Introduction:
A Guide to the Reader

We summarize here the contents of the following chapters by describing the results that are essential. Since most of the terms are not precisely defined in this introduction, these lines can be used as guidelines to find a given result in the text.

In *Chapter 1* the definitions of interpolating, smoothing and mixed (smoothing for a part of data and interpolating for the rest) splines are given here in Atteia style for the case of abstract Hilbert space. The existence and uniqueness theorems for all cases are proved and corresponding properties of orthogonality are obtained. For every type of splines, Euler's equation is presented. Furthermore, the general theorem on equivalent normalization of Hilbert space is proved. Special norm is connected with the operators of uniquely solvable spline problem. The main theorems in this part of the chapter are discussed through examples of one-dimensional splines and multidimensional splines at the scattered meshes both in analytic and in finite element cases. In the last part of the chapter the exact expressions for the interpolating spline-projector and smoothing spline operator are obtained in the special scalar product.

Chapter 2 is devoted to the characterization of splines. In general form, the structure of spline is connected with the "reproducing mapping" in Hilbert space with some semi-norm. Reproducing mapping is the generalization of the well-known notion of the reproducing kernel introduced by N.Aronszjain. The existence of the reproducing mapping is proved and characterization of the interpolating, smoothing and mixed splines are obtained. Examples of the concrete splines are also presented here.

Chapter 3 contains general convergence techniques for interpolating splines in Hilbert space and error estimation for splines in Sobolev spaces. In the 1-st part the definition of "right system" of interpolating operators is given and general convergence theorem for the abstract interpolating splines is proved. On this base the convergence theorem for the splines in Hilbert functional space is proved when the interpolating points form the condensated ε-net in domain. Particular case is D^m-splines at the scattered meshes. In the 2-nd part the general error estimation technique is presented. Error estimate can be obtained in various norms or semi-norms by few steps: error estimation for generalized Lagrangian polynomials, special covers of domains with the cone condition and summation of the local errors. For the particular case of D^m-splines in unbounded domain the error estimate were obtained by J.Duchon.

The same error estimates in bounded domain in L_p-norm are obtained here. These problems in more detailed form are also considered in the chapter 5.

Chapter 4 is devoted to finite dimensional analogues of interpolating and smoothing splines. In many important practical situations (multidimensional D^m-splines at the scattered meshes) structure of the analytical spline is too complicated for calculations because we have the global representation formulae connected with the reproducing mapping (or kernel) for suitable Hilbert space. Moreover this mapping is often unknown in analytical form. These reasons lead to very natural idea: application of the finite element technique for splines construction. We obtain as a result sparse algebraic systems and local piece-wise polynomial representation formulae. In 4-th chapter the finite dimensional analogues of interpolating and smoothing splines are considered, the general convergence theorems are proved and special technique for error estimation is presented. In the second part of chapter the special questions concerning with spline-approximations of discontinuous functions are considered. The discontinuity lines or surfaces are apriori given or can be automatically determined at the scattered mesh. On the base of algorithm for the construction of spline the special finite elements with discontinuities are lying.

Chapter 5 consists of two parts. The 1-st part is connected with the fundamental inequalities for the functions from Sobolev spaces with condensated zeros. On the base of this fundamental result the various types of D^m-splines at the scattered meshes are considered: D^m-splines in the bounded domains, D^m-splines in the whole R^n, see also (J. Duchon, 1976), D^m-splines with the boundary conditions. For all types of splines the error estimates in L_p-norms for splines and their derivatives are obtained. The 2-nd part of chapter is the detailed consideration of multidimensional B-splines as a finite elements for the construction of the analogues of D^m-splines at the scattered meshes. Original algorithms for the assembling of matrices and storage distribution are presented here. Special equivalent normalization of Sobolev space provides numerically effective structure of energy matrices (multidimensional "cross" instead "box") for iterative processes for the solution of linear systems. Splines in anisotropical Sobolev spaces are also considered here and corresponding error estimates are given.

In *Chapter 6* new object in variational spline theory is considered: the traces of D^m-splines on the manifolds. The aim of this consideration is to find numerical algorithm for the interpolation of the function which is known in the interpolation points of scattered condensated mesh on the manifold. The existence and uniqueness theorem for the traces of splines on algebraic and non-algebraic manifolds is proved here. As in usual case the fundamental inequality for Sobolev space of functions with fractional index is proved when function vanishes at condensated mesh on manifold. On this base error estimates for the traces of D^m-splines are obtained. From the numerical point of view the traces of D^m-splines on the unit sphere are considered in details. Second part of the chapter is connected with the approximation in analytical and finite element forms of function given on manifold by the trace of D^m-spline from the "thin layer" near the manifold. By the special normalization of the Hilbert functional space in the thin layer it is possible to preserve error estimates for traces of splines.

Chapter 7 is devoted to vector splines. Some examples of this objects are known in spline theory, for example as tools for approximation of the curves in the plane or in the space (parametric splines). The feature of our consideration is the following: vector spline is the solution of the variational problem and functionals which are given with respect to vector spline may be connect not only its separate scalar components but vector spline at whole. For this case we apply the general notion of reproducing mapping and obtain the representation formulae for analytical and finite element cases. The most important application of variational vector splines is the general numerical algorithm for calculation of multidimensional rational splines at the scattered meshes (variational formulation for one-dimensional rational splines was introduced by J. Rozhenko) which are extremely useful for the approximations of pole singularities.

In *Chapter 8* tensor and blending splines are presented. Tensor product of two linear methods for the approximations of mesh function is reduced to the consequent solutions of "pseudo-one-dimensional" problems and leads to the solution of "two-dimensional" problem at the Decart product of two meshes. Chapter contains the variational formulation for tensor spline for the case of two or more abstract spline problems. Error estimates are proved here for tensor interpolating splines. Smoothing tensor splines are also considered from variational point of view. The generalization of tensor splines is blending interpolations where instead of tensor product operation we use more general Boolean product. In this case we obtain other kinds of interpolating meshes but the variational principle and error estimates are the same.

Chapter 9 is devoted to optimal approximation of operators and functionals. In any sense the approximation of operators is more general problem than approximation of function. At first in this Chapter we prove that spline-operator helps to reach the optimal (in some semi-norm) approximation of any operator when the initial information is the value of another bounded linear operator. Then with the help of reproducing mappings or kernels particular case of given values for initial functionals (instead of operators) is considered and corresponding algebraic systems for optimal coefficients are obtained. Moreover the exact error estimates with the help of the reproducing mapping are proved. There is some alternative with respect to error estimation by error analysis in approximation of functions. After that the optimization procedure is proposed for finite dimensional case and natural connection between optimal approximation of functionals and finite-dimensional analogues of splines is founded. Furthermore the optimal quadratures on the unit sphere with spherical weight functions at the scattered meshes is presented as important example. Another example is the well-known Sobolevs optimal quadratures, which are obtained from D^m-spline theory.

The *Chapter 10* summarizes the descriptions of variational spline objects and methods presented in previous chapters. These are splines in subspaces, splines on manifolds, vector splines and tensor splines, spline-methods for optimal approximation of functionals. This chapter gives the principle of the classification on these groups which is connected with the fundamental operations over Hilbert spaces: formation of the closed subspaces, factorization by the closed subspaces, Decart and tensor products, conjugation. The combinations of these operations lead us to new Hilbert spaces. Like that the various combinations of spline methods lead us to the various kinds of mixed spline objects: splines on manifolds mixed with tensor splines, vector splines mixed with tensor splines and so on.

Chapter 11 is devoted to so called sum-product approximations. General consideration for the abstract problem of optimal $\Sigma\Pi$-approximation in the tensor product of two Hilbert spaces is given in the numerical finite dimensional sense. The generalized eigen value problem for any matrix arises to determine the optimal sum of product of "one-dimensional" functions to approximate "two-dimensional" function. Examples of these approximations based on B-splines and Fourier expansions are presented here. Numerical results for data compressing in digital image processing and decompositions of two-dimensional filters into one-dimensional row-column filters are also given.

Chapter 12 concern the choice of optimal smoothing parameter from the residual principle. The properties of residual function are obtained in most general case on the base of spectral decomposition of smoothing spline operator. The various tricks to accelerate the calculations in the Newton's method are discussed here and many useful practical formulas are obtained.

There are two Appendices in this book. Appendix 1 contains (without proofs) the main theorems on functional analysis used in our book; Appendix 2 contains the brief description of software LIDA-3 (library on data approximation and digital signal and image processing) which is produced by authors and their colleges in Computing Center of USSR Academy of Science in Novosibirsk.