Bull. Nov. Comp. Center, Comp. Science, 26 (2007), 103–110

Generalized decomposability notions for first-order theories

Denis Ponomaryov*

Abstract. This paper introduces the notion of decomposability in an extension and relative decomposability for first-order theories. We describe several basic facts connected with these notions and formulate a criterion of relative decomposability.

Keywords: decomposable theories, modular terminological systems

1. Introduction

The interest in studying compositional properties of first-order theories has its roots in research on formal terminological systems and automated reasoning. One of these properties is decomposability, which means that a theory can be represented as a union of several signature-disjoint theories. From the applications point of view, this allows for reducing the search space and using multiple reasoners in automated theorem proving. In the context of formal terminological systems, e.g. ontologies, the decomposability serves as a tool for component-based development and distributed processing of large terminological knowledge bases. Both application scenarios were already discussed in [1] and [2].

The study of the decomposability property has been set in [10]. It has been proved that every first-order theory has a unique decomposition into indecomposable theories. This fact has laid the foundation for algorithmic approach to the decomposability property and turned out to have many valuable applications [9]. On the other hand, it appeared necessary to study generalized notions of decomposability related to extensions and modularity of terminological systems. At present, there is a large body of model-theoretical research on this subject (see [6] for an overview) concentrated mostly around Description Logics [2, 3, 4, 5, 7].

In this paper, we introduce the notion of *decomposability in an extension* and *relative decomposability* for first-order theories. These definitions and the properties proven around them are motivated by the following questions. What happens to the decomposability property under extensions of theories?

*The author was supported by the COMO project of Russian Foundation for Basic Research (project No. 05-01-04003-NNIO_a) and Deutsche Forschungsgemeinschaft (GZ: 436 RUS 113/829/0-1).

Can a decomposable theory become indecomposable under extensions (or vice versa)? These questions are natural for formal terminological systems developed incrementally. It is not hard to demonstrate that each of the cases is possible. However, we formulate a condition on extensions, which, in some sense, guarantees the preservation of the decomposability property. The next question is: given an indecomposable theory $\mathcal{T}$, can we find subtheories in $\mathcal{T}$ that are decomposable? Depending on a concrete reasoning task, it may be possible to omit one or several axioms of $\mathcal{T}$ in order to obtain a decomposable theory.

For studying the new decomposability notions, we introduce in Section 2 all necessary definitions and notations; we also prove an auxiliary lemma, which will be used to formulate examples. Section 3 contains all the basic results of this paper. Section 4 concludes.

2. Preliminaries

Throughout this paper, we assume that all theories considered are deductively closed. For theories $\mathcal{S}_1$ and $\mathcal{S}_2$ of signatures Σ_1 and Σ_2 we denote by $\langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ the deductive closure of all sentences of $\mathcal{S}_1$ and $\mathcal{S}_2$ in the predicate calculus of $\Sigma_1 \cup \Sigma_2$. A similar abbreviation will be used in examples to denote the deductive closures of more than two theories.

Definition 1 [10]. *A theory $\mathcal{T}$ of signature Σ is called* ***decomposable****, if there exist theories $\mathcal{S}_1$ and $\mathcal{S}_2$ of disjoint signatures $\Sigma_1 \cap \Sigma_2 = \varnothing$, $\Sigma_1 \cup \Sigma_2 = \Sigma$ such that $\mathcal{T} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$.*

The pair $[\mathcal{S}_1, \mathcal{S}_2]$ is called ***decomposition*** *of $\mathcal{T}$ and the theories $\mathcal{S}_1$, $\mathcal{S}_2$ are called* ***decomposition components*** *of $\mathcal{T}$.*

This definition is based on the notion of decomposability, which has been first formulated in [8] in connection with the study of formal ontologies.

Only non-trivial decompositions, with the components $\mathcal{S}_1$ and $\mathcal{S}_2$ having non-empty signatures, are of interest for consideration. Therefore, we will study decomposable theories assuming the existence of non-trivial decompositions, as well as their absence for the case of indecomposability. Trivial cases will be admitted only in Theorem 3, where they are needed for generality.

