)$. $\square$

2. Variational statement

We define the weak solution to the first boundary-value problem for equation (1) in the domain D as a function $u \in \overset{\circ}{H}{}^1_y(D)$, for which the following integral equality holds:

$$\forall v \in \overset{\circ}{H}{}^1_y(D) \quad a_1(u,v) = l_1(v), \tag{4}$$

where

$$a_1(u,v) = \int_D y \nabla u \cdot \nabla v\, dD, \qquad l_1(v) = \int_D y f v\, dD, \quad f \in L_{2,y}(D),$$

$$L_{2,y}(D) = \{f : \ y^{1/2} f \in L_2(D)\}, \qquad \nabla w = \Big(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial x}\Big).$$

Theorem 1. *Let $f \in L_{2,y}(D)$. Then the solution of the variational problem* (4) *exists, is unique, and the following estimate holds*:

$$\|u\|_{H^1_y(D)} \leq C\|f\|_{L_{2,y}(D)}. \tag{5}$$

Proof. To prove the theorem, let us check the conditions of the Lax–Milgram lemma on elements of the dense set $C_0^\infty(D)$ in the space $\overset{\circ}{H}{}^1_y(D)$.

First we check $\overset{\circ}{H}{}^1_y(D)$-coercivity of the bilinear form $a_1(\cdot,\cdot)$. By using inequality (3) with $\alpha = 1$, which is analogous to the Friedrichs inequality, we obtain

$$\begin{aligned} a_1(u,u) = \int_D y|\nabla u|^2 dD &\geq C_1\left[\int_D y|\nabla u|^2\, dD + \int_D y^2|\nabla u|^2\, dD\right] \\ &\geq C_2\left[\int_D y|\nabla u|^2\, dD + \int_D u^2\, dD\right] \geq C_3\, \|u\|^2_{H^1_y(D)}. \end{aligned} \tag{6}$$

Therefore, the bilinear form $a_1(u,v)$ is $\overset{\circ}{H}{}^1_y(D)$-coercive.

Continuity of the bilinear form $a_1(u,v)$ in the norm of the space $H^1_y(D)$ on $C_0^\infty(D)\times C_0^\infty(D)$ and the linear form $l_1(v)$ on $C_0^\infty(D)$ are obtained from the Cauchy–Bunyakovskii inequality:

$$\begin{aligned} |a_1(u,v)| = \left|\int_D y\nabla u\cdot\nabla v\, dD\right| &\leq \left(\int_D y|\nabla u|^2 dD\right)^{1/2}\left(\int_D y|\nabla v|^2 dD\right)^{1/2} \\ &\leq \|u\|_{H^1_y(D)}\|v\|_{H^1_y(D)}. \end{aligned} \tag{7}$$

$$|l_1(v)| = \left|\int_D yfv\, dD\right| \leq \|y^{1/2}f\|\|y^{1/2}v\| \leq C_4\|v\|_{H^1_y(D)}. \tag{8}$$

Finally, all the conditions of the Lax–Milgram lemma are checked on elements of the dense set $C_0^\infty(D)$ in $\overset{\circ}{H}{}^1_y(D)$. The correctness of these conditions on the elements of $\overset{\circ}{H}{}^1_y(D)$ is determined by a common closure procedure in inequalities (6)–(8). □

We define the strong solution to the first boundary value problem in the domain D for equation (1) as a function $u \in \overset{\circ}{\tilde{H}}{}^2_y(D)$, for which the following equality holds:

$$\forall v \in \overset{\circ}{\tilde{H}}{}^2_y(D) \quad a_2(u,v) = l_2(v), \tag{9}$$

where

$$a_2(u,v) = \int_D y\mathcal{A}(u)\mathcal{A}(v)\, dD, \quad l_2(v) = \int_D y\mathcal{A}(v)\, f dD, \quad f \in L_{2,y}(D).$$

Theorem 2. *Let $f \in L_{2,y}(D)$. Then the solution u_0 of variational problem* (9) *exists, is unique, and the following estimate holds*:

$$\|u_0\|_{\tilde{H}^2_y(D)} \le C\,\|f\|_{L_{2,y}(D)}. \tag{10}$$

Proof. We again check the conditions of the Lax–Milgram lemma for problem (9) on elements of the space $E(D)$ dense in $\overset{\circ}{\tilde{H}}{}^2_y(D)$.

