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Spectral models of vector-valued random fields

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Numerical models of vector-valued random fields are extensively used in solving applied problems [1–5]. These models have become the subject of many investigations [6–13]. The paper deals with methods of numerical modeling of homogeneous vector-valued random fields based on the spectral decomposition. General relations for spectral models are obtained and particular algorithms for simulation are presented here.

1. Spectral representations

1.1. Let $w(x)$ be a complex vector-valued homogeneous field

$$w(x) = \begin{bmatrix} w_1(x) \\ \vdots \\ w_s(x) \end{bmatrix}, \quad w_r(x) \in \mathbf{C}, \quad r = 1, \dots, s, \quad x \in \mathbf{R}^k,$$

with mean zero and (matrix-valued) correlation function

$$\begin{aligned} K(x) &= Mw(x+y)w^*(y) \\ &= M\big[\operatorname{Re} w(x+y)(\operatorname{Re} w(y))^T + \operatorname{Im} w(x+y)(\operatorname{Im} w(y))^T \\ &\quad + i\{-\operatorname{Re} w(x+y)(\operatorname{Im} w(y))^T + \operatorname{Im} w(x+y)(\operatorname{Re} w(y))^T\}\big]. \end{aligned}$$

For a vector $c \in \mathbf{C}^s$ the scalar random field $c^*w(x)$ is homogeneous with the correlation function $c^*K(x)c$.

If the field $w(x)$ is continuous in mean square, then [14]

$$K(x) = \int_{\mathbf{R}^k} \exp(i\langle x, \lambda\rangle) F(d\lambda), \tag{1.1}$$

where $F(A)$ is the matrix-valued spectral measure in $\mathbf{R}^k$, i.e., for any measurable set $A \subset \mathbf{R}^k$ the complex-valued (s,s)-matrix $F(A)$ is positive definite (hence, $F(A) = F^*(A)$, $\operatorname{Re} F(A) = (\operatorname{Re} F(A))^T$, $\operatorname{Im} F(A) =$

$-(\mathrm{Im}\, F(A))^T)$, and for any vector $c \in \mathbf{C}^s$ the function of sets $c^*F(A)c$ is a finite measure in $\mathbf{R}^k$ (evidently, it is the spectral measure of the field $c^*w(x)$).

Some properties of a correlation function are presented below:

1) $K(0) = F(\mathbf{R}^k)$,
2) $K(x) = K^*(-x)$,
3) $K(x)K^*(x) \le K^2(0)$,
4) $[K(x) - K(y)][K(x) - K(y)] \le K(0)[2K(0) - (K(x-y) + K^*(x-y))]$.

Properties 2)–4) may be obtained as a sequence of positive definiteness of the correlation function. If the measure $F(A)$ is absolutely continuous with respect to the Lebesgue measure in $\mathbf{R}^k$, then

$$f(\lambda) = [f_{rt}(\lambda)] = [F_{rt}(d\lambda)/d\lambda]$$

is called the matrix-valued spectral density and

$$f(\lambda) = (2\pi)^{-k} \int\limits_{\mathbf{R}^k} \exp(-i(x,\lambda))K(dx). \tag{1.2}$$

The spectral representation of the field $w(x)$ is of the form (see [14])

$$w(x) = \int\limits_{\mathbf{R}^k} \exp(i(x,\lambda)z(d\lambda), \tag{1.3}$$

where $z(A)$ is the vector-valued spectral stochastic measure in $\mathbf{R}^k$. The spectral stochastic measure satisfies the following properties:

1) $Mz(A) = 0$,
2) if $A_1 \cap A_2 = \emptyset$, then $z(A_1 + A_2) = z(A_1) + z(A_2)$,
3) $Mz(A)z^*(B) = F(A \cap B)$, i.e.,
$\mathrm{Re}\, F(A \cap B) = M\mathrm{Re}\, z(A)(\mathrm{Re}\, z(B))^T + M\mathrm{Im}\, z(A)(\mathrm{Im}\, z(B))^T$,
$\mathrm{Im}\, F(A \cap B) = M[-\mathrm{Re}\, z(A)(\mathrm{Im}\, z(B))^T + \mathrm{Im}\, z(A)(\mathrm{Re}\, z(B))^T]$.

