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Modeling of the dynamic behaviour of anisotropic media containing parallel cracks ensembles*

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Abstract. The present paper deals with the dynamic problem of linearly deformable elastic media with an arbitrary anisotropy, containing flat parallel cavities-cracks — the so-called viruses of vibration strength of class 2. The new method with allowance for specific features of integrand functions is used for calculation of multi-dimensional inverse Fourier transform, representing a displacement vector in a near zone is briefly described.

With mathematical modeling of geophysical structures, a class of fundamental problems, which includes studying the dynamic processes in complicated anisotropic elastic medium containing local heterogeneities is often considered. In case when a heterogeneous formation is exposed to influence of concentrated or volume forces, there are processes connected with development of zones of nonlinear demultiplexing (dilatancy zones). In publications by academician V.A. Babeshko, it is shown that the origin and development of dilatancy zones can be associated with activization of the so-called "vibration strength viruses," that is, plural cracks or inclusions which form dynamic dilatancy structure. Today it is possible to consider that the origin and forming of a local structures dilatancy type always take place at the stage of preparation and development of disastrous processes in complicated anisotropic elastic media. Therefore it is necessary to mention publications by academician A.S. Alekseev and his disciples, who were the first to develop this problem in seismology. They managed to show that forming of dilatancy zones in a geological structure always occurs on the eve of a large seismic event and generates a number of characteristic wave precursors in seismic, acoustic and electromagnetic areas.

In the present work, on the basis of classical approaches used in mechanics, the boundary value problem of long-period fluctuations of an anisotropic layered elastic medium containing a limited zone with raised fracturing is considered. Fluctuations of a medium are described by the differential equations of the following form:

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_l \partial x_j} - \rho\frac{\partial^2 u_i}{\partial t^2} + \rho Q_i = 0,$$

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where $u_i(x_1, x_2, x_3, t)$ are displacements of the medium, t is time, c_{ijkl} is tensor of elastic modules, ρ is density of the medium, Q are volume forces.

As an elastic medium, we can choose an elastic space, a semi-space or an elastic layer (a number of layers). The dilatancy zone is characterized by the high concentration of vibration strength viruses of classes 1 and 2 type S, with level L. For the sake of definiteness, we will consider that heterogeneities are cracks, S are areas-carriers of cracks, L is the quantity of cracks, can be as much as possible, but final. In the conditions of the fluctuation established with frequency ω borders $x_3 = h_l$ of cracks are exposed to the influence of stresses with the amplitudes $\boldsymbol{\tau}_l = (\tau_{l1}, \tau_{l2}, \tau_{l3})$. We mark amplitudes of displacements in sections $x_3 \to \pm h_l$ as $\boldsymbol{u}_l^{\pm} = t(u_{l1}^{\pm}, u_{l2}^{\pm}, u_{l3}^{\pm})$. Displacements of medium on cracks are discontinuous.

For medium of layer (or a package of layers) type on the bottom border, various boundary conditions can be set. For a homogeneous semi-space, it is necessary to add conditions of decreasing a component of the displacement vector with depth, in case of a layered medium as well as conditions of coupling on interfaces between layers. For space the condition of decrease of amplitudes of displacement on infinity is set.

By means of Fourier integral transformations and a formalism based on the Betty theorem [1],

$$\int_\Omega (u\Delta^0 v - v\Delta^0 u)\, d\Omega = \int_S (uTv - vTu)\, dS$$

the boundary value problem is reduced to the following system of integral equations:

$$\sum_{l=0}^{L+1} \sum_{m=1}^{3} \iint\limits_{S_l} L_{km}(\alpha_1, \alpha_2, \alpha_3, \kappa) u_{lm}(\xi_1, \xi_2, \xi_3) \exp\big(i(\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3)\big)\, dS$$
$$= \sum_{l=0}^{L+1} \sum_{m=1}^{3} \iint\limits_{S_l} D_{km}(\alpha_1, \alpha_2, \alpha_3, \kappa) \tau_{lm}(\xi_1, \xi_2, \xi_3) \exp\big(i(\alpha_1\xi_1 + \alpha_2\xi_2 + \alpha_3\xi_3)\big)\, dS,$$
$$|\alpha_1| \le \infty, \qquad |\alpha_2| \le \infty, \qquad \alpha_3 = \sigma_k^{\pm}(\alpha_1, \alpha_2, \kappa), \quad k = 1, 2, 3. \tag{1}$$