The $\overset{\circ}{\tilde{H}}{}^2_y(D)$-coercivity of the bilinear form $a_2(u,v)$ is proved by the inequality

$$\forall u \in \overset{\circ}{\tilde{H}}{}^2_y(D) \quad a_2(u,u) \ge \gamma \|u\|^2_{\tilde{H}^2_y(D)}, \quad \gamma > 0. \tag{11}$$

Let $u \in E(D)$. Then

$$\begin{aligned} a_2(u,u) &= \int_D y(\mathcal{A}(u))^2\,dD = \int_D y\big[(y^{-1}(yu_y)_y)^2 + 2y^{-1}(yu_y)_y u_{xx} + u_{xx}^2\big]\,dD \\ &= \int_D y(y^{-1}(yu_y)_y)^2\,dD + 2\int_D (yu_y)_y u_{xx}\,dD + \int_D y u_{xx}^2\,dD. \end{aligned} \tag{12}$$

We transform the first term in (12) using integration by parts and the fact that all the derivatives of u on Γ_1 are equal to zero and u_y vanishes on Γ_0:

$$\begin{aligned} \int_D y(y^{-1}(yu_y)_y)^2\,dD &= \int_D y[u_{yy}^2 + 2y^{-1}u_y u_{yy} + (y^{-1}u_y)^2]\,dD \\ &= \int_D y u_{yy}^2\,dD + \int_D [(u_y^2)_y + y^{-1}u_y^2]\,dD \\ &= \int_D y u_{yy}^2\,dD + \int_D y^{-1}(yu_y^2)_y\,dD \\ &= \int_D y u_{yy}^2\,dD + \int_D y^{-1}u_y^2\,dD. \end{aligned}$$

By integrating the second term in (12) twice by parts, at first with respect to y and then with respect to x, we arrive at

$$2\int_D (yu_y)_y u_{xx}\,dD = -2\int_D y u_y u_{xxy}\,dD = 2\int_D y u_{xy}^2\,dD.$$

Finally, we obtain

$$a_2(u,u) = \int_D y(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)\,dD + \int_D y^{-1}u_y^2\,dD. \tag{13}$$

This formula shows that the value $\sqrt{a_2(u,u)}$ is equal to the seminorm of the function u in the space $\overset{\circ}{\tilde{H}}{}^2_y(D)$.

Let us prove that in the space $\overset{\circ}{\tilde{H}}{}^2_y(D)$ the seminorm defined in (13) is equivalent to the norm. It is easy to notice that inequality (3) holds for the functions $u \in E(D)$ in the case $2\alpha - 1 > 0$. Applying it with $2\alpha = 3$ to $u \in E(D)$ we obtain

$$\int_D y^{-1} u_y^2 \, dD \geq C_1 \int_D y^3 u_y^2 \, dD \geq C_2 \int_D y u^2 \, dD, \tag{14}$$

$$\int_D y^{-1} u_y^2 \, dD \geq C_3 \int_D y u_y^2 \, dD. \tag{15}$$

To finish the proof of coercivity inequality (11), we need to prove that

$$\int_D y u_x^2 \, dD \leq \frac{1}{2}\Big(\int_D y u_{xx}^2 \, dD + \int_D y u^2 \, dD\Big). \tag{16}$$

Indeed,

$$\begin{aligned}\int_D y u_x u_x \, dD &= -\int_D y u_{xx} u \, dD = -\int_D (y^{1/2} u_{xx})(y^{1/2} u) \, dD \\ &\leq \Big(\int_D y u_{xx}^2 \, dD\Big)^{1/2} \Big(\int_D y u^2 \, dD\Big)^{1/2} \\ &\leq \frac{1}{2}\Big(\int_D y u_{xx}^2 \, dD + \int_D y u^2 \, dD\Big).\end{aligned}$$

Using inequalities (14)–(16) to attain the lower estimate for seminorm defined in (13), we obtain (11) for each $u \in E(D)$.