To formulate several examples in Section 3, we need to give the definition of a decomposable sentence from [10] and prove Lemma 1 below.

Definition 2 [10]. *Let $\mathcal{T}$ be a theory. A sentence $\varphi \in \mathcal{T}$ is called* ***decomposable in*** *$\mathcal{T}$ if there exist sentences $\theta \in \mathcal{T}$ and $\psi \in \mathcal{T}$ with the following properties:*

1. *θ and ψ contain symbols only from the signature of φ;*

2. θ and ψ do not have signature symbols in common;

3. neither θ nor ψ is an equality formula;

4. $\theta, \psi \vdash \varphi$.

If there are no these sentences in $\mathcal{T}$ then we call φ ***indecomposable in $\mathcal{T}$****.*

Lemma 1. *Let $\mathcal{T}$ be a theory of signature Σ, which has only infinite models. Let $\Sigma_1 \subset \Sigma$ and $\Sigma_2 \subset \Sigma$ be disjoint signatures and each Σ_i, $i = 1, 2$ consist of exactly one element.*

Consider a formula $\Pi \in \mathcal{T}$ of the form $\Pi = \xi_1 \vee \xi_2$, where $\xi_1 \notin \mathcal{T}$ and $\xi_2 \notin \mathcal{T}$ are sentences of signatures Σ_1 and Σ_2, respectively. Then Π is indecomposable in $\mathcal{T}$.

Proof. Suppose that Π is decomposable in $\mathcal{T}$. Then there exist sentences $\varphi \in \mathcal{T}$ and $\psi \in \mathcal{T}$ of signatures Σ_1 and Σ_2, respectively, such that $\varphi, \psi \vdash \Pi$. From the definition of Π, we obtain $\varphi, \psi \vdash \neg\xi_1 \to \xi_2$ and $\varphi, \neg\xi_1 \vdash \psi \to \xi_2$.

As φ and ψ (as well as ξ_1 and ξ_2) do not have signature symbols in common, by Craig's interpolation theorem, there exists an equality formula θ, for which $\varphi, \neg\xi_1 \vdash \theta$ and $\theta \vdash \psi \to \xi_2$ hold. As $\mathcal{T}$ has only infinite models, we have either $\theta \in \mathcal{T}$ or $\neg\theta \in \mathcal{T}$.

If $\theta \in \mathcal{T}$ then $\psi, \theta \vdash \xi_2$ and $\xi_2 \in \mathcal{T}$. If $\neg\theta \in \mathcal{T}$ then $\varphi, \neg\theta \vdash \xi_1$ and $\xi_1 \in \mathcal{T}$. Both cases contradict the condition of lemma, hence Π is indecomposable in $\mathcal{T}$. ∎

In Section 3, we will use the abbreviation $\Pi(a, b)$ to denote formulas of the form $\xi_1 \vee \xi_2$, where ξ_1 and ξ_2 are consistent non-tautological sentences of disjoint signatures $\Sigma_1 = \{a\}$ and $\Sigma_2 = \{b\}$, respectively. We will also use the decomposability criterion from [10] formulated below.

Definition 3. *Let S be a set of sentences in signature Σ.*

The ***signature graph*** *over S is the graph $G = (\Sigma, I)$ with the incidence relation $I \subseteq \Sigma \times \Sigma$ defined as follows: for each $a \in \Sigma$ and $b \in \Sigma$ we have $(a, b) \in I$ iff there exists a sentence $\varphi \in S$ including the symbols a and b.*

The ***adjoint signature graph*** *over S is the graph $G' = (S, I')$ with the incidence relation $I' \subseteq S \times S$ defined as follows: for each $\varphi \in S$ and $\psi \in S$ we have $(\varphi, \psi) \in I'$ iff the signatures of φ and ψ have a non-empty intersection.*

Decomposability criterion([10]) *Let $\mathcal{T}$ be a theory and Ψ be a system of axioms for $\mathcal{T}$, with each sentence $\psi \in \Psi$ indecomposable in $\mathcal{T}$. Then $\mathcal{T}$ is decomposable iff the signature graph over Ψ is not connected.*

The decomposability criterion together with Lemma 1 and the assumption of infinite models will be used in the next section to formulate examples.

3. Generalized notions of decomposability

Definition 4. *Let $\mathcal{T}$ and $\mathcal{T}'$ be theories of signatures Σ and Σ', respectively. Let $\mathcal{T}'$ be an extension of $\mathcal{T}$. We call $\mathcal{T}$* ***decomposable in the extension*** *$\mathcal{T}'$, if there exist theories $S_1' \subseteq \mathcal{T}'$ and $S_2' \subseteq \mathcal{T}'$ of disjoint signatures $\Sigma_1' \cup \Sigma_2' \supseteq \Sigma$, $\Sigma_1' \cap \Sigma \neq \varnothing \neq \Sigma_2' \cap \Sigma$ such that $\langle S_1', S_2' \rangle \vdash \mathcal{T}$.*

For brevity, we will often omit the word *extension* and say that $\mathcal{T}$ is **decomposable in $\mathcal{T}'$**. When pointing to theories S_1' and S_2' is necessary, we will use a longer formulation and say $\mathcal{T}$ is decomposable in the extension $\mathcal{T}'$ with the components S_1' and S_2'.

Remark 1. *If $\mathcal{T}$ is a decomposable theory, then $\mathcal{T}$ is decomposable in any extension $\mathcal{T}'$. On the other hand, each of the following four situations may take place depending on $\mathcal{T}$ and $\mathcal{T}'$:*

Table 1. The possible cases under extensions

$\mathcal{T}$ is decomposable	$\mathcal{T}'$ is decomposable	Reference
+	+	Example 1
-	-	Example 2
+	-	Example 3
-	+	Example 4

Example 1. *Let $\mathcal{T} = \langle \Pi(a,b), \Pi(c,d) \rangle$ and $\mathcal{T}' = \langle \Pi(a,b),\ \Pi(c,d),\ \Pi(b,e),\ \Pi(c,f) \rangle$. Both theories $\mathcal{T}$ and $\mathcal{T}'$ are decomposable.*

Example 2. *Consider theories $\mathcal{T} = \langle \Pi(a,b), \Pi(b,c) \rangle$, $\mathcal{T}' = \langle \Pi(a,b), \Pi(b,c), \Pi(c,d) \rangle$. The theories $\mathcal{T}$ and $\mathcal{T}'$ are indecomposable.*

Example 3. *Let $\mathcal{T} = \langle \Pi(a,b), \Pi(c,d) \rangle$ and $\mathcal{T}' = \langle \Pi(a,b), \Pi(b,c), \Pi(c,d) \rangle$. The theory $\mathcal{T}$ is decomposable, but $\mathcal{T}'$ is not. However, $\mathcal{T}$ is decomposable in $\mathcal{T}'$.*

Example 4. *Let $\mathcal{T} = \langle \Pi(a,b) \rangle$ and let the formula $\Pi(a,b)$ have the form $\xi_1 \vee \xi_2$, where ξ_1, ξ_2 are consistent non-tautological sentences of signatures $\Sigma_1 = \{a\}$ and $\Sigma_2 = \{b\}$. Let $\mathcal{T}' = \langle \Pi(a,b),\ \xi_1 \rangle$. The theory $\mathcal{T}$ is originally indecomposable, but decomposable in $\mathcal{T}'$. The theory $\mathcal{T}'$ is decomposable, $\mathcal{T}' = \langle \{\xi_1\}, \{\theta\} \rangle$, where θ is a tautological sentence of signature Σ_2.*

Regarding these examples, it is important to determine the types of extensions, under which the decomposability property is preserved. The following theorem gives one of the conditions on extensions.

Theorem 1. *Let $\mathcal{T}$ be a theory of signature Σ. If $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$, then $\mathcal{T}$ is decomposable iff $\mathcal{T}$ is decomposable in $\mathcal{T}'$.*

Proof. ($\Rightarrow$) is mentioned in Remark 1.
($\Leftarrow$) As $\mathcal{T}$ is decomposable in $\mathcal{T}'$, we have theories $S_1' \subseteq \mathcal{T}'$ and $S_2' \subseteq \mathcal{T}'$ of disjoint signatures $\Sigma_1' \cup \Sigma_2' \supseteq \Sigma$ such that $\langle S_1', S_2' \rangle \vdash \mathcal{T}$. Therefore, for each $\varphi \in \mathcal{T}$ there exist sentences $\psi_1 \in S_1'$ and $\psi_2 \in S_2'$ such that $\psi_1, \psi_2 \vdash \varphi$; thus, $\psi_1 \vdash \psi_2 \to \varphi$.

Let $\Sigma_i = \Sigma_i' \cap \Sigma,\ i = 1, 2$ be the decomposition of the signature of $\mathcal{T}$. By Craig's interpolation theorem, there exists a sentence θ_1 of signature $\Sigma_{\theta_1} \subseteq \Sigma_1' \cap (\Sigma_2' \cup \Sigma) = \Sigma_1$, for which $\psi_1 \vdash \theta_1$ and $\theta_1 \vdash \psi_2 \to \varphi$ hold. In particular, we have $\psi_2 \vdash \theta_1 \to \varphi$. By applying the interpolation theorem again, we obtain a sentence θ_2 of signature $\Sigma_{\theta_2} \subseteq \Sigma_2' \cap (\Sigma_1 \cup \Sigma) = \Sigma_2$ such that $\psi_2 \vdash \theta_2$ and $\theta_2 \vdash \theta_1 \to \varphi$ hold; hence, $\theta_1, \theta_2 \vdash \varphi$.

Besides, we have $\psi_1 \in \mathcal{T}'$, $\psi_2 \in \mathcal{T}'$, $\psi_1 \vdash \theta_1$, $\psi_2 \vdash \theta_2$, $\Sigma_{\theta_1} \cup \Sigma_{\theta_2} \subseteq \Sigma$, and $\mathcal{T}'$ is a conservative extension of $\mathcal{T}$. Therefore $\mathcal{T} \vdash \theta_1, \theta_2$.

Let S_i, $i = 1, 2$, denote the set of all sentences θ_i constructed as above for each $\varphi \in \mathcal{T}$ (note that the sets S_1, S_2 are signature disjoint). Due to the above-mentioned and the arbitrary selection of $\varphi \in \mathcal{T}$, we obtain that $\mathcal{T} = \langle S_1, S_2 \rangle$. ■

It follows from this theorem that extensions by definitions ([11], Ch.4) completely preserve the decomposability property of the initial theory.

Remark 2. *There exists a class $\mathcal{C}$ of theories such that for any $\mathcal{T} \in \mathcal{C}$ and consistent extension $\mathcal{T}'$ of $\mathcal{T}$, the theory $\mathcal{T}$ is indecomposable in $\mathcal{T}'$. The class $\mathcal{C}$ includes all theories of signatures consisting of exactly one element. We call the theories from $\mathcal{C}$* ***indecomposable in extensions****.*

Remark 3. *If a theory $\mathcal{T}$ is indecomposable in extensions, then there exists an extension $\mathcal{T}'$ for $\mathcal{T}$ such that $\mathcal{T}'$ is decomposable.*

Definition 5. *A theory $\mathcal{T}'$ is called* ***relatively decomposable****, if there exists a theory $\mathcal{T}$ decomposable in $\mathcal{T}'$ with non-tautological components. If no such theory exists in $\mathcal{T}'$, then we call $\mathcal{T}'$* ***essentially indecomposable****.*

The class of essentially indecomposable theories includes all theories of signatures consisting of exactly one element.

The property of relative decomposability can be studied by methods similar to that applied to decomposable theories. The decomposability criterion formulated in Section 2 shows that the key set, which determines whether a theory $\mathcal{T}$ is decomposable, is a set of axioms for $\mathcal{T}$ that are indecomposable in $\mathcal{T}$. Here it suffices to find any two non-tautological sentences $\varphi \in \mathcal{T}'$ and $\psi \in \mathcal{T}'$ of non-empty disjoint signatures to demonstrate that $\mathcal{T}'$ is relatively decomposable. We show, however, that there exists a "smaller" set of sentences, which determines the property of relative decomposability.

Notation For a theory $\mathcal{T}$, let $\Delta(\mathcal{T})$ denote the following set of sentences: if $\mathcal{N} \subset \mathcal{T}$ is the set of all sentences in $\mathcal{T}$ that are not equality formulas and are non-tautological, then $\Delta(\mathcal{T}) \subseteq \mathcal{N}$ is the set of sentences having a minimal number of signature symbols among the sentences of $\mathcal{N}$.

Theorem 2. *Let $\mathcal{T}'$ be a theory of signature Σ'. Then $\mathcal{T}'$ is relatively decomposable iff there exist two sentences $\varphi \in \Delta(\mathcal{T}')$ and $\psi \in \Delta(\mathcal{T}')$ having disjoint signatures.*

Proof. ($\Leftarrow$) is straightforward: the theory $\langle\{\varphi\}, \{\psi\}\rangle$ is decomposable in $\mathcal{T}'$. ($\Rightarrow$) Let $\mathcal{T}$ be a theory decomposable in $\mathcal{T}'$ and $S'_1 \subseteq \mathcal{T}'$, $S'_2 \subseteq \mathcal{T}'$ be non-tautological theories of disjoint signatures $\Sigma'_1 \cup \Sigma'_2 \supseteq \Sigma$, $\Sigma'_1 \cap \Sigma \neq \varnothing \neq \Sigma'_2 \cap \Sigma$ such that $\langle S'_1, S'_2\rangle \vdash \mathcal{T}$. Let $\varphi \in S'_1$ and $\psi \in S'_2$ be two sentences of signatures $\Sigma_1 \subseteq \Sigma'_1$ and $\Sigma_2 \subseteq \Sigma'_2$, respectively, such that φ and ψ are not equality formulas and are non-tautological.

Consider the partial order $\leq$ on subsets of Σ' defined as follows: for each $\sigma_1 \subseteq \Sigma'$ and $\sigma_2 \subseteq \Sigma'$ we have $\sigma_1 \leq \sigma_2$ iff $\sigma_1 \subseteq \sigma_2$ and there exist sentences $\phi_1 \in \mathcal{T}'$ and $\phi_2 \in \mathcal{T}'$ of signatures σ_1, σ_2.

For the signatures Σ_1 and Σ_2 there exist minimal σ_1 and σ_2 such that $\sigma_1 \leq \Sigma_1$, $\sigma_2 \leq \Sigma_2$, and the corresponding sentences θ_1, θ_2 are not equality formulas and are non-tautological. Thus, there exist θ_1 and θ_2, which are a pair of signature-disjoint sentences from $\Delta(\mathcal{T})$. ■

Criterion of relative decomposability. *A theory $\mathcal{T}$ is relatively decomposable iff the adjoint signature graph over the set $\Delta(\mathcal{T})$ is not complete.*

Finally, we formulate an important property of decomposable theories, which gives an insight on how they are constructed. The following theorem is the direct consequence of the statements proved in [10]. Note that here we admit the existence of trivial decomposition components for the sake of generality.

Theorem 3. *Let $\mathcal{T}$ be a theory and $\Omega \subseteq \mathcal{P}(\mathcal{T})$ be the set of all decomposition components of $\mathcal{T}$. Consider the relation $\prec \subseteq \Omega \times \Omega$ defined as follows: for each $\mathcal{S} \in \Omega$ and $\mathcal{U} \in \Omega$ we have $\mathcal{S} \prec \mathcal{U}$ iff $\mathcal{S}$ is a decomposition component of $\mathcal{U}$.*

Then $(\Omega, \prec)$ is a boolean algebra.

Proof. It follows from Theorem 1 in [10] that $\mathcal{T}$ has a unique decomposition into components that have only trivial decompositions. Let K denote the set of these components. It is known that each decomposition component of $\mathcal{T}$ is a union of some sets from K (Lemma 3 in [10]). Therefore, $< \Omega, \cup, \cap, \varnothing, \mathcal{T} >$ is a boolean algebra.

For the rest of the proof, let us demonstrate that for each $\mathcal{S} \in \Omega$ and $\mathcal{U} \in \Omega$ we have $\mathcal{S} \prec \mathcal{U}$ iff $\mathcal{S} \subseteq \mathcal{U}$.

If $\mathcal{S} \prec \mathcal{U}$ then, clearly, $\mathcal{S} \subseteq \mathcal{U}$ according to Definition 1. Now assume that $\mathcal{S} \subseteq \mathcal{U}$. Denote $\mathcal{S}' = \mathcal{U} \setminus \mathcal{S}$. As $\mathcal{S}$ is a decomposition component of $\mathcal{T}$, we have that $\mathcal{S}$ is a union of some sets from K. The same is true for $\mathcal{U}$. This means that $\mathcal{S}$ (as well as $\mathcal{U}$) contains entirely some sets from K. Therefore, $\mathcal{S}'$ is a union of some sets from K, different from that for $\mathcal{S}$. As all theories in K are signature-disjoint, we conclude that $\mathcal{U}$ is decomposable into $\mathcal{S}$ and $\mathcal{S}'$; hence, $\mathcal{S} \prec \mathcal{U}$. ∎

Note that the set of atoms of $(\Omega, \prec)$ is exactly the set K from the proof above. Having that some atoms can be relatively decomposable, one can understand the structure of $\mathcal{T}$ as built of "nested" decomposition lattices in which essentially indecomposable theories are elements that do not allow further nesting.

4. Conclusion

We have formulated two generalizations for the decomposability property of first-order theories, which has been first introduced in [8] and studied in [10]. The two new definitions (*decomposability in an extension* and *relative decomposability*) have been motivated by the classical notion of an extension of theories. The definitions are much related to each other, yet serve different purposes. The first one is connected with the question of "stability" of the decomposability property under extensions, while the second one is aimed at the task of identifying decomposable parts of indecomposable theories. We have described several basic facts around these properties and formulated a criterion of relative decomposability, which is, in some sense, similar to the "pure" decomposability criterion. A detailed investigation of the generalized decomposability properties is the subject of our further research.

References

[1] Amir E., McIlraith S. Partition-Based Logical Reasoning for First-Order and Propositional Theories // Artificial Intelligence. — 2005. — Vol. 162, N 1–2. — P. 49–88.

[2] Cuenca Grau B., Horrocks I., Kazakov Y., Sattler U. Extracting Modules from Ontologies: Theory and Practice. — 2007. — (Tech. rep. / Univ. of Manchester). — Available from http://www.cs.man.ac.uk/~bcg/Publications.html

[3] Cuenca-Grau B., Parsia B., Sirin E., Kalyanpur A. Modularity and Web Ontologies // Proc. of KR2006. — 2006. — P. 198–208.

[4] Cuenca-Grau B., Horrocks I., Kutz O., Sattler U. Will My Ontologies Fit Together? // Proc. of DL06. — 2006. — P. 189.

[5] Ghilardi S., Lutz C., Wolter F. Did I damage my ontology? A Case for Conservative Extensions in Description Logic // Proc. of KR2006. — 2006. — P. 187–197.

[6] Loebe F. Requirements for Logical Modules // Proc. of the First Internat. Workshop on Modular Ontologies (WoMO 2006).

[7] Lutz C., Walther D., Wolter F. Conservative Extensions in Expressive Description Logics // Proc. of IJCAI 2007. — 2007. — P. 453–458.

[8] Palchunov D. GABEK for Ontology Generation. — GABEK. Contributions to Knowledge Organization, 2. — Wien: LIT-publishing Company, 2005.

[9] Ponomaryov D. Semantic Web Basics in Logical Consideration // Lecture Notes in Informatics. — 2006. — P-94. — P. 337–344.

[10] Ponomaryov D. Syntactic unambiguity with respect to knowledge representation in first-order logic / PhD Thes. — Institute of Informatics Systems, Novosibirsk (Russia), 2006. (in Russian)

[11] Shoenfield J. Mathematical logic. — Addison-Wesley Publishing Co., 1967.