Here, the stresses T_k and the displacements u_m are assigned to the areas S_l. Taking into consideration that S_l (in case of plane-cracks) are straight parallel lines and using the mark "+" for the upper boundary of each crack and "−" for the lower boundary, let us present (1) in the form:

$$\boldsymbol{L}_l^{\pm} \boldsymbol{U}_l^{\pm} - \boldsymbol{L}_{l+1}^{\pm} \boldsymbol{U}_{l+1}^{\pm} = \boldsymbol{D}_l^{\pm} \boldsymbol{T}_l - \boldsymbol{D}_{l+1}^{\pm} \boldsymbol{T}_{l+1},$$

where $\boldsymbol{U}_l^{\pm}$ are 2D Fourier transform of $u_l^{\pm}$ — the displacement vector on the upper and the lower boundaries of crack number l, $\boldsymbol{T}_l^{+} = \boldsymbol{T}_l^{-} = \boldsymbol{T}_l$ are

2D Fourier transforms of τ_l — vector of stresses on the upper and the lower boundaries of crack number l, matrices $\boldsymbol{L}_l^+$ and $\boldsymbol{D}_l^+$ represent the influence of elastic waves propagating in a medium in the vertical direction and $\boldsymbol{L}_l^-$ and $\boldsymbol{D}_l^-$ — down in the vertical directions. The matrices $\boldsymbol{L}$, $\boldsymbol{D}$ are constructed in the following way:

$$L_{ni} = L_{ni}^{\pm} = \sum_{m=1}^{3} f_m \tau_{mi}(\alpha_1, \alpha_2, \sigma_n^{\pm}, \kappa, l), \quad D_{nm}^{=} D_{nm}^{\pm} = f_m, \quad i, m = 1, 2, 3$$

where

$$\tau_{km} = i \sum_{n=1}^{3} \sum_{j=1}^{3} C_{njkm} \alpha_m l_j, \quad m, k = 1, 2, 3,$$
$$f_m = c_{pm}^0(\alpha_1, \alpha_2, \sigma_n^{\pm}, \kappa),$$

$\sigma_n^{\pm}$ is the root of determinant of $\boldsymbol{Y}$ with entries

$$Y_{nm} = -\sum_{j=1}^{3} \sum_{k=1}^{3} {}_{mjnk} \alpha_j \alpha_k + \delta_{nm} \kappa^2, \quad n, m = 1, 2, 3,$$

$\kappa = \omega\sqrt{\rho}$ is a unit frequency, δ_{nm} is the Kronecker symbol, α_j are parameters of the Fourier integral transform.

Here U_{nm} and T_{nk} are specified as

$$U_{nm}(\alpha_1, \alpha_2) \exp(i\alpha_3 h_n) = \iint\limits_{S_n} u_{nm}(\xi_1, \xi_2, h_n) \exp\big(i(\alpha_1 \xi_1 + \alpha_2 \xi_2)\big) d\xi_1 d\xi_2,$$
$$T_{nk}(\alpha_1, \alpha_2) \exp(i\alpha_3 h_n) = \iint\limits_{S_n} \tau_{nk}(\xi_1, \xi_2, h_n) \exp\big(i(\alpha_1 \xi_1 + \alpha_2 \xi_2)\big) d\xi_1 d\xi_2.$$