Continuity of the bilinear form $a_2(\cdot,\cdot)$ and of the linear functional $l_2(\cdot)$ follows from equality (13) and the Cauchy–Bunyakovskii inequality:

$$|a_2(u,v)| \leq |a_2(u,u)|^{1/2} |a_2(v,v)|^{1/2} \leq \|u\|_{\tilde{H}^2_y(D)} \|v\|_{\tilde{H}^2_y(D)}, \tag{17}$$

$$\begin{aligned}|l_2(v)| &= \Big|\int_D y f \mathcal{A}(v) \, dD\Big| \leq \Big(\int_D y f^2 \, dD\Big)^{1/2} \Big(\int_D y (\mathcal{A}(v))^2 \, dD\Big)^{1/2} \\ &\leq M \|v\|_{\tilde{H}^2_y(D)},\end{aligned} \tag{18}$$

where $u, v \in E(D)$.

Passing to the limit proves inequality (10) for any $u \in \overset{\circ}{\tilde{H}}{}^2_y(D)$, inequality (18) for any $v \in \overset{\circ}{\tilde{H}}{}^2_y(D)$, and inequality (17) for any $u, v \in \overset{\circ}{\tilde{H}}{}^2_y(D)$. □

3. Approximation with linear functions

Let for the domain D, the grid domains D_h^{ex} ($D \subset D_h^{\text{ex}}$) and D_h^{in} ($D_h^{\text{in}} \subset D$) with sets of nodes $\bar{R}_h^{\text{ex}}$ and $\bar{R}_h^{\text{in}}$ be built [10], D_h^{ex} is a minimal set of triangulation elements containing D with its boundary and D_h^{in} is a maximal set of triangulation elements containing D without its boundary. We assume that in the grid domains D_h^{ex} and D_h^{in}, the triangles are regular adjacent to Γ_0. In the case of an irregular grid, we assume that the grid domains D_h^{ex} and D_h^{in} satisfy additional regularity conditions (e.g. see [10, p. 69]).

Note that any function $u \in H_y^2(D)$ can be considered continuous on $\overline{D}$. Indeed, a function $u \in H_y^2(D)$, considered for any $\varepsilon > 0$ in the subdomain $D_\varepsilon = D \setminus (y \le \varepsilon)$ is an element of Sobolev's space $H^2(D_\varepsilon)$, whose elements are continuous in D_ε according to Sobolev's embedding theorem. This and Lemma 2 bring about continuity of the function u in $\overline{D}$ and the fact that $u = 0$ on Γ_0.

For a function $u \in \overset{\circ}{H}{}_y^2(D)$, we define a piecewise-linear function $\tilde{u}$

$$\tilde{u}(x,y) = \begin{cases} \sum_{(x_i,y_i)\in \bar{R}_h^{\mathrm{ex}}} u(x_i,y_i)\varphi_i(x,y), & y > h, \\ \sum_{(x_i,y_j)\in \bar{R}_h^{\mathrm{ex}}} u(x_i,y_j)\varphi_i(x,y), & y \le h, \end{cases} \tag{19}$$

where the function $\varphi_i(x,y)$ is linear in each triangulation element and equal to 1 in the node (x_i,y_i) and to 0 in all other nodes. The function $\tilde{u}(x,y)$ is continuous and belongs to $\overset{\circ}{\tilde{H}}{}_y^2(D_h^{\mathrm{ex}})$.

Assume that for the domain D considered, it is possible to continue any function from the weight spaces $\overset{\circ}{H}{}_y^1(D)$ and $\overset{\circ}{\tilde{H}}{}_y^2(D_h^{\mathrm{ex}})$ saving its norm and class of smoothness on the whole half-plane $\mathbb{R}_+^2 = \{(x,y):\ y > 0\}$.

Theorem 3. *Let $u \in \overset{\circ}{\tilde{H}}{}_y^2(D_h^{\mathrm{ex}})$ be continued on $\mathbb{R}_+^2$ with its norm and smoothness class saved. Let also the grid domain D_h^{ex} be regular. Then the following inequalities hold*:

$$\|u - \tilde{u}\|_{L_{2,y}(D_h^{\mathrm{ex}})} \le Ch^2 \|u\|_{\tilde{H}_y^2(D)}, \tag{20}$$

$$\|u - \tilde{u}\|_{H_y^1(D_h^{\mathrm{ex}})} \le Ch \|u\|_{\tilde{H}_y^2(D)}. \tag{21}$$

The proof of this theorem is analogous to the appropriate proof in [10] except in this case the result should be first proven for the dense set $E(D)$ and then completed with passing to the limit.

