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Variational rational splines of many variables

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The purpose of this paper is to construct the interpolating function as a ratio of two splines. The numerator and the denominator of this ratio minimize some combined variational functional on the set of pairs of functions which satisfy interpolating conditions and some additional restrictions. Such a construction was proposed by author in [7] for one-dimensional case. In present paper it is generalized to the multi-dimensional case and some convergence results are obtained.

1. Class of interpolated functions

Let Ω be a bounded domain in R^n and $X(\Omega)$ be a Hilbert space of real functions continuously embedded into $C(\overline{\Omega})$. We assume that the interpolated function may be represented in the form $f(t) = f_1(t)/f_2(t)$, where f_1 and f_2 belong to $X(\Omega)$. This representation is not unique since the numerator and the denominator can be multiplied by any sufficiently smooth function having no zeroes on $\overline{\Omega}$.

The function f on $\overline{\Omega}$ takes finite or infinite values (we identify $+\infty$ and $-\infty$). It is determined everywhere in $\overline{\Omega}$ except the points where the numerator and the denominator are simultaneously equal to zero.

Example 1. Consider the function

$$f(t) = \begin{cases} -1, & t \in [-1, 0), \\ +1, & t \in (0, 1]. \end{cases}$$

It is not determined at the point 0. Let $X(\Omega)$ be the Sobolev space $W_2^m[-1, 1]$, $m > 0$. Then this function can be presented in the form

$$f(t) = \frac{\operatorname{sign} t \cdot t^m}{t^m}$$

with the numerator and the denominator belonging to $W_2^m[-1, 1]$.

Example 2. Define on the square $[-2,2]\times[-2,2]$ the function

$$f(x,y)=\frac{y-\sin x}{y+\sin x}.$$

On the curves $y=C\cdot\sin x$ it is equal to the constant $(1-C)/(1+C)$. At the point $(0,0)$ the numerator and the denominator are equal to zero. Since all these curves pass through the point $(0,0)$, we can not determine the function at that point.

Let us consider the Hilbert space $X^2(\Omega)$ consisting of pairs $[x_1,x_2]$, where $x_i\in X(\Omega)$, $i=1,2$.

Definition 1. *The point* $t\in\overline{\Omega}$ *is called regular for the pair* $[x_1,x_2]\in X^2(\Omega)$, *if*

$$x_1^2(t)+x_2^2(t)\neq 0.$$

The set of all regular points of the pair $[x_1,x_2]$ will be denoted by $\mathrm{Reg}[x_1,x_2]$.

2. Variational setting of a problem

Let $Y(\Omega)$ be a Hilbert space and $T\in\mathcal{L}(X(\Omega),Y(\Omega))$, i.e., T is a bounded linear operator. We assume also that the operator T has the finite-dimensional kernel $\mathcal{N}(T)$ and its image is closed in $Y(\Omega)$.

Let $\omega\subset\overline{\Omega}$ be a set of interpolating points. We have to construct the rational spline $\sigma(t)=\sigma_1(t)/\sigma_2(t)$, which satisfies the interpolating conditions

$$\sigma(t)=f(t),\quad t\in\omega. \tag{1}$$

Replace (1) to the weaker conditions

$$f_2(t)\sigma_1(t)-f_1(t)\sigma_2(t)=0,\quad t\in\omega,$$

where $f=f_1/f_2$ is some representation of the interpolated function.

Fix a subset π from $\omega\cap\mathrm{Reg}[f_1,f_2]$ and call it the *tie set* of the rational spline.

Definition 2. *The function* $\sigma(t)=\sigma_1(t)/\sigma_2(t)$ *is called a rational interpolating spline for the function* f, *if*

$$[\sigma_1,\sigma_2]=\arg\min_{[x_1,x_2]\in I_{\pi,\omega}}\|Tx_1\|^2+\|Tx_2\|^2, \tag{2}$$

where

$$I_{\pi,\omega} = I_\omega \cap G_\pi,$$
$$I_\omega = \left\{[x_1, x_2] \in X^2(\Omega): \quad f_2 x_1 - f_1 x_2|_\omega = 0\right\},$$
$$G_\pi = \left\{[x_1, x_2] \in X^2(\Omega): \quad [x_1, x_2]|_\pi = [f_1, f_2]|_\pi\right\}.$$

Since $[f_1, f_2] \in I_{\pi,\omega}$, we have $I_{\pi,\omega} \neq \emptyset$. Any pair $[x_1, x_2] \in I_{\pi,\omega}$ is regular in some neighbourhood of the set π and

$$\frac{x_1(t)}{x_2(t)} = f(t), \quad \forall t \in \omega \cap \mathrm{Reg}[f_1, f_2] \cap \mathrm{Reg}[x_1, x_2].$$

Hence, the rational spline σ interpolates the function f on the set

$$\omega \cap \mathrm{Reg}[f_1, f_2] \cap \mathrm{Reg}[\sigma_1, \sigma_2]$$

and this set is not empty (it includes at least the set π).

Remark 1. The set $I_{\pi,\omega}$ is a closed afine subspace of $X^2(\Omega)$ not containing zero.

Remark 2. The equality

$$I_{\pi,\omega} = I_{\pi,\omega \cap \mathrm{Reg}[f_1, f_2]}$$

is valid, i.e., the non-regular points of the pair $[f_1, f_2]$ does not affect on the solution of the problem (2).

Remark 3. If the set π consists of one point, then the rational spline $\sigma(t)$ does not depend on the choice of representation of the interpolated function $f(t)$ as a ratio f_1/f_2.

3. Existence and uniqueness of the rational spline

The problem (2) is always solvable but the solution may be not unique. Here we obtain the sufficient conditions for its uniqueness.

Lemma 1 [5]. *Let X, Y, Z be some Hilbert spaces and let $T \in \mathcal{L}(X, Y)$, $A \in \mathcal{L}(X, Z)$ have closed images and the kernels $\mathcal{N}(T)$ and $\mathcal{N}(A)$ respectively. Then the following statements are equivalent:*

(i) *The subspace $\mathcal{N}(T) + \mathcal{N}(A)$ is closed in X and $\mathcal{N}(T) \cap \mathcal{N}(A) = \emptyset$.*

(ii) *The norm $\|u\|_* \overset{df}{=} \left(\|Tu\|_Y^2 + \|Au\|_Z^2\right)^{1/2}$ is equivalent to the norm $\|u\|_X$.*

Theorem 1. *Let $\mathcal{N}(T) \times \mathcal{N}(T) \cap I_\omega = \emptyset$. Then the problem (2) is uniquely solvable.*

Proof. Since the subspace I_ω is closed in $X^2(\Omega)$, there exists the orthoprojector $A \in \mathcal{L}(X^2(\Omega), X^2(\Omega))$ onto $I_\omega^\perp$. Let us define an operator $B \in \mathcal{L}(X^2(\Omega), Y^2(\Omega))$ by the identity

$$B[x_1, x_2] = [Tx_1, Tx_2].$$

Then

$$\mathcal{N}(A) = I_\omega, \qquad \mathcal{N}(B) = \mathcal{N}(T) \times \mathcal{N}(T).$$

By finite-dimensionality of $\mathcal{N}(T)$, the subspace $\mathcal{N}(A) + \mathcal{N}(B)$ is closed in $X^2(\Omega)$. This implies the item (i) of Lemma 1 due to the assumption of the theorem. Hence, the norm

$$|||[x_1, x_2]|||_T \stackrel{df}{=} \left(\|Tx_1\|^2 + \|Tx_2\|^2\right)^{1/2} \tag{3}$$

is equivalent to the norm of the subspace I_ω induced by the original norm of X. Thus, the problem (2) is reduced to finding an element of the afine subspace $I_{\pi,\omega} \subset I_\omega$ least deviated from zero with respect to the norm $\|\cdot\|_T$. It is well-known that this problem is uniquely solvable. □

Definition 3. *Let* $\dim \mathcal{N}(T) = k$. *A set* $\tilde{\omega} \subset \overline{\Omega}$ *consisting of* $2k$ *points is called the L-set, if the system of equations*

$$f_2 x_1 - f_1 x_2|_{\tilde{\omega}} = 0, \quad x_1, x_2 \in \mathcal{N}(T)$$

has only the null solution.

Remark 4. The condition of the unique solvability of the problem (2) can be written in the following form: if the set ω contains an L-set, then the problem (2) is uniquely solvable.

Denote

$$\mathcal{S}_{\pi,\omega}[f_1, f_2] = [\sigma_1, \sigma_2], \quad \mathcal{R}_{\pi,\omega}[f_1, f_2] = \sigma_1/\sigma_2,$$

where the pair $[\sigma_1, \sigma_2]$ is the solution of the problem (2).

4. Convergence of the pairs

Here we will show the strong convergence of the pairs $[\sigma_1, \sigma_2]$ to the limited pair

$$[g_1, g_2] \stackrel{df}{=} \mathcal{S}_{\pi,\overline{\Omega}}[f_1, f_2].$$

on the sequence of the nested meshes, which are condensated everywhere in $\mathrm{Reg}[f_1, f_2]$.

Lemma 2. *Let H be a Hilbert space and let $\{M_i\}_{i\in N}$ be a family of non-empty closed convex sets, such that $M_{i+1} \subset M_i$. Then the sequence of the elements*

$$x_i = \arg\min_{x\in M_i} \|x\|$$

strongly converges to the element

$$x_\infty = \arg\min_{x\in M_\infty} \|x\|, \quad M_\infty = \bigcap_{i=1}^{\infty} M_i.$$

Proof. The set M_∞ is not empty by virtue of the closeness of the space H. Obviously, it is closed and convex. By [4], the norm minimization problem on the closed convex set is uniquely solvable. Hence, the elements x_i and x_∞ are uniquely determined.

Let us prove that the sequence (x_i) is fundamental. Since $M_{i+1} \subset M_i$, the sequence $(\alpha_i \stackrel{df}{=} \|x_i\|)$ monotonously increases. By definition of the set M_∞ the element x_∞ belongs to M_i, therefore, the sequence (α_i) is upper bounded with respect to $\|x_\infty\|$, and, thus, it is convergent. With respect to [4, Theorem 2.2.2], the element x_i satisfies the inequality

$$(x_i, x - x_i) \geq 0, \quad \forall x \in M_i,$$

i.e.,

$$(x_i, x) \geq \|x_i\|, \quad \forall x \in M_i.$$

Hence,

$$\|x - x_i\|^2 = \|x\|^2 - 2(x, x_i) + \|x_i\|^2 \leq \|x\|^2 - \|x_i\|^2.$$

Substituting $x = x_j$ $(j > i)$, we obtain

$$\|x_j - x_i\|^2 \leq \alpha_j^2 - \alpha_i^2$$

and from the convergence of the sequence (α_i) follows that the sequence (x_i) is fundamental.

Since the set M_i is closed and for $j > i$ we have $x_j \in M_i$, it is obvious that the element $x_* \stackrel{df}{=} \lim_{i\to\infty} x_i$ belongs to M_i. Therefore, $x_* \in M_\infty$. It remains to show, that $x_* = x_\infty$. This fact follows from the inequality $\|x_i\| \le \|x_\infty\|$ and from the uniqueness of the element from M_∞ with minimal norm. □

Theorem 2. *Let $\{\omega_i \subset \overline{\Omega}\}_{i\in N}$ be a family of sets, such that $\omega_i \subset \omega_{i+1}$, the sets ω_i include an L-set and the tie set π. If the sets w_i are condensated everywhere in* $\mathrm{Reg}[f_1, f_2]$, *then the sequence of the splines $[\sigma_{1,i}, \sigma_{2,i}] = \mathcal{S}_{\pi,\omega_i}[f_1, f_2]$ strongly converges to the spline $[g_1, g_2] = \mathcal{S}_{\pi,\overline{\Omega}}[f_1, f_2]$.*

Proof. The condition $\omega_i \subset \omega_{i+1}$ implies that $I_{\omega_{i+1}} \subset I_{\omega_i}$. This means that the family of the sets $\{M_i \stackrel{df}{=} I_{\pi,\omega_i}\}$ satisfies the condition $M_{i+1} \subset M_i$. Obviously, they are closed and convex. Further, let us consider the subspace I_{ω_1} and introduce the norm $\|\cdot\|_T$ by formula (3). This norm is equivalent to the original one on the subspace I_{ω_1} (see the proof of Theorem 1).

Setting $H = (I_{\omega_1}, \|\cdot\|_T)$ and using Lemma 2 we derive that the sequence $[\sigma_{1,i}, \sigma_{2,i}]$ strongly converges to the solution of the problem

$$\min_{[x_1,x_2]\in I_\infty} \|Tx_1\|^2 + \|Tx_2\|^2 ,$$

where

$$I_\infty = \bigcap_{i=1}^{\infty} I_{\pi,\omega_i}.$$

It remains to show that the set I_∞ is equal to $I_{\pi,\overline{\Omega}}$. Since $I_{\pi,\omega_i} = I_{\omega_i} \cap G_\pi$ and $I_{\pi,\overline{\Omega}} = I_{\overline{\Omega}} \cap G_\pi$, it is sufficient to prove that

$$I_{\overline{\Omega}} = \bigcap_{i=1}^{\infty} I_{\omega_i}.$$

Obviously, $I_{\overline{\Omega}} \subset \bigcap_{i=1}^\infty I_{\omega_i}$. Prove the inverse embedding. Let $[x_1, x_2] \in \bigcap_{i=1}^\infty I_{\omega_i}$, i.e., the function $y \stackrel{df}{=} f_2x_1 - f_1x_2$ vanishes on the set $\bigcup_{i=1}^\infty \omega_i$. This set is dense in $\mathrm{Reg}[f_1, f_2]$ due to the assumption of the theorem. Further, the function y vanishes on the set $\overline{\Omega} \setminus \mathrm{Reg}[f_1, f_2]$ by definition of the set $\mathrm{Reg}[f_1, f_2]$. Therefore, the function y vanishes on the dense subset of $\overline{\Omega}$, which by embedding of the space $X(\Omega)$ into $C(\overline{\Omega})$ implies that the function y is identically equal to zero. □

5. Convergence of the rational splines

Definition 4. *The function $g = g_1/g_2 = \mathcal{R}_{\pi,\overline{\Omega}}[f_1, f_2]$ is called the limited rational spline for the function $f = f_1/f_2$ on the tie set π.*

Denote

$$Q = \mathrm{Reg}[f_1, f_2] \cap \mathrm{Reg}[g_1, g_2].$$

The set Q contains a neighbourhood of π and, thus, has non-empty interiority. By definition of the set $I_{\pi,\overline{\Omega}}$ the function f coincides with the limited rational spline g on Q.

Let the sequence of the sets $\{\omega_i\}$ satisfy the conditions of Theorem 2. Then the rational splines $\sigma_i = \mathcal{R}_{\pi,\omega_i}[f_1, f_2]$ point-wisely converge to the function f on the set Q with respect to topology of the space $\overline{R} = R \cup \{\infty\}$ supplemented by neighbourhood of the point ∞. Let us show that on the special subsets of Q the uniform convergence takes place.

Fix $\varepsilon > 0$ and define the subset $Q_\varepsilon \subset Q$ by

$$Q_\varepsilon = \left\{t \in Q : \quad \mathrm{dist}(t, \overline{\Omega} \setminus Q) \geq \varepsilon\right\}.$$

The points of the set Q_ε stay away from the non-regular points of the functions f and g by at least ε-distance. For a sufficiently small ε the set Q_ε is not empty and is compact.

Further, fix any constant $M > 0$ and cover the set Q_ε by two subsets

$$\begin{aligned} Q^M_{1,\varepsilon} &= \{t \in Q_\varepsilon : \quad |f(t)| \leq M\}, \\ Q^M_{2,\varepsilon} &= \{t \in Q_\varepsilon : \quad |f(t)| \geq M\}. \end{aligned} \tag{4}$$

It is obvious that the sets $Q^M_{i,\varepsilon}$ are compact and at least one of them is not empty.

Theorem 3. *Let the sequence of the sets $\{\omega_i\}$ satisfy the conditions of Theorem 2. Then for $i \to \infty$ the sequence of the rational splines $\sigma_i = \sigma_{1,i}/\sigma_{2,i} = \mathcal{R}_{\pi,\omega_i}[f_1, f_2]$ uniformly converges to the function f on the set $Q^M_{1,\varepsilon}$ and the sequence σ_i^{-1} uniformly converges to f^{-1} on the set $Q^M_{2,\varepsilon}$.*

Proof. Since the function f coincides with the limited rational spline $g = g_1/g_2 = \mathcal{R}_{\pi,\overline{\Omega}}[f_1, f_2]$ on the set Q and $Q^M_{i,\varepsilon} \subset Q_\varepsilon \subset Q$, we may prove the convergence to the functions g and g^{-1}.

Let the set $Q^M_{1,\varepsilon}$ be not empty. Since the space $X(\Omega)$ is continuously embedded into $C(\overline{\Omega})$ and the set $Q^M_{1,\varepsilon}$ is compact, the function $|g_2|$ attains the minimum at some point of $Q^M_{1,\varepsilon}$. Denote the value of this minimum by

θ. By definition of the set $Q^M_{1,\epsilon}$ we have $\theta > 0$. Further, applying Theorem 2 and continuous embedding of the space $X(\Omega)$ into $C(\overline{\Omega})$ we obtain that the sequence $(\sigma_{2,i})$ converges to the function g_2 with respect to the norm of the space $C(\overline{\Omega})$. Hence, there exists an index $i_1 \in N$, such that

$$\|\sigma_{2,i} - g_2\|_{C(\overline{\Omega})} \le \theta/2, \quad \forall i \ge i_1.$$

Therefore, for $i \ge i_1$

$$\min_{t \in Q^M_{1,\epsilon}} |\sigma_{2,i}(t)| \ge \theta/2.$$

At last,

$$\begin{aligned}\|\sigma_i - g\|_{C(Q^M_{1,\epsilon})} &= \left\| \frac{\sigma_{1,i}g_2 - \sigma_{2,i}g_1}{\sigma_{2,i}g_2} \right\|_{C(Q^M_{1,\epsilon})} \\ &\le \frac{2}{\theta^2} \|\sigma_{1,i}g_2 - \sigma_{2,i}g_1\|_{C(Q^M_{1,\epsilon})} \longrightarrow 0,\end{aligned}$$

for $i \to \infty$.

Similarly, the sequence (σ_i^{-1}) uniformly converges to g^{-1}. □

6. Rational D^m-splines

Let $\Omega \subset R^n$ be a bounded, simply connected domain with the Lipschitz boundary and let $X(\Omega)$ be the Sobolev space $W_2^m(\Omega)$, where $m > n/2$ is an integer. Under these assumptions the space $W_2^m(\Omega)$ is compactly embedded into $C(\overline{\Omega})$. Define the operator T by

$$Tu = D^m u \stackrel{df}{=} \left\{ \sqrt{m!/\alpha!} D^\alpha u : \ |\alpha| = m \right\}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_n)$ are multi-indices with non-negative integer components,

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad \alpha! = \alpha_1! \cdot \dots \cdot \alpha_n!,$$
$$D^\alpha u = \partial^m u / \partial^{\alpha_1} t_1 \dots \partial^{\alpha_n} t_n.$$

The image of the operator D^m coincides with the space $[L_2(\Omega)]^\kappa$, where κ is the number of various multi-indices α, such that $|\alpha| = m$. The norm of the element Tu is defined by

$$\|Tu\|_{Y(\Omega)} = \|D^m u\|_{L_2(\Omega)} \stackrel{df}{=} \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha u)^2 d\Omega \right)^{1/2}$$

The kernel of the operator D^m consists of the polinomials of degree less than m.

Definition 5. *The rational spline $\sigma = \mathcal{R}_{\pi,\omega}[f_1, f_2]$ for $X(\Omega) = W_2^m(\Omega)$ and $T = D^m$ is called the rational interpolating D^m-spline.*

In the next sections we will prove the convergence of the rational D^m-splines for any condensed (not necessarily nested) meshes, and obtain the asymptotic error estimates on the compact sets $Q_{1,\epsilon}^M$ and $Q_{2,\epsilon}^M$ which are of the same order as for the ordinary D^m-splines. Convergence of the rational splines takes place only on Q. Therefore, it is interesting to investigate the non-regular points distribution for the pairs $[f_1, f_2]$ and $[g_1, g_2]$. The question is to determine the structure of the set $\mathrm{Reg}[g_1, g_2]$ for the set $\mathrm{Reg}[f_1, f_2]$ given. In simple cases we can obtain an analitical representation of the limited rational D^m-spline.

Example 3. Let $f = \mathrm{sign}x$ and $\pi \subset [-1, 1]$ be a tie set. Let $[f_1, f_2]$ be a representation of the function f, and $\mathrm{Reg}[f_1, f_2] = [-1, 1] \setminus \{0\}$. Then the numerator g_1 and the denominator g_2 of the limited rational D^m-spline on the segment $[-1, 1]$ are described as follows

$$[g_1, g_2] = \begin{cases} [\varphi(t), \varphi(t)], & t \in [0, 1], \\ [\varphi(t), -\varphi(t)], & t \in [-1, 0], \end{cases}$$

where the function φ is the solution of the problem

$$\begin{cases} \varphi|_\pi = f_1|_\pi, \\ \varphi^{(k)}(0) = 0, \quad k = 0, \ldots, m-1, \\ \int\limits_{-1}^{1} \left(\varphi^{(m)}(t)\right)^2 dt = \min. \end{cases}$$

If we fix only one tie node, then the non-regular points of the limited pair will occupy a whole segment. For example, if $\pi = \{1\}$ then the function φ on the segment $[-1, 0]$ is equal to zero.

7. Convergence of the D^m-pairs

Definition 6. *For $X(\Omega) = W_2^m(\Omega)$ and $T = D^m$ the pair $\sigma = \mathcal{S}_{\pi,\omega}[f_1, f_2]$ is called the D^m-pair.*

We will need the following

Lemma 3 [3]. *Let X, Y be some Hilbert spaces and $T \in \mathcal{L}(X, Y)$ have a closed image and a finite-dimensional kernel $\mathcal{N}(T)$. Let*

$$\{k_p \in X^* : \quad p \in B\}$$

be a parametric family of the linear bounded functionals k_p on X, where B is a compact subset of R^m. Let B_ε be a finite ε-net in B. Further, let x_* be a fixed element of X and σ_ε be a solution of the following interpolation problem

$$\sigma_\varepsilon = \arg \min_{x \in M_{B_\varepsilon, x_*}} \|Tx\|_Y, \tag{5}$$

$$M_{B_\varepsilon, x_*} = \{x \in X : \quad k_p(x) = k_p(x_*), \ p \in B_\varepsilon\}.$$

If

$$\mathcal{N}(T) \cap \{x \in X : \ k_p(x) = 0, \ p \in B\} = \{0\} \tag{6}$$

and the mapping $p \to k_p$ is continuous on B, then for sufficiently small $\varepsilon > 0$ the problem (5) *is uniquely solvable and for $\varepsilon \to 0$ the splines σ_ε strongly converge to the solution of the problem*

$$\sigma = \arg \min_{x \in M_{B, x_*}} \|Tx\|_Y,$$

$$M_{B, x_*} = \{x \in X : \quad k_p(x) = k_p(x_*), \ p \in B\}.$$

Theorem 4. *Let the interpolated function f cannot be presented on the set* $\mathrm{Reg}[f_1, f_2]$ *as a ratio of two polynomials of degree less than m. If a finite set ω_ε form an ε-net in* $\mathrm{Reg}[f_1, f_2]$ *and the tie set π is a subset of ω_ε, then for sufficiently small $\varepsilon > 0$ the D^m-pair $[\sigma_{1,\varepsilon}, \sigma_{2,\varepsilon}] = \mathcal{S}_{\pi,\omega_\varepsilon}[f_1, f_2]$ is uniquely determined and for $\varepsilon \to 0$ the D^m-pairs $[\sigma_{1,\varepsilon}, \sigma_{2,\varepsilon}]$ strongly converge to the limited D^m-pair $\mathcal{S}_{\pi,\overline{\Omega}}[f_1, f_2]$.*

Proof. In order to use Lemma 3 for D^m-pairs we have to construct the continuous parametric family of functionals and verify the validity of (6).

Let us take the family consisting of the interpolating functionals

$$\varphi_t[x_1, x_2] = f_2(t)x_1(t) - f_1(t)x_2(t), \quad t \in \overline{\Omega}$$

and the tie functionals

$$\pi_{1,t}[x_1, x_2] = x_1(t), \ \pi_{2,t}[x_1, x_2] = x_2(t), \quad t \in \pi.$$

The condition that the interpolated function f can not be presented on $\mathrm{Reg}[f_1, f_2]$ as a ratio of two polynomials of degree less than m implies that the set $\mathrm{Reg}[f_1, f_2]$ includes an L-set and, thus, (6) is valid.

Prove that the mapping $t \to \varphi(t)$ is continuous. By definition

$$\|\varphi_t - \varphi_{t'}\| = \sup_{\|x_1\|^2_{W_2^m} + \|x_2\|^2_{W_2^m} \le 1} |(\varphi_t - \varphi_{t'})[x_1, x_2]|. \tag{7}$$

Estimate the right-hand side of (7). We have

$$|(\varphi_t-\varphi_{t'})[x_1,x_2]| \leq |f_2(t)x_1(t)-f_2(t')x_1(t')|+|f_1(t)x_2(t)-f_1(t')x_2(t')|. \quad (8)$$

Further, using the continuous embedding of the Sobolev space $W_2^m(\Omega)$ into the Hölder class H^α with $\alpha \in (0, m-n/2)$ and into $C(\overline{\Omega})$ we can write down for the first term of the right-hand side of (8) the following estimate

$$\begin{aligned}
&|f_2(t)x_1(t) - f_2(t')x_1(t')| \\
&\leq |f_2(t)| \cdot |x_1(t) - x_1(t')| + |x_1(t')| \cdot |f_2(t) - f_2(t')| \\
&\leq \|f_2\|_{C(\overline{\Omega})}|x_1(t) - x_1(t')| + \|x_1\|_{C(\overline{\Omega})}|f_2(t) - f_2(t')| \\
&\leq C_1\|t-t'\|_2^\alpha \cdot (\|f_2\|_C + \|x_1\|_C) \\
&\leq C_1C_2\|t-t'\|_2^\alpha \cdot \left(\|f_2\|_{W_2^m} + \|x_1\|_{W_2^m}\right) \\
&\leq C_1C_2\|t-t'\|_2^\alpha(\|f_2\|_{W_2^m} + 1).
\end{aligned}$$

Here $\|t-t'\|_2$ is the Euclidean distance between t and t'.
Similarly,

$$|f_1(t)x_2(t) - f_1(t')x_2(t')| \leq C_1C_2\|t-t'\|_2^\alpha(\|f_1\|_{W_2^m} + 1).$$

Finally,

$$\|\varphi_t - \varphi_{t'}\| \leq C\|t-t'\|_2^\alpha$$

and, therefore, the mapping $t \to \varphi(t)$ is continuous. □

8. Algebraic properties of the Sobolev spaces

Lemma 4. *Let $\Omega \subset R^n$ be a bounded, simply connected domain with the Lipschitz boundary and let $W_p^m(\Omega)$ be the Sobolev space, where $1 \leq p \leq \infty$ and $m > n/p$ is an integer. Let the set of multi-indices $\{\alpha_1, \ldots, \alpha_k\}$ be such that $\sum_{i=1}^k |\alpha_i| \leq m$. Then for any system of functions $\{f_i \in W_p^m(\Omega);\ i = 1, \ldots, k\}$*

$$\left\|\prod_{i=1}^k D^{\alpha_i} f_i\right\|_{L_p(\Omega)} \leq C\prod_{i=1}^k \|f_i\|_{W_p^m(\Omega)}, \quad (9)$$

where the constant C does not depend on f_i.

Proof. Define the norm of the space $W_p^m(\Omega)$ by

$$\|f\|_{W_p^m(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L_p(\Omega)}.$$

For $p = \infty$ inequality (9) is obvious with $C = 1$.

The proof of (9) for $p < \infty$ is based on the embedding theorems and the generalized Hölder inequality [6]

$$\left\| \prod_{i=1}^{k} f_i \right\|_{L_p} \leq \prod_{i=1}^{k} \|f_i\|_{L_{q_i}}, \quad \forall f_i \in L_{q_i}. \tag{10}$$

Here $1 \leq p < \infty$, $p \leq q_i \leq \infty$ and $\sum_{i=1}^{k} 1/q_i = 1/p$.

If the parameters q_i are choosen in such a way, that $\sum_{i=1}^{k} 1/q_i = 1/p$ and the sets $D^{\alpha_i} W_p^m(\Omega)$ are embedded into $L_{q_i}(\Omega)$, then

$$\left\| \prod_{i=1}^{k} D^{\alpha_i} f_i \right\|_{L_p} \leq \prod_{i=1}^{k} \|D^{\alpha_i} f_i\|_{L_{q_i}} \leq \prod_{i=1}^{k} C_i \|f_i\|_{W_p^m} = \prod_{i=1}^{k} C_i \cdot \prod_{i=1}^{k} \|f_i\|_{W_p^m},$$

which was to be proved. Here the first inequality follows from (10) and the second one from the embedding of $D^{\alpha_i} W_p^m(\Omega)$ into $L_{q_i}(\Omega)$.

Thus, we have to choose the parameters q_i. Define

$$q_i = pm/|\alpha_i|, \quad i = 1, \ldots, k-1,$$

$$q_k = pm/\left(m - \sum_{i=1}^{k-1} |\alpha_i|\right).$$

Obviously, $\sum_{i=1}^{k} 1/q_i = 1/p$. To show the embedding of $D^{\alpha_i} W_p^m(\Omega)$ into $L_{q_i}(\Omega)$ we will prove [1] that

$$m - |\alpha_i| - n/p + n/q_i \geq 0$$

and if $q_i = \infty$, then this inequality have to be strict.

If $i < k$, then

$$m - |\alpha_i| - n/p + n/q_i = m - |\alpha_i| - \frac{n}{p} \cdot \frac{m - |\alpha_i|}{m} = \frac{m - |\alpha_i|}{m} \cdot (m - n/p) \geq 0.$$

If $q_i = \infty$, then $|\alpha_i| = 0$ and this inequality is strict.

If $i = k$, then

$$m - |\alpha_k| - n/p + n/q_k \geq \sum_{i=1}^{k-1} |\alpha_i| - n/p + n/q_k$$

$$= \sum_{i=1}^{k-1} |\alpha_i| - \frac{n}{p} \cdot \frac{\sum_{i=1}^{k-1} |\alpha_i|}{m} = \frac{\sum_{i=1}^{k-1} |\alpha_i|}{m} \cdot (m - n/p) \geq 0.$$

If $q_k = \infty$, then $\sum_{i=1}^{k-1} |\alpha_i| = m$ and this inequality is strict. □

Lemma 5. *Let $\Omega \subset R^n$ be a bounded, simply connected domain with the Lipschitz boundary and let $1 \le p \le \infty$, $m > n/p$. Then the Sobolev space $W_p^m(\Omega)$ is an algebra, i.e., for any $f, g \in W_p^m(\Omega)$ the function fg belongs to $W_p^m(\Omega)$ and*

$$\|fg\|_{W_p^m(\Omega)} \le C \|f\|_{W_p^m(\Omega)} \|g\|_{W_p^m(\Omega)},$$

where the constant C does not depend on f and g.

Proof. Let the multi-index α be such that $|\alpha| \le m$. Consider the function $D^\alpha(fg)$. Using the Leibniz rule and Lemma 4 we obtain

$$\begin{aligned}\|D^\alpha(fg)\|_{L_p} &\le \sum_{\beta\le\alpha} \frac{m!}{\beta!(\alpha-\beta)!} \left\|D^\beta f D^{\alpha-\beta} g\right\|_{L_p} \\ &\le \sum_{\beta\le\alpha} \frac{m!}{\beta!(\alpha-\beta)!} C_{\alpha,\beta} \cdot \|f\|_{W_p^m} \|g\|_{W_p^m} \\ &\stackrel{df}{=} C_\alpha \|f\|_{W_p^m} \|g\|_{W_p^m}.\end{aligned}$$

Thus, the function fg belongs to $W_p^m(\Omega)$ and

$$\|fg\|_{W_p^m} = \sum_{|\alpha|\le m} \|D^\alpha(fg)\|_{L_p} \le \sum_{|\alpha|\le m} C_\alpha \cdot \|f\|_{W_p^m} \|g\|_{W_p^m}.$$

□

Lemma 6. *Let $\Omega \subset R^n$ be a bounded, simply connected domain with the Lipschitz boundary and let $1 \le p \le \infty$, $m > n/p$. Let $f, g \in W_p^m(\Omega)$ and*

$$\operatorname{ess}\inf_{t\in\overline{\Omega}} |g(t)| \ge \theta > 0.$$

Then the function f/g belongs to $W_p^m(\Omega)$ and

$$\|D^\alpha(f/g)\|_{L_p(\Omega)} \le \frac{C_\alpha}{\theta^{|\alpha|+1}} \cdot \|f\|_{W_p^m(\Omega)} \|g\|_{W_p^m(\Omega)}^{|\alpha|}, \tag{11}$$

where the multi-index α such that $|\alpha| \le m$ and the constant C_α does not depend on f and g.

Proof. Denote $k = |\alpha|$. Differentiating the fraction f/g we obtain

$$D^{\alpha}(f/g) = \sum_{|\alpha_0|+\ldots+|\alpha_k|=k} C_{\alpha_0,\ldots,\alpha_k} D^{\alpha_0} f \prod_{i=1}^{k} D^{\alpha_i} g / g^{k+1},$$

where $\alpha_0, \ldots, \alpha_k$ are multi-indices and the constants $C_{\alpha_0,\ldots,\alpha_k}$ do not depend on f and g. Further, using Lemma 4 we get

$$\begin{aligned} \left\| D^{\alpha_0} f \prod_{i=1}^{k} D^{\alpha_i} g / g^{k+1} \right\|_{L_p} &\leq \frac{1}{\theta^{k+1}} \left\| D^{\alpha_0} f \prod_{i=1}^{k} D^{\alpha_i} g \right\|_{L_p} \\ &\leq \frac{c_{\alpha_0,\ldots,\alpha_k}}{\theta^{k+1}} \|f\|_{W_p^m} \|g\|_{W_p^m}^{k}. \end{aligned}$$

Hence, we may put

$$C_{\alpha} = \sum_{|\alpha_0|+\ldots+|\alpha_k|=k} C_{\alpha_0,\ldots,\alpha_k} \cdot c_{\alpha_0,\ldots,\alpha_k}.$$

Finally, for $|\alpha| \leq m$ the function $D^{\alpha}(f/g)$ belongs to $L_p(\Omega)$ and, therefore, the function f/g belongs to $W_p^m(\Omega)$. □

9. Special covering of set

Denote by $\mathcal{B}(\omega, \delta)$ the opened δ-neighbourhood of the set $\omega \subset R^n$, i.e.,

$$\mathcal{B}(\omega, \delta) \stackrel{df}{=} \bigcup_{t \in \omega} B(t, \delta),$$

where $B(t, \delta)$ is an open ball of the radius δ centered at the point t. It is easy to verify that

(a) $\mathcal{B}(\omega, \delta) = \mathcal{B}(\overline{\omega}, \delta)$,

(b) $\mathcal{B}(\mathcal{B}(\omega, \delta_1), \delta_2) = \mathcal{B}(\omega, \delta_1 + \delta_2), \quad \forall \delta_1, \delta_2 > 0$,

(c) $\mathcal{B}(\bigcup_{a \in A} \omega_a, \delta) = \bigcup_{a \in A} \mathcal{B}(\omega_a, \delta)$, for any family $\{\omega_a \subset R^n\}_{a \in A}$.

Definition 7. *Say that a family $\{\omega_a \subset R^n\}_{a \in A}$ satisfies the L-condition, if for any two sets ω_a, ω_b, $a, b \in A$, either $\overline{\omega_a} \cap \overline{\omega_b} = \emptyset$ or the set $\omega_a \cap \omega_b$ is a domain with the Lipschitz boundary.*

It is easy to prove that a family of balls $\{B(t_a, \varepsilon_a) : a \in A\}$ satisfies the L-condition, if and only if for any two balls $B(t_a, \varepsilon_a)$, $B(t_b, \varepsilon_b)$, $a, b \in A$ the inequality

$$\operatorname{dist}(t_a, t_b) \neq \varepsilon_a + \varepsilon_b$$

is valid. Moreover, if the number of this balls is finite, then their union is a finite set of domains with the Lipschitz boundary.

Lemma 7. *Let $\Omega \subset R^n$ be a bounded domain with the Lipschitz boundary and let $\omega \subset \overline{\Omega}$. Then for any $\delta > 0$ there exists the set $\omega_\delta \subset \Omega$ consisting of a finite number of domains with the Lipschitz boundary, such that*

$$\omega \subset \overline{\omega_\delta} \quad \text{and} \quad \omega_\delta \subset \mathcal{B}(\omega, \delta). \tag{12}$$

Proof. The set ω_δ will be constructed by the following scheme. Firstly, the near-boundary points of the set ω are covered by a finite number of the small balls centered at the points of the boundary Γ of the domain Ω, such that their intersections with the domain Ω have the Lipschitz boundary. Secondly, the rest points of the set ω are covered by a finite number of the small balls centered at the points of ω. The balls radii have to be choosen in such a way that the family of balls satisfies the L-condition. This choice is possible if the balls radii may be varied in some limits.

So, let $t \in \Gamma$. Since the boundary Γ satisfies the Lipschitz condition, there exists $0 < \varepsilon_t \leq \delta$, such that for any $0 < \varepsilon \leq \varepsilon_t$ the minimal angle of the intersection of the manifold Γ and the boundary of $B(t, \varepsilon)$ is greater than zero. In other words, the pair of sets $\{\Omega, B(t, \varepsilon)\}$ satisfies the L-condition. Construct the covering

$$\{B(t, \varepsilon_t/4) : \quad t \in \Gamma\}$$

of the compact Γ and select the finite subcovering

$$\{B(t_i, \varepsilon_{t_i}/4) : \quad i = 1, \ldots, N\}.$$

Further, choose the parameters ε_i, such that $3\varepsilon_{t_i}/4 \leq \varepsilon_i \leq \varepsilon_{t_i}$ and the family

$$\{B(t_i, \varepsilon_i) : \quad i = 1, \ldots, N\}$$

satisfies the L-condition. Denote

$$\varepsilon = \min_{i=1,\ldots,N} \varepsilon_{t_i}/2.$$

Then

$$\mathcal{B}(\Gamma, \varepsilon) \subset \mathcal{B}(\bigcup_{i=1}^{N} B(t_i, \varepsilon_{t_i}/4), \varepsilon) = \bigcup_{i=1}^{N} B(t_i, \varepsilon_{t_i}/4 + \varepsilon) \subset \bigcup_{i=1}^{N} B(t_i, \varepsilon_i),$$

i.e., the balls

$$\{B(t_i, \varepsilon_i) : \quad i = 1, \ldots, N\} \tag{13}$$

cover the ε-neighbourhood of the set Γ. Finally, select from the family (13) the balls contaning the points of the set ω and denote them by $B_1, \dots, B_M$. Set

$$\omega_\Gamma \stackrel{df}{=} \bigcup_{i=1}^{M} B_i.$$

By construction, the set $\Omega \cap \omega_\Gamma$ consists at most of M domains with the Lipschitz boundary. Further, the set ω_Γ covers all points of ω which are disposed at the ε-neighbourhood of Γ, i.e.,

$$\mathcal{B}(\omega \setminus w_\Gamma, \varepsilon) \subset \Omega. \tag{14}$$

Since the radii of the balls B_i are not greater than δ, we have

$$\omega_\Gamma \subset \mathcal{B}(\omega, \delta). \tag{15}$$

Now we construct the set $\omega_{\varepsilon/2}$ including $\omega \setminus \omega_\Gamma$, such that

$$\omega_{\varepsilon/2} \subset \mathcal{B}(\omega \setminus \omega_\Gamma, \varepsilon/2). \tag{16}$$

Let us take the covering of the set $\overline{\omega \setminus \omega_\Gamma}$ with the balls of the radius $\varepsilon/4$, after that select the finite subcovering and increase the radii of each ball in the limits from $\varepsilon/4$ to $\varepsilon/2$ in such a way that the family of this balls and the balls B_i satisfy the L-condition. The union of these balls is the required set $\omega_{\varepsilon/2}$.

Define

$$\omega_\delta \stackrel{df}{=} \Omega \cap (\omega_\Gamma \cup \omega_{\varepsilon/2})$$

and prove that it is the set required. From (14) and (16) we conclude, that

$$\mathcal{B}(\omega_{\varepsilon/2}, \varepsilon/2) \subset \Omega,$$

i.e., the set $\omega_{\varepsilon/2}$ stays away from Γ by at least $\varepsilon/2$-distance. Hence,

$$\omega_\delta = (\Omega \cap \omega_\Gamma) \cup \omega_{\varepsilon/2}$$

and, by construction, the set ω_δ consists of a finite number of domains with the Lipschitz boundary. It satisfies (12) due to $\omega_{\varepsilon/2} \subset \mathcal{B}(\omega, \delta)$ and (15). □

10. Assimptotic error estimates for the rational D^m-splines

Lemma 8 [2]. *Let $\Omega \subset R^n$ be a bounded, simply connected domain with the Lipschitz boundary and let $m > n/2$ be an integer,*

$$2 \leq p \leq \infty, \quad |\alpha| - \frac{n}{p} \leq m - \frac{n}{2} \quad \text{(with strict inequality for } p = \infty). \quad (17)$$

Then, there exists constants $C, h_0 > 0$, such that for any function $u \in W_2^m(\Omega)$ with an h-set of zeroes in the domain Ω

$$\|D^\alpha u\|_{L_p(\Omega)} \leq C h^{m-|\alpha|-n/2+n/p} \|D^m u\|_{L_2(\Omega)}, \quad h \leq h_0. \quad (18)$$

The constant C depends on the parameters mentioned and does not depend on u.

Theorem 5. *Let the interpolated function f cannot be presented on the set* $\mathrm{Reg}[f_1, f_2]$ *as a ratio of two polynomials of degree less than m and let the sets $Q_{i,\varepsilon}^M$, $i = 1, 2$ be defined by* (4) *with $X(\Omega) = W_2^m(\Omega)$ and $T = D^m$. If the set ω forms an h-net in* $\mathrm{Reg}[f_1, f_2]$*, then for the rational D^m-spline $\sigma = \sigma_1/\sigma_2 = \mathcal{R}_{\pi,\omega}[f_1, f_2]$ and for the multi-index α and the parameter p satisfying* (17) *we have*

$$\|D^\alpha(\sigma - f)\|_{L_p(Q_{1,\varepsilon}^M)} + \|D^\alpha(\sigma^{-1} - f^{-1})\|_{L_p(Q_{2,\varepsilon}^M)} = o(h^{m-|\alpha|-n/2+n/p}), \quad (19)$$

for sufficiently small $h > 0$.

Proof. We will derive the estimate (19) for the first term of the left-hand side. For the second term the proof is provided in a similar way.

Since the function f coincides with the limited rational D^m-spline $g = g_1/g_2 = \mathcal{R}_{\pi,\overline{\Omega}}[f_1, f_2]$ on the set Q and $Q_{1,\varepsilon}^M \subset Q_\varepsilon \subset Q$, we can replace f in (19) by g.

1. Denote

$$\theta = \min_{t \in Q_{1,\varepsilon}^M} |g_2(t)|.$$

From definition of the set $Q_{1,\varepsilon}^M$ it follows that $\theta > 0$. Prove that there exists the parameter $0 < \delta < \varepsilon/2$, such that

$$|g_2(t)| \geq \theta/2, \quad \forall t \in \overline{\Omega \cap \mathcal{B}(Q_{1,\varepsilon}^M, \delta)}. \quad (20)$$

The function g_2 is continuous on $\overline{\Omega}$ due to the embedding of $W_2^m(\Omega)$ into $C(\overline{\Omega})$. Hence, for any $t \in \overline{\Omega}$ there exists the ball $B(t, \delta_t)$, $0 < \delta_t \leq \varepsilon/2$, such that

$$|g_2(t)| \geq \theta/2, \quad \forall t \in \overline{\Omega \cap \mathcal{B}(t, \delta_t)}.$$

Construct the covering $\{B(t,\delta_t/2):\ t\in Q^M_{1,\varepsilon}\}$ and select the finite subcovering

$$\{B(t_i,\delta_{t_i}/2):\ \ i=1,\dots,N\}.$$

Set

$$\delta=\min_{i=1,\dots,N}\delta_{t_i}/2.$$

Then,

$$\mathcal{B}(Q^M_{1,\varepsilon},\delta)\subset\mathcal{B}\left(\bigcup_{i=1}^N B(t_i,\delta_{t_i}/2),\delta\right)=\bigcup_{i=1}^N B(t_i,\delta_{t_i}/2+\delta)\subset\bigcup_{i=1}^N B(t_i,\delta_{t_i})$$

and, therefore, with such δ relation (20) is valid.

2. From Lemma 7 there exists the set ω_δ consisting of a finite number K of domains with the Lipschitz boundary such that

$$Q^M_{1,\varepsilon}\subset\overline{\omega_\delta}\quad\text{and}\quad\omega_\delta\subset\mathcal{B}(Q^M_{1,\varepsilon},\delta)\cap\Omega. \tag{21}$$

Suppose for simplicity that $K=1$.

3. By Theorem 4 and the imbedding of $W_2^m(\Omega)$ into $C(\overline{\Omega})$ we can find the sufficiently small constant h_1, such that for $h<h_1$

$$|\sigma_2(t)-g_2(t)|\le\theta/4,\quad\forall t\in\overline{\omega_\delta}.$$

Therefore, taking into account (20) and (21) we obtain that for $h\le h_1$

$$|\sigma_2(t)|\ge\theta/4,\quad\forall t\in\overline{\omega_\delta}.$$

Thus, $\sigma,g\in W_2^m(\omega_\delta)$ due to Lemma 6.

4. From the relations

$$\omega_\delta\ \subset\ \mathcal{B}(Q^M_{1,\varepsilon},\delta)\cap\Omega\ \subset\ \mathcal{B}(Q^M_{1,\varepsilon},\varepsilon)\cap\Omega\ \subset\ Q\subset\ \mathrm{Reg}[f_1,f_2]$$

and by the condition of the Theorem, the domain ω_δ has an h-net of the interpolating points. Therefore, due to Lemma 8 for $h<\min\{h_0,h_1\}$

$$\begin{aligned}\|D^\alpha(\sigma-g)\|_{L_p(Q^M_{1,\varepsilon})}&\le\|D^\alpha(\sigma-g)\|_{L_p(\omega_\delta)}\\&\le C_1h^{m-|\alpha|-n/2+n/p}\|D^m(\sigma-g)\|_{L_2(\omega_\delta)}.\end{aligned}$$

5. To complete the proof we have to show that for $h\to0$

$$\|D^m(\sigma-g)\|_{L_2(\omega_\delta)}\to0.$$

By Lemmas 5, 6

$$\begin{aligned}\|D^m(\sigma-g)\|_{L_2(\omega_\delta)} &= \left\|D^m\frac{\sigma_1g_2-\sigma_2g_1}{\sigma_2g_2}\right\|_{L_2(\omega_\delta)}\\ &\le \frac{C_2}{\theta^{m+1}}\cdot\|\sigma_1g_2-\sigma_2g_1\|_{W_2^m(\omega_\delta)}\cdot\|\sigma_2g_2\|^m_{W_2^m(\omega_\delta)}.\end{aligned}$$

The last multiplier is bounded due to Lemma 5 and the convergence of σ_2 to g_2. For the second one we have that for $h\to 0$

$$\|\sigma_1g_2-\sigma_2g_1\|_{W_2^m(\omega_\delta)} \le \|\sigma_1g_2-g_1g_2\|_{W_2^m(\omega_\delta)}+\|g_2g_1-\sigma_2g_1\|_{W_2^m(\omega_\delta)}$$

$$\le C_3\left(\|\sigma_1-g_1\|_{W_2^m(\omega_\delta)}\|g_2\|_{W_2^m(\omega_\delta)}+\|\sigma_2-g_2\|_{W_2^m(\omega_\delta)}\|g_1\|_{W_2^m(\omega_\delta)}\right)\to 0.$$

□

11. Representation of the limited D^m-pair

Theorem 6. *Let* $\mathrm{Reg}[f_1,f_2]=\overline{\Omega}$ *and let* J *be defined on* $W_2^m(\Omega)$ *by the rule*

$$Jx=[xf_1,xf_2],\quad x\in W_2^m(\Omega).$$

Then the operator J *realizes the linear continuous isomorfism from* $W_2^m(\Omega)$ *onto the space*

$$I_{\overline{\Omega}}=\left\{[x_1,x_2]\in[W_2^m(\Omega)]^2:\quad x_1f_2-x_2f_1|_{\overline{\Omega}}=0\right\}.$$

Proof. By Lemma 5

$$\begin{aligned}\|Jx\|_{[W_2^m(\Omega)]^2} &\overset{df}{=} \left(\|xf_1\|^2_{W_2^m(\Omega)}+\|xf_2\|^2_{W_2^m(\Omega)}\right)^{1/2}\\ &\le C\|x\|_{W_2^m(\Omega)}\cdot\left(\|f_1\|^2_{W_2^m(\Omega)}+\|f_2\|^2_{W_2^m(\Omega)}\right)^{1/2}.\end{aligned}$$

Therefore, the operator J is bounded.

Obviously, the image $\mathcal{R}(J)$ of the operator J is included to $I_{\overline{\Omega}}$. To complete the proof we have only to show that

$$\mathcal{N}(J)=\{0\} \tag{22}$$

and

$$\mathcal{R}(J)=I_{\overline{\Omega}}. \tag{23}$$

1. Let $u\in\mathcal{N}(J)$. By the imbedding of $W_2^m(\Omega)$ into $C(\overline{\Omega})$ we have that for any $t\in\overline{\Omega}$

$$x(t)f_1(t)=0\quad\text{and}\quad x(t)f_2(t)=0.$$

Since the pair $[f_1, f_2]$ is regular on $\overline{\Omega}$, we have that

$$f_1(t) \neq 0 \quad \text{or} \quad f_2(t) \neq 0. \tag{24}$$

Therefore, $x(t) = 0$ and (22) is valid.

2. In order to show (23) we will construct for any pair $[x_1, x_2] \in I_{\overline{\Omega}}$ the element $x \in W_2^m(\Omega)$, such that $Jx = [x_1, x_2]$.

Define the function x by

$$x(t) = \begin{cases} x_1(t)/f_1(t), & \text{if } f_1(t) \neq 0, \\ x_2(t)/f_2(t), & \text{if } f_2(t) \neq 0. \end{cases}$$

Since (24) is valid and $[x_1, x_2] \in I_{\overline{\Omega}}$, this definition is correct. By the imbedding of $W_2^m(\Omega)$ into $C(\overline{\Omega})$, for any point $t \in \overline{\Omega}$ there exists a ball $B(t, \delta_t)$, such that

$$f_1|_{\overline{\Omega \cap B(t,\delta_t)}} \neq 0 \quad \text{or} \quad f_2|_{\overline{\Omega \cap B(t,\delta_t)}} \neq 0.$$

Further, by Lemma 7 there exists a domain $\omega_t \subset \Omega$ with the Lipschitz boundary, such that

$$t \in \overline{\omega_t} \quad \text{and} \quad \omega_t \subset B(t, \delta_t).$$

Taking into account Lemma 6 we derive that $x \in W_2^m(\omega_t)$.

Finally, selecting a finite subcovering from the covering

$$\{\omega_t : \quad t \in \overline{\Omega}\}$$

we obtain that

$$\|x\|_{W_2^m(\Omega)} \leq \sum_{i=1}^{N} \|x\|_{W_2^m(\omega_{t_i})} < \infty,$$

i.e., the function x belongs to $W_2^m(\Omega)$. □

Corollary. *Let* $\mathrm{Reg}[f_1, f_2] = \overline{\Omega}$. *Then the limited D^m-pair is represented by*

$$\mathcal{S}_{\pi,\overline{\Omega}}[f_1, f_2] = [uf_1, uf_2],$$

where $u \in W_2^m(\Omega)$ is the solution of the problem

$$u = \arg\min_{x \in U_\pi} \|D^m(f_1 x)\|^2_{W_2^m(\Omega)} + \|D^m(f_2 x)\|^2_{W_2^m(\Omega)}, \tag{25}$$

$$U_\pi = \{x \in W_2^m(\Omega) : \quad x|_\pi = 1\}.$$

Thus, the set of non-regular points of the pair $[g_1, g_2]$ coincides with the set of points, where the function u vanishes. For one-dimensional case it was shown [8] that for $f_1, f_2 \in C^{2m}[a, b]$ and for the finite tie set π the solution u of the problem (25) vanishes on a finite subset of $[a, b]$ and zero multiplicity of u at every point of this subset is less than m.

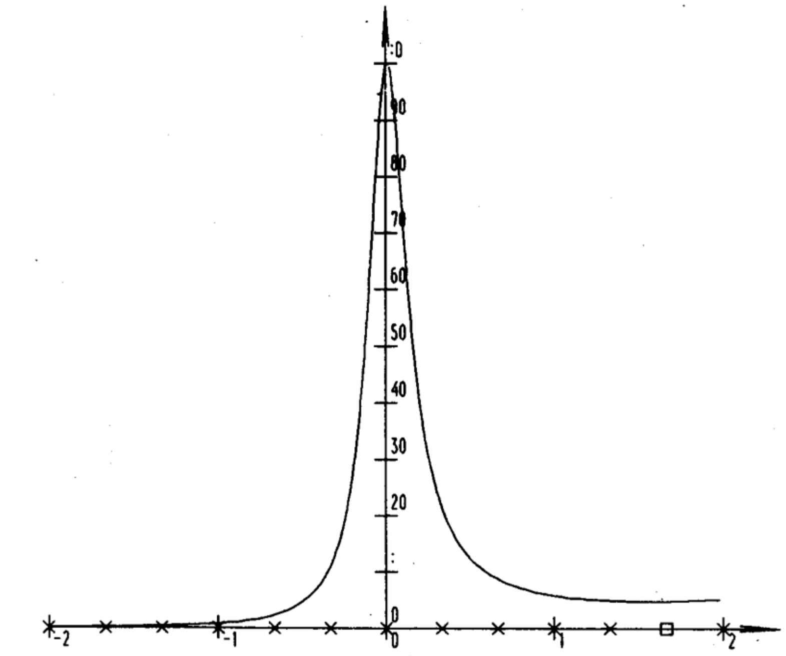

Figure 1

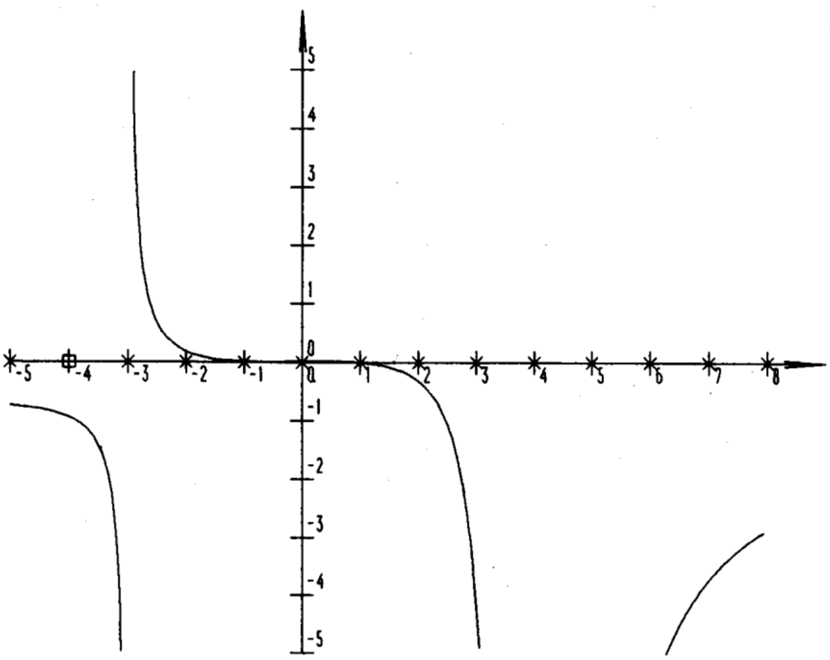

Figure 2

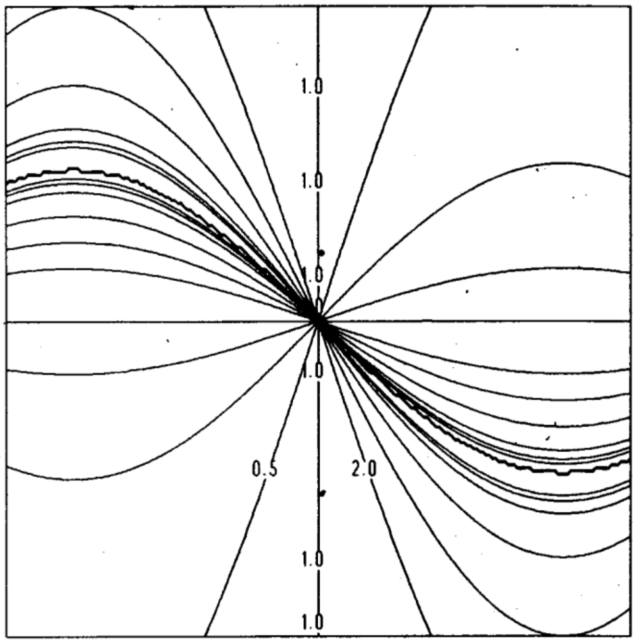

Figure 3