Similar to [2, 3], we transform (1) to traditional functional-matrix form

$$\boldsymbol{U} = \boldsymbol{K}\boldsymbol{T},$$

where the matrix $\boldsymbol{K}$ has a block structure

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{G}_{11}^{(0)} & \boldsymbol{G}_{12}^{(0)} & 0 & \dots & 0 & 0 & 0 \\ -\boldsymbol{G}_{21}^{(0)} & \boldsymbol{G}_{11}^{(1)} - \boldsymbol{G}_{22}^{(0)} & \boldsymbol{G}_{12}^{(1)} & \vdots & \vdots & \vdots & \vdots \\ 0 & -\boldsymbol{G}_{21}^{(1)} & \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \boldsymbol{G}_{12}^{(L-2)} & 0 & \vdots \\ \vdots & \vdots & \vdots & -\boldsymbol{G}_{21}^{(L-2)} & \boldsymbol{G}_{11}^{(L-1)} - \boldsymbol{G}_{22}^{(L-2)} & \boldsymbol{G}_{12}^{(L-1)} & 0 \\ \vdots & \vdots & \vdots & 0 & -\boldsymbol{G}_{21}^{(L-1)} & \boldsymbol{G}_{11}^{(L)} - \boldsymbol{G}_{22}^{(L-1)} & \boldsymbol{G}_{12}^{(L)} \\ 0 & 0 & \dots & \dots & 0 & \boldsymbol{G}_{21}^{(L)} & \boldsymbol{G}_{22}^{(L)} \end{pmatrix}, \quad (2)$$

$$\boldsymbol{G}_n = \begin{pmatrix} \boldsymbol{L}^- & -\boldsymbol{F}^-_{n,n+1}\boldsymbol{L}^- \\ \boldsymbol{L}^+ & -\boldsymbol{F}^+_{n,n+1}\boldsymbol{L}^+ \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{D}^- & -\boldsymbol{F}^-_{n,n+1}\boldsymbol{D}^- \\ \boldsymbol{D}^+ & -\boldsymbol{F}^+_{n,n+1}\boldsymbol{D}^+ \end{pmatrix},$$

$$\|\boldsymbol{F}^{(m,k)\pm}\| = F^{(m,k)\pm}_{pr} = \delta_{pr}\exp\left(i\sigma^\pm_p(h_k - h_m)\right), \quad p, r = 1, 2, 3,$$

p, r are linearly independent rows of

$$\boldsymbol{c}^0 = V\boldsymbol{c} = \begin{pmatrix} (Y_{22}Y_{33} - Y_{23}Y_{32}) & (Y_{13}Y_{32} - Y_{12}Y_{33}) & (Y_{12}Y_{23} - Y_{13}Y_{22}) \\ (Y_{23}Y_{31} - Y_{21}Y_{33}) & (Y_{11}Y_{33} - Y_{13}Y_{31}) & (Y_{13}Y_{21} - Y_{11}Y_{23}) \\ (Y_{21}Y_{32} - Y_{22}Y_{31}) & (Y_{12}Y_{31} - Y_{11}Y_{32}) & (Y_{11}Y_{22} - Y_{12}Y_{21}) \end{pmatrix}.$$

Components of $\boldsymbol{U}$ have the form $\boldsymbol{U}_l = \boldsymbol{U}^+_l - \boldsymbol{U}^-_l$ and represent a discontinuity of displacement vector components on cracks.

Representation of the solution to the boundary value problem in the integral form describes a field of fluctuations in the vicinity of heterogeneous formations of the dilatancy type. As an approximate solution to contact problem, the source of fluctuations can be set in the form of a concentrated or a distributed load onto the known law in the set area. An advantage of the obtained representation of the Green matrix is that the presented formalism can be easily generalized to the case when elastic properties of layers are various. For this purpose, when calculating matrices, it is sufficient to use for each layer elastic constants of a certain layer.

The integrated representation of wave processes by means of the Green matrix allows us to construct a wave field in a near and a far zone, to study processes of supplying energy to a medium, to set and to solve a problem of formation of the oriented radiation of a set of sources, with allowance for their mutual influence; and, also, to study the influence of cracks on a wave field in an elastic medium, in particular, resonant features of a multilayered medium and in a medium containing systems of cracks or inclusions.

It should be noted that analysis of a contribution of features of the matrix $\boldsymbol{K}$ allows allocation of some regular structures which can be identified with volume, superficial or channel waves from the integral representation of a wave field.

The essence of the method of transformation of data integral equations systems to the equations of the second kind by the means of factorization consists in transformation of the initial integral operator in the vicinity of normals to the border of the definition domain of the integral equation in the identical operator and the Fredholm operator. This is done by means of special transformations of the equations system with information about a real singularity of the matrix entries of the system and its determinant. For one or two cracks, this problem can be easily solved, because the properties of a matrix, i.e., functions of a symbol of the kernel enable us to take advantage of known algorithms of calculation of determinants of block matrices. With the growth of the number of various levels with heterogeneities, the use of the given approach looses its effectiveness because of the necessity of matrices inversion of high order, generally of $(L+1)\cdot 6$ dimensions. Then the solution to linear system (1) concerning its right-hand or left-hand side can be obtained, inverting the same blocks of matrices of dimension 6 which is much more preferable from the standpoint of computation.

In order to obtain a solution to a direct problem of the theory of vibration strength viruses it is necessary to develop special numerical methods of studying the functions presented in the form of multi-dimensional Fourier integrals. In the general case of areas of an arbitrary form, the specified integrals can be presented as functions on coverings of topological decomposition of unit. In cases, when areas under study have plainly-parallel borders of a section, the Fourier integrals become simpler.

In spite of the existence of numerous methods and programs for calculation of similar integrals, the elaboration of new effective methods with allowance for specific features of integrand functions, is still relevant.

A new method of an approximate calculation of the inverse Fourier transform of the displacement vector $\boldsymbol{w} = F^{-1}_{\alpha_1,\alpha_2}[\boldsymbol{W}]$ with a known vector of discontinuity on the crack borders $\boldsymbol{F}$ is presented below. In the cylindrical co-ordinates (r, β, z), the vector w can be presented in the following form:

$$\begin{aligned}\boldsymbol{w} = \frac{1}{4\pi^2}\int\limits_0^{\pi}\int\limits_{\Gamma} \boldsymbol{K}(\alpha,\gamma,z)\boldsymbol{F}(\alpha,\gamma)\exp(i\alpha r\sin\tau)\alpha d\alpha d\tau + \\ \frac{1}{4\pi^2}\int\limits_0^{2\pi}\int\limits_{\Gamma} \boldsymbol{K}(\alpha,\gamma,z)\boldsymbol{F}(\alpha,\gamma)\exp(i\alpha r\sin\tau)\alpha d\alpha d\tau, \qquad (3)\end{aligned}$$

where $\tau = \gamma - \beta - \pi/2$.

A large number of demanded operations in the matrix calculation, strong oscillation and singularity of an integrand expression essentially complicates a direct account of the specified integral. For overcoming this difficulty, a new approach that allows with insignificant computing expenses to estimate a contribution of the material poles and the complex poles nearest to a real axis into the value of integral (3) is elaborated.

In the first integral (3), $\sin\tau \geq 0$, therefore, the contour Γ can be deformed and looped in the first quadrant, in the second integral $\sin\tau \leq 0$ and Γcan be similarly looped in the fourth quadrant. Closed loops have integrals along the imaginary axis $\Gamma_i^+ \leftrightarrow (i\infty, 0]$, $\Gamma_i^- \leftrightarrow [0, -i\infty)$, on the arcs of the quarter-circles R^+ in the first and R^- in the fourth quadrants:

$$\tilde{\boldsymbol{w}}(r,\beta,z) = \frac{1}{4\pi^2}\int\limits_0^{\pi}\int\limits_{\Gamma_R+R^++\Gamma_i^+} \tilde{\boldsymbol{W}}(r,\beta,\alpha,\gamma,z)\,d\alpha\,d\tau + \frac{1}{4\pi^2}\int\limits_{\pi}^{2\pi}\int\limits_{\Gamma_R+R^-+\Gamma_i^-} \tilde{\boldsymbol{W}}(r,\beta,\alpha,\gamma,z)\,d\alpha\,d\tau \tag{4}$$

where Γ_R is the contour Γ in the case when $\alpha \le R$. Then $\boldsymbol{KF}$ is diminishing at $\alpha \to \infty$ rather than α^{-1}, integrals on R^+, R^- according to Jordan's lemma at $R \to \infty$ tend to zero. Then

$$\boldsymbol{w} = \boldsymbol{w}^+ + \boldsymbol{w}^- - \boldsymbol{w}_I^+ - \boldsymbol{w}_I^- \tag{5}$$

where

$$\boldsymbol{w}^\pm = \frac{\pm i}{2\pi}\sum_k \int\limits_{\sigma^\pm}^{\psi^\pm} \boldsymbol{b}^{(k)}(\gamma,z)\exp\big(i\xi^{(k)}(\gamma)r\sin\tau\big)\,d\tau,$$

$$\sigma^+ = 0, \quad \sigma^- = \pi, \quad \psi^+ = \pi, \quad \psi^- = 2\pi,$$

$$\boldsymbol{w}_I^\pm = \frac{\mp 1}{4\pi^2}\int\limits_{\sigma^\pm}^{\psi^\pm}\int\limits_{\Gamma_i^\pm} \boldsymbol{K}(\alpha,\gamma,z)\boldsymbol{F}(\alpha,\gamma)\alpha\exp(-|\alpha r\sin\tau|)\,d\alpha\,d\tau,$$

$$b_j^{(n)}(\gamma,z) = \operatorname{res} K_{jm}(\alpha,\gamma,z)f_m(\alpha,\gamma)\alpha|_{\alpha=\xi^{(n)}(\gamma)}, \quad j,m = 1,2,3.$$

From the computational point of view, the integrals $\boldsymbol{w}_I^\pm$ converge rather than $\boldsymbol{w}^\pm$. In addition, it has been experimentally found that $|\boldsymbol{w}_I^\pm| \ll |\boldsymbol{w}^\pm|$. If the poles of $\boldsymbol{K}$ lie on an imaginary axis, it is possible to round them arbitrarily (at the corresponding quadrant) in the process of computation of $\boldsymbol{w}_I^\pm$ but in this case, some oscillating parts in the integrand exponent function can appear.

Using the presented numerical procedure and its modifications for the case of vibration strength viruses of class 1 (systems of inclusions), it is possible to vary its updatings structure of the anisotropic layered elastic medium (a number of layers with different mechanical properties and an arbitrary anisotropy, arrangement of vibration strength viruses in the layered medium) and to investigate configuration of a zone with nonlinear properties. The borders of dilatancy zones can be defined according to the condition of excess of the greatest tangents of pressure of some threshold value.

A classical factorization method used in a number of publications, such as [4, 5] is effective only for one-coherent convex areas, only some zones of plane borders being possible. With some essential updatings, the method of factorization is applicable for the research of boundary value problems in unlimited and multicoherent areas. The generalized factorization method, presented in [6], is applied for the research into boundary value problems in multicoherent areas with borders, provided a change of signs of the curvature of a surface. In its basis lies a representation of a group of any movements which brings about Bessel and spherical functions. In this connection, projections are carried out not on a plane and a semispace as in the case of classical factorization, but on an area of a more complicated structure, in particular, on circles, cylinders and spheres, thus considerably expanding the geometry of areas of the boundary value problems. These can be multicoherent areas and the areas having relief borders.

The newest updatings of the generalized factorization method [5] enable us to approach as much as possible the problem geometry to a real geological complex structure. In the process of performing the research there usually are complexities dictated by specific features of a problem, causing a considerable simplification of models, such as the presence of concentrators of pressure, a big number of parameters, the necessity of introduction of additional criteria of destruction (integrated temperature criteria and averaging mechanical criteria for a system of cracks). These problems can be overcome by a set of techniques of applied topology, namely, a powerful tool of modern mathematics when studying concrete applied questions of physics and mechanics — the theory of differential forms on manifolds.

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