4. A numerical example

As a numerical example, we present the results of numerical solution of the first boundary value problem for equation (1) in a unit square $D = (0,1) \times (0,1)$. In this case $\Gamma_0 = \big\{(x,0),\ x \in [0,1]\big\}$ and $\Gamma_1 = \Gamma \setminus \Gamma_0$.

Let the domain D be partitioned into squares with vertices

$$\big\{(x_i, y_j):\ x_i = ih,\ y_j = jh,\ i, j \in \overline{0, N+1}\big\}.$$

We prepare triangulation by splitting the mesh cells into triangles along the lines parallel to $x = y$. Next, we define finite elements $\phi_{i,j}$, $i \in \overline{1, N}$, $j \in \overline{2, N}$, as functions linear in each of the triangulation elements, whose values are equal to $1/h$ at the node (x_i, y_j) and to zero at all other nodes. Elements adjacent to the axis $y = 0$, i.e., $\phi_{i,1}, i \in \overline{1, N}$ we define as linear functions, whose values are equal to $1/h$ in (x_i, y_1), (x_i, y_0) and to zero in all other nodes.

To find the approximate solution

$$u_h(x, y) = \sum_{j=1}^{N} \sum_{i=1}^{N} h u_{i,j} \phi_{i,j}(x, y), \tag{22}$$

we need to calculate the coefficients $u_{i,j}$ from the system of linear equations

$$a_1(u_h, \phi_{k,l}) = l_1(\phi_{k,l}), \quad k, l \in \overline{1, N}. \tag{23}$$

The matrix of the linear system is the following:

$$A\bar{u} = f, \tag{24}$$

where $\bar{u} = (u_{i,j})$, $A = (a_{i,j,k,l}) = (a_1(\phi_{i,j}, \phi_{k,l}))$, $f = (f_{k,l}) = (l_1(\phi_{k,l}))$, $i, j, k, l \in \overline{1, N}$.

Finally, we have the following finite difference scheme for problem (23):

$$\begin{aligned}
-&\frac{y_{j+1}^3 - 2y_j^3 + y_{j-1}^3}{6h^4}(u_{i+1,j} - 2u_{i,j} - u_{i-1,j}) - \\
&\qquad \frac{y_{j+1}^2 - y_j^2}{2h^3}(u_{i,j+1} - u_{i,j}) - \frac{y_j^2 - y_{j-1}^2}{2h^3}(u_{i,j} - u_{i,j-1}) = f_{i,j}, \quad j > 1, \\
-&\frac{y_{j+1}^3 - 2y_j^3 + y_{j-1}^3}{6h^4}(u_{i+1,j} - 2u_{i,j} - u_{i-1,j}) - \\
&\qquad \frac{y_{j+1}^2 - y_j^2}{2h^3}(u_{i,j+1} - u_{i,j}) = f_{i,j}, \quad j = 1.
\end{aligned}$$

Note that the matrix of linear system (24) is symmetric and has 5-diagonal structure. In addition, it belongs to the class of the so-called M-matrices (see [11, p. 41]).

As a test problem, we have solved the first boundary-value problem for equation (1) with

$$f(x, y) = -(4 - 16y^2)x^2(1 - x^2) - y^2(1 - y^2)(2 - 12x^2).$$

The exact solution in this case is $u(x, y) = x^2(1 - x^2)y^2(1 - y^2)$.

Linear system (24) was solved using the conjugate gradients method with a preconditioner. The preconditioner is based on an incomplete block factorization method with a row sum compensation [11, p. 162]. The stop criterion for iterations was the reduction of the residual by 10 orders in magnitude. To check the convergence rate, the problem was solved on a sequence of nested grids. The first grid had $N = 10$ and the parameter N doubled each following one.

The table below shows the values $\delta_N = \|u - u_{2h}\|_{L_{2,y}(D)} / \|u - u_h\|_{L_{2,y}(D)}$, where u_h is solution of the problem on a grid with the step $h = 1/N$ and u is the exact solution.

N	20	40	80	160	320	640	1280
δ_N	3.78	3.88	3.91	3.94	3.94	3.96	3.96

The results are in correspondence with theoretical estimate (20) and demonstrate the second order of convergence in the norm of the space $L_{2,y}(D)$.

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