From 3) follows that if $A_1 \cap A_2 = \emptyset$, then

$$Mz(A_1)z^*(A_2) = 0. \tag{1.4}$$

1.2. For the vector field $w(x)$ to be real-valued it is necessary and sufficient that

$$z(A) = \bar{z}(-A), \text{ and in this case } F(A) = \bar{F}(-A) = F^T(-A). \tag{1.5}$$

From the orthogonality property (see [4]) of the stochastic spectral measure $z(A)$ it follows that if $A \cap -A = \emptyset$, then

$$Mz(A)z^T(A) = 0. \tag{1.6}$$

Thus, under assumption $A \cap -A = \emptyset$, we have

$$\begin{aligned} M\mathrm{Re}\, z(A)(\mathrm{Re}\, z(A))^T - M\mathrm{Im}\, z(A)(\mathrm{Im}\, z(A))^T &= 0, \\ M\mathrm{Re}\, z(A)(\mathrm{Im}\, z(A))^T + M\mathrm{Im}\, z(A)(\mathrm{Re}\, z(A))^T &= 0, \end{aligned}$$

and, therefore,

$$\begin{aligned} M\mathrm{Re}\, z(A)(\mathrm{Re}\, z(A))^T = M\mathrm{Im}\, z(A)(\mathrm{Im}\, z(A))^T &= \mathrm{Re}\, F(A)/2, \\ -M\mathrm{Re}\, z(A)(\mathrm{Im}\, z(A))^T = M\mathrm{Im}\, z(A)(\mathrm{Re}\, z(A))^T &= \mathrm{Im}\, F(A)/2. \end{aligned}$$

Spectral representations of the real vector-valued field $w(x)$ and its correlation function may be written in the form

$$\begin{aligned} w(x) &= z\{0\} + 2\int_{\mathbf{P}} \cos\langle x,\lambda\rangle \mathrm{Re} > z(d\lambda) - \sin\langle x,\lambda\rangle \mathrm{Im} > z(d\lambda) \\ &= \int_{\mathbf{R}^k} \cos\langle x,\lambda\rangle \mathrm{Re} > z(d\lambda) - \sin\langle x,\lambda\rangle \mathrm{Im} > z(d\lambda), \end{aligned} \tag{1.7}$$

$$\begin{aligned} K(x) &= F\{0\} + 2\int_{\mathbf{P}} \cos\langle x,\lambda\rangle \mathrm{Re}\,, F(d\lambda) - \sin\langle x,\lambda\rangle \mathrm{Im}\,, F(d\lambda) \\ &= \int_{\mathbf{R}^k} \cos\langle x,\lambda\rangle \mathrm{Re} > F(d\lambda) - \sin\langle x,\lambda\rangle \mathrm{Im} > F(d\lambda). \end{aligned} \tag{1.8}$$

Here $\mathbf{P}$ is a half-space of $\mathbf{R}^k$, i.e., $\mathbf{P}$ is a measurable set such that $\mathbf{P} \cap (-\mathbf{P}) = \emptyset$, $\mathbf{P} + (-\mathbf{P}) + \{0\} = \mathbf{R}^k$. For the spectral density of the real vector field we have

$$f(\lambda) = (2\pi)^{-k}\Big(\int_{\mathbf{R}^k} \cos\langle x,\lambda\rangle K(x)dx + i\int_{\mathbf{R}^k} \sin\langle x,\lambda\rangle K(x)dx\Big). \tag{1.9}$$

Remark. If a random field has a real-valued correlation function, then it does not mean that the field is a real-valued one. A simple example is presented below. Let z be a complex-valued random value such that $Mz = 0$, $Mz\bar{z} = 0$

$$M(\operatorname{Re} z)^2 = M(\operatorname{Im} z)^2 = A, \quad M \operatorname{Re} z \operatorname{Im} z = 0.$$

Consider the random processes

$$\begin{aligned} u(x) &= \exp(i\lambda x)z + \exp(-i\lambda x)\bar{z} = 2(\operatorname{Re} z\cos(\lambda x) - \operatorname{Im} z\sin(\lambda x)), \\ v(x) &= \exp(i\lambda x)z - \exp(-i\lambda x)\bar{z} = 2i(\operatorname{Im} z\cos(\lambda x) + \operatorname{Re} z\sin(\lambda x)). \end{aligned}$$

One of the processes is purely imaginary and the other is real-valued, while both of them have the same correlation function

$$K(x) = A\cos(\lambda x).$$

2. Isotropic fields

The homogeneous complex vector-valued random field $w(x)$ with correlation function $K(x)$ is said to be isotropic if

$$K(x) = K(Vx) = B(\|x\|), \quad x \in \mathbf{R}^k, \tag{2.1}$$

for any orthogonal transformation V (i.e., v is a combination of rotations and reflections). For the isotropic field we have $F(V^{-1}A) = F(A)$ and $K(x) = K^*(x)$.

The spectral representation (1) of the correlation function of the isotropic homogeneous field may be written in the form (see [14, 15]).

$$\begin{aligned} B(\rho) &= \int_0^\infty Y_k(\gamma\rho)G(d\gamma), \\ Y_k(\alpha) &= 2^{(k-2)/2}\Gamma(k/2)(\alpha)^{-(k-2)/2}J_{(k-2)/2}(\alpha), \end{aligned} \tag{2.2}$$

here J_m are the Bessel functions of the first kind and $G(B) = F(\|\lambda\| \in B)$ is a matrix-valued measure in $\mathbf{R}$. Note that

$$\begin{aligned} Y_1(a) &= \cos(\alpha), & Y_2(\alpha) &= J_0(\alpha), \\ Y_3(\alpha) &= \sin(\alpha)/\alpha, & Y_4(\alpha) &= 2\alpha^{-1}J_1(\alpha). \end{aligned}$$

If the spectral measures F and G are absolutely continuous with respect to the Lebesgue measure and f, g are the corresponding spectral densities, then

$$g(\gamma) = S_k(\gamma)f(\gamma e),$$

where $S_k(\gamma) = (2\pi^{k/2}/\Gamma(k/2))\gamma^{k-1}$ is the square of a sphere in $\mathbf{R}^k$ with radius γ and e is a unit vector.

The transformations inverse to (2.2) are of the form [16, 17]

$$
\begin{aligned}
G[0,\gamma) &= 2^{-(k-2)/2}\Gamma^{-1}(k/2)\int\limits_0^\infty (\gamma\rho)^{k/2}J_{k/2}(\gamma\rho)\rho^{-1}K(\rho)d\rho, \\
g(\gamma) &= 2^{-(k-2)/2}\Gamma^{-1}(k/2)\int\limits_0^\infty (\gamma\rho)^{k/2}J_{(k-2)/2}(\gamma\rho)K(\rho)d\rho.
\end{aligned} \tag{2.3}
$$

For real-valued homogeneous isotropic vector fields the matrix-valued spectral measures G and F are real-valued ones ($F(A) = F(-A) = \bar{F}(A)$) and $K(x) = K^T(x)$.

3. Modeling of random harmonics

3.1. A complex-valued vector random harmonic

$$
\xi(x) = \exp(i\langle x,\lambda\rangle)z, \quad \xi(x), \quad z \in \mathbf{C}^s, \quad x,\lambda \in \mathbf{R}^k, \tag{3.1}
$$

where z is a complex random vector with zero mean, has the correlation function

$$
K(x) = \exp(i\langle x,\lambda\rangle)M(zz^*)
$$

and the spectral measure is concentrated at the point λ

$$
\begin{aligned}
F(A) &= I\{\lambda \in A\}M(zz^*), \\
\operatorname{Re} F\{\lambda\} &= M\operatorname{Re} z(\operatorname{Re} z)^T + M\operatorname{Im} z(\operatorname{Im} z)^T, \\
\operatorname{Im} F\{\lambda\} &= -M\operatorname{Re} z(\operatorname{Im} z)^T + M\operatorname{Im} z(\operatorname{Re} z)^T.
\end{aligned}
$$

The field $\xi(x)$ is homogeneous (in other words, homogeneous in narrow sense, i.e., the finite-dimensional distributions are invariant with respect to shifts) if and only if the random vector z has the structure

$$
z = \exp(i\theta)z_0, \tag{3.2}
$$

where z_0 is an arbitrary complex-valued random vector, while θ is a random value independent of z_0 and uniformly distributed on $[0,2\pi]$. Obviously, we have

$$
Mzz^* = Mz_0z_0^*.
$$

A harmonic with a random frequency λ and amplitude z_λ, which depends on the frequency

$$\xi(x) = \exp(i\langle x, \lambda\rangle)z_\lambda, \tag{3.3}$$
$$Mz_\lambda = 0, \quad Mz_\lambda z_\lambda^* = F(d\lambda)/\mu(d\lambda),$$

where λ is a random vector distributed in $\mathbf{R}^k$ according to the probability measure μ, absolutely continuous with respect to F, has the correlation function

$$K(x) = \int\limits_{\mathbf{R}^k} \exp(i\langle x, \lambda\rangle)F(d\lambda)$$

and the spectral measure $F(d\lambda)$.

Remark. Basically in the capacity of the measure μ one may take an arbitrary probability measure in $\mathbf{R}^k$ absolutely continuous with respect to F. The authors of paper [11] propose

$$\mu(d\lambda) = trF(d\lambda)/trF(\mathbf{R}^k), \tag{3.4}$$

where tr denotes the trace of a matrix. The value $trF(\mathbf{R}^k)$ may be interpreted as the "full energy" of the field and $trF(d\lambda)$ as the "energy" of the frequencies $d\lambda$.

If $z_\lambda = z$ does not depend on λ, then

$$K(x) = M[zz^*] \int\limits_{\mathbf{R}^k} \exp(i\langle x, \lambda\rangle)\mu(d\lambda).$$

3.2. Let us consider the real-valued case. The vector harmonic

$$\begin{aligned} \xi(x) &= \exp(i\langle x, \lambda\rangle)z + \exp(-i\langle x, \lambda\rangle)\bar{z} \\ &= 2(\cos\langle x, \lambda\rangle \operatorname{Re} z - \sin\langle x, \lambda\rangle \operatorname{Im} z), \end{aligned} \tag{3.5}$$

where $\lambda \neq 0$, $Mz = 0$, $Mzz^T = 0$, has the correlation function

$$K(x) = 2\cos\langle x, \lambda\rangle \operatorname{Re} M[zz^*] - 2\sin\langle x, \lambda\rangle \operatorname{Im} M[zz^*] \tag{3.6}$$

and the spectral measure being concentrated at the points λ, $-\lambda$

$$F\{\lambda\} = M[zz^*] = \bar{F}\{-\lambda\}.$$

The equality $Mzz^T = 0$ means

$$\begin{aligned} M\operatorname{Re} z(\operatorname{Re} z)^T - M\operatorname{Im} z(\operatorname{Im} z)^T &= 0, \\ M\operatorname{Re} z(\operatorname{Im} z)^T + M\operatorname{Im} z(\operatorname{Re} z)^T &= 0, \end{aligned}$$

and, hence,

$$\begin{aligned} M\mathrm{Re}\, z(\mathrm{Re}\, z)^T &= M\mathrm{Im}\, z(\mathrm{Im}\, z)^T = \mathrm{Re}\, F\{\lambda\}/2, \\ -M\mathrm{Re}\, z(\mathrm{Im}\, z)^T &= M\mathrm{Im}\, z(\mathrm{Re}\, z)^T = \mathrm{Im}\, F\{\lambda\}/2. \end{aligned} \tag{3.7}$$

If

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix} = \begin{bmatrix} \rho_1 \exp(i\varphi_1) \\ \vdots \\ \rho_s \exp(i\varphi_s) \end{bmatrix}, \quad \xi(x) = \begin{bmatrix} \xi(x) \\ \vdots \\ \xi(x) \end{bmatrix},$$

then (3.5) may be rewritten in the form

$$\xi_m(x) = 2\rho_m \cos(\langle x, \lambda\rangle + \varphi_m), \quad m = 1, \ldots, s. \tag{3.8}$$

Formula (3.8) proves to be more effective for the numerical modeling than (3.5). A strictly homogeneous harmonic may be presented in the form (see (3.8))

$$\xi_m(x) = 2\rho_m \cos(\langle x, \lambda\rangle + \varphi_m^0 + \Theta), \quad m = 1, \ldots, s, \tag{3.9}$$

where the value of Θ is independent of ρ_m and φ_m^0 is uniformly distributed on $[0, 2\pi]$.

The harmonic

$$\begin{aligned} \xi(x) &= \cos\langle x, \lambda\rangle \mathrm{Re}\, z(\lambda) - \sin\langle x, \lambda\rangle \mathrm{Im}\, z(\lambda) \\ &= \begin{bmatrix} \rho_1(\lambda)\cos(\langle x, \lambda\rangle + \varphi_1(\lambda)) \\ \vdots \\ \rho_s(\lambda)\cos(\langle x, \lambda\rangle + \varphi_s(\lambda)) \end{bmatrix}, \end{aligned} \tag{3.10}$$

$$z(\lambda) = \begin{bmatrix} z_1(\lambda) \\ \vdots \\ z_s(\lambda) \end{bmatrix} = \begin{bmatrix} \rho_1(\lambda)\exp(i\varphi_1(\lambda)) \\ \vdots \\ \rho_s(\lambda)\exp(i\varphi_s(\lambda)) \end{bmatrix},$$

$$M z(\lambda) z^*(\lambda) = \begin{cases} 2F(d\lambda)/\mu(d\lambda), & \lambda \neq 0, \\ F\{0\}/\mu\{0\}, & \lambda = 0, \end{cases}$$

$$M z(\lambda) = 0, \quad M z(\lambda) z^T(\lambda) = 0,$$

where λ is a random value in $\mathbf{R}^k$ distributed according to the symmetric ($\mu(d\lambda) = \mu(-d\lambda)$) probability measure μ, absolutely continuous with respect to F, has the correlation function (1.8) and the spectral measure $F(d\lambda)$. Moreover, the following is fulfilled:

$$\begin{aligned} M\mathrm{Re}\, z(\lambda)(\mathrm{Re}\, z(\lambda))^T &= M\mathrm{Im}\, z(\lambda)(\mathrm{Im}\, z(\lambda))^T = \mathrm{Re}\, F\{d\lambda\}/\mu(d\lambda), \\ -M\mathrm{Re}\, z(\lambda)(\mathrm{Im}\, z(\lambda))^T &= M\mathrm{Im}\, z(\lambda)(\mathrm{Re}\, z(\lambda))^T = \mathrm{Im}\, F\{d\lambda\}/\mu(d\lambda), \end{aligned}$$

$$M\begin{bmatrix}\operatorname{Re} z(\lambda)\\ \operatorname{Im} z(\lambda)\end{bmatrix}\begin{bmatrix}\operatorname{Re} z(\lambda)\\ \operatorname{Im} z(\lambda)\end{bmatrix}^T=\frac{1}{\mu(d\lambda)}\begin{bmatrix}\operatorname{Re} F\{d\lambda\} & -\operatorname{Im} F\{d\lambda\}\\ \operatorname{Im} F\{d\lambda\} & \operatorname{Re} F\{d\lambda\}\end{bmatrix} \qquad (3.11)$$

If $F\{0\} = 0$, then in the capacity of the measure μ one may take the measure concentrated at a halfspace. In this case

$$Mz(\lambda)z^*(\lambda) = 4F(d\lambda)/\mu(d\lambda),$$

and in the right-hand side of (3.11) coefficient 2 will appear.

3.3. Modeling of the complex-valued random vector z with mean zero and specified covariance matrix $F = Mzz^*$ (with the additional condition $Mzz^T = 0$ for the real-valued harmonics) is the basic problem for constructing homogeneous vector-valued harmonics (3.1), (3.3), (3.5), (3.10). The conventional solution of this problem consists of two stages: modeling of the orthonormal vector ε, $M\varepsilon\varepsilon^* = E$ and subsequent linear trasformation

$$z = A\varepsilon, \text{ where } AA^* = F. \qquad (3.12)$$

In the general case, in the capacity of the orthonormal vector ε one may take a real-valued vector. In the case of real-valued harmonics, to obtain $Mzz^T = 0$, we must require that $M\varepsilon\varepsilon^T = 0$, i.e.,

$$M\operatorname{Re}\varepsilon(\operatorname{Re}\varepsilon)^T = M\operatorname{Im}\varepsilon(\operatorname{Im}\varepsilon)^T = E/2,$$
$$M\operatorname{Re} z(\operatorname{Im} z)^T = 0.$$

So, the real and the imaginary parts of vector ε must be mutually orthogonal, and the covariance matrices of these parts must be the same and equal to $E/2$.

It will be assumed that the positive-definite matrix F is not singular. The linear transformation A in (3.12) is uniquely defined up to a unitary operator U: vectors $A\varepsilon$ and $AU\varepsilon$ have the same covariance matrix F.

If we require that A, then $A = (F)^{1/2}$. In this case the matrix A may be found by the following recurent procedure:

$$A_0 = 0, \quad A_{n+1} = A_n + (2\|F\|^{1/2})^{-1}(F - A_n^2)$$

(see, e.g., [18]), or by the formula $A = WD^{1/2}W^*$, where D is the diagonal matrix with the eigenvalues of the matrix F as diagonal entries, and W is the matrix, whose columns are the eigenvectors of matrix F ($F = WDW^*$).

If we require that a matrix A be a lower triangular one, then it gives us the well-known recurrent formulas for the matrix entries A_{rt}

$$
\begin{aligned}
A_{11} &= (F_{11})^{1/2}\exp(i\varphi_1),\\
A_{rt} &= \Big(F_{rt} - \sum_{k=1}^{t-1} A_{rk}\bar{A}_{tk}\Big)/A_{tt}, \quad t = 1,\ldots,r-1,\\
A_{rr} &= \Big(F_{rr} - \sum_{t=1}^{r-1} |A_{rt}|^2\Big)^{1/2}\exp(i\varphi_r), \quad r = 2,\ldots,s,
\end{aligned}
$$

(the diagonal entries of the matrix F are real), where $\varphi_1,\ldots,\varphi_s$ are arbitrary real constants. The ambiguity of constructing the matrix A is explained by the fact that the matrix

$$
A \times \begin{bmatrix} \exp(i\varphi_1) & 0 & \ldots & 0 \\ 0 & \exp(i\varphi_2) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & \exp(i\varphi_s) \end{bmatrix}
$$

is also a lower triangular one.

4. Spectral models of homogeneous Gaussian vector fields

Let $w(x)$ be a Gaussian homogeneous vector field with mean zero, correlation function $K(x)$ and spectral measure $F(d\lambda)$. An approximate model $w_n(x)$ of the form

$$w_n(x) = \xi_1(x) + \ldots + \xi_n(x),$$

where $\xi_j(x)$ are the random harmonics considered in Section 3, will be called a spectral model. We assume that

$$M\xi_i(x)\xi_j^*(y) = 0, \text{ as } i \neq j, \quad x, y \in \mathbf{R}^k. \tag{4.1}$$

Then

$$K_{(n)}(x) = \sum_{j=1}^{n} K_j(x), \quad F_{(n)}(x) = \sum_{j=1}^{n} F_j(x).$$

Here $K_{(n)}$, $F_{(n)}$, K_j, F_j are the corresponding correlation functions and spectral measures of the fields w_n and ξ_j.

One may use a lot of various procedures to construct spectral models $w_n(x)$, which ensure convergence to the Gaussian vector field $w(x)$, as $n \to \infty$. These procedures are well-studied for scalar fields (see [19–26] and Appendix).

Let us present two typical models

A. $F_j(d\lambda) = F(d\lambda)/n, \quad j = 1, \ldots, n;$

B. $\mathbf{R}^k = \sum_{j=1}^{n} \Lambda_j, \quad F_j(d\lambda) = \begin{cases} F(d\lambda), & d\lambda \subset \Lambda_j, \\ 0, & d\lambda \subset \mathbf{R}^k - \Lambda_j. \end{cases}$

Note that the fields w_n are asymptotically Gaussian as $n \to \infty$ for model **A**, while for model **B** the following condition is sufficient:

$$\max_{j \leq n} F(\Lambda_j) \to 0, \text{ as } n \to \infty.$$

Obviously, for both models the spectral measures $F_{(n)}(d\lambda)$ of the approximate fields $w_n(x)$ coincide with the spectral measure $F(d\lambda)$ of the limit field $w(x)$. This is attained by the randomized choice of the harmonic frequencies.

5. Examples of modeling

5.1. Let $v(x)$ be a homogeneous, scalar, differentiable in the mean square complex-valued random field $x \epsilon \mathbf{R}^k$, with the correlation function $R(x)$ and spectral measure $\mu(d\lambda)$. The potential vector field

$$w(x) = \text{grad } v(x) = \left[\frac{\partial}{\partial x_1} v(x), \ldots, \frac{\partial}{\partial x_k} v(x)\right]^T \tag{5.1}$$

is homogeneous with the correlation function

$$K(x) = \left[\frac{\partial R(x)}{\partial x_r \partial x_t}\right]_{r,t=1,\ldots,k}$$

and spectral measure

$$F(d\lambda) = [-\lambda_r \lambda_t \mu(d\lambda)]_{r,t=1,\ldots,k}.$$

If the field $v(x)$ is isotropic,

$$R(x) = B(\|x\|), \quad \mu(d\lambda) = S_k^{-1}(\gamma) d\sigma(\gamma) \nu(d\lambda), \tag{5.2}$$

where $\gamma = \|\lambda\|$, $S_k(\gamma)$ is the square of a sphere in $\mathbf{R}^k$ with the radius γ, $d\sigma(\gamma)$ is the area element of this sphere, then field (5.1) is isotropic in the following sense [27]:

$$Mw(x) = 0, \quad Mw(x)w^*(y) = V[Mw(Vx)w^*(Vy)]V^*, \tag{5.3}$$

where V is an arbitrary orthogonal transformation in $\mathbf{R}^k$.

5.2. In this section, the fields which are isotropic in the sense (5.3) will be called bi-isotropic in order to distinguish this concept from the definition in Section 2. So, the homogeneous vector field $w(x)\epsilon\mathbf{R}^k$, $x\epsilon\mathbf{R}^k$, is called bi-isotropic if the first and the second moments of the field $w(x)$ coincide with the field $Vw(Vx)$. It means that if $K(x)$ is the correlation function of the field $w(x)$, then $K(x) = VK(Vx)V^*$ for any orthogonal transformation V.

The spectral measure of the bi-isotropic field is of the form [27]

$$F(d\lambda) = S_k^{-1}(\lambda)d\sigma(\lambda)\big[L\varphi(d\lambda) + (E - L)\psi(d\lambda)\big], \tag{5.4}$$

where $\gamma = \|\lambda\|$, $S_k(\gamma)$ is the square of a sphere in $\mathbf{R}^k$ with the radius γ, $d\sigma(\gamma)$ is the area element of this sphere, E is the unit (k,k)-matrix, $L = \lambda\lambda^T/\|\lambda\|^2$, φ and ψ are some finite measures on $[0,\infty)$. It is easy to verify that

$$M\|w\|^2 = \int_0^\infty \varphi(d\gamma) + (k-1)\int_o^\infty \psi(d\gamma).$$

For potential bi-isotropic field (5.1) the following is fulfilled:

$$\varphi(d\gamma) = \gamma^2\nu(d\gamma), \quad \psi(d\gamma) = 0.$$

If $w(x)$ is a bi-isotropic field with spectral measure (5.4), then the scalar field div $w(x)$ is isotropic with the "radial" spectral measure $\gamma^2\varphi(d\gamma)$. So, the field $w(x)$ is solenoidal (div $w(x) = 0$) if and only if $\varphi(d\gamma) = 0$.

The general representation of correlation functions of bi-isotropic random fields is obtained in [27].

5.3. In two-dimensional case any correlation function of potential bi-isotropic field may be written in the form

$$K(x) = \int_0^\infty [-J_2(\gamma\rho)X + J_1(\gamma\rho)(\gamma\rho)^{-1}E]\varphi(d\gamma),$$

where $\rho = \|x\|$, $X = xx^*/\|x\|$, E is the unit (2,2)-matrix.

Realizations of spectral models of the scalar isotropic field on the plane with the correlation function $J_0(cx)$ and the gradient of this field are presented in Figures 1, 2. This case corresponds to the spectral measures ν, φ concentrated at the point $\gamma = c$. More general models may be obtained by summation of such fields of different scale (see Figures 3, 4).

A correlation function of any solenoidal bi-isotropic two-dimensional vector field may be written in the form

$$K(x) = \int_0^\infty \{J_2(\gamma\rho)X + [J_0(\gamma\rho) - J_1(\gamma\rho)(\gamma\rho)^{-1}]E\}\psi(d\gamma).$$

Some realizations of spectral models of solenoidal fields on the plane are presented in Figures 5, 6.

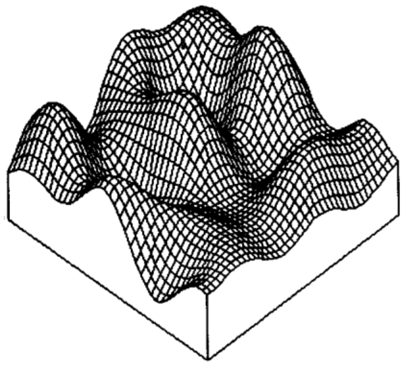

Figure 1

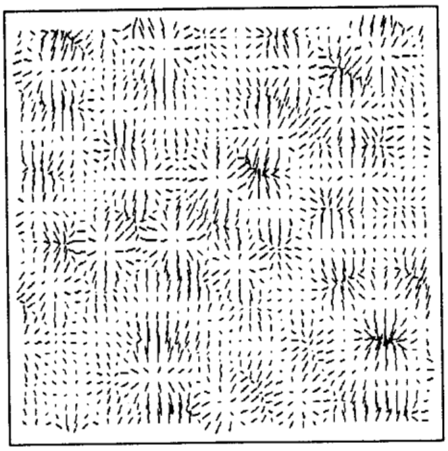

Figure 2

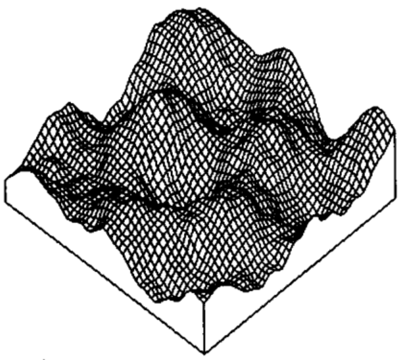

Figure 3

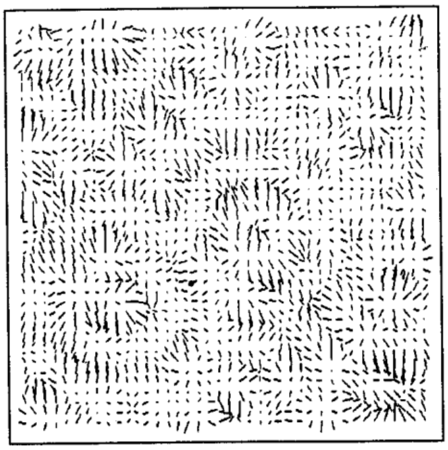

Figure 4

Note. Programs for modeling on a Q basic language of scalar-valued and vector-valued homogeneous isotropic Gaussian fields on the plane are presented in paper [28].

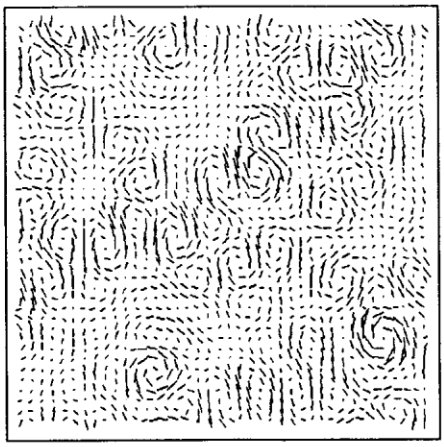

Figure 5

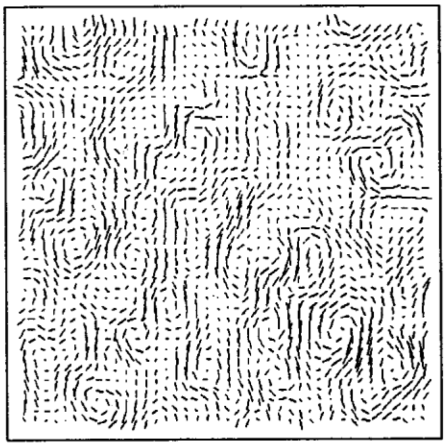

Figure 6

Appendix. Spectral models of isotropic Gaussian fields on the plane

Algorithms of modeling of homogeneous isotropic Gaussian fields on the plane, based on the the spectral models [29] and parametric models [8] are considered here.

A correlation function of any homogeneous isotropic field $w(x,y)$ may be written in the form

$$B(r) = \sigma^2 \int_0^\infty J_0(\rho r)\nu(d\rho), \tag{5.5}$$

where $r^2 = x^2 + y^2$, σ^2 is a variance of the field, J_0 is the Bessel function of the first kind and $\nu(d\rho)$ is a "radial" spectral measure on $[0,\infty)$. Further we shall assume that measure ν has a density $\nu(d\rho) = g(\rho)d\rho$. (A table of spectral densities and corresponding correlation functions of isotropic fields is presented in [30]).

Assume that $0 = R_0 < R_1 < \ldots < R_{N-1} < R_N = \infty$. We shall consider the following modification of the spectral model for isotropic case:

$$\begin{aligned} w^*(x,y) = \sigma \sum_{n=1}^{N} c_n M_n^{-1/2} \sum_{m=1}^{M_n} (-2\ln\alpha_{nm})^{1/2} \\ \times \cos[(x\rho_n\cos\omega_{nm} + y\rho_n\sin\omega_{nm}) + 2\pi\beta_{nm}]. \end{aligned} \tag{5.6}$$

Here $c_n^2 = \int\limits_{R_n-1}^{R_n} g(\rho)d\rho$; $\omega_{nm} = \pi(m - \gamma_{nm})/M_n$; ρ_n are random values distributed on $[R_{n-1}, R_n)$ according to the density $g(\rho)/c_n^2$; α_{nm}, β_{nm}, γ_{nm} are independent random values uniformly distributed on $[0,1]$.

The algorithm of the modeling is in the creating of the following arrays:

$$\begin{aligned} A(n,m) &= \sigma c_n(-2(\ln \alpha_{nm})/M_n)^{1/2}, \quad D(n,m) = 2\pi\beta_{nm}, \\ B(n,m) &= \rho_n \cos\omega_{nm}, \quad C(n,m) = \rho_n \sin\omega_{nm}, \end{aligned}$$

and the value of the field at point (x,y) is calculated by formula

$$w^*(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M_n} A(n,m)\cos[B(n,m)x + C(n,m)y + D(n,m)].$$

Model (5.6) corresponds to the splitting of the spectral space into the rings and into the segments of equal magnitude. The correlation function of model (5.6) is equal to (5.5) and if $\sum\limits_{n=1}^{N} M_n \to \infty$ and $\max\limits_{n\leq N}(c_n^2/M_n) \to \infty$, then the field $w^*(x,y)$ is asymptotically Gaussian.

Some other versions of the spectral model are allowed. In particular the following changings in (5.6) (individually and in combination) are acceptable

1. $\rho_n = \rho_{nm}$,
2. $\gamma_{nm} = \gamma_n$,
3. $\gamma_{nm} = \gamma$,

where ρ_{nm}, γ_n, γ are independent random values with correspondent distributions. Replacement 1. may be specifically appropriate for $n = N$.

Algorithmically more simple, but not flexible is model (5.6), where $c_n = N^{-1/2}$ and ρ_n are independent and distributed on the whole semi-axis $[0,\infty)$ with density $g(\rho)$. The programme realization of such an algorithm is presented in [28]. The choice of version of the simulation algorithm and its parameters is specified by the aim to represent more or less in detail the corresponding parts of spectrum and influence on the character of realizations $w^*(x,y)$.

The spectral models of isotropic fields were applied in [31] for heap cloudiness simulation for statistical modeling of the radiation transfer in the cloudy atmosphere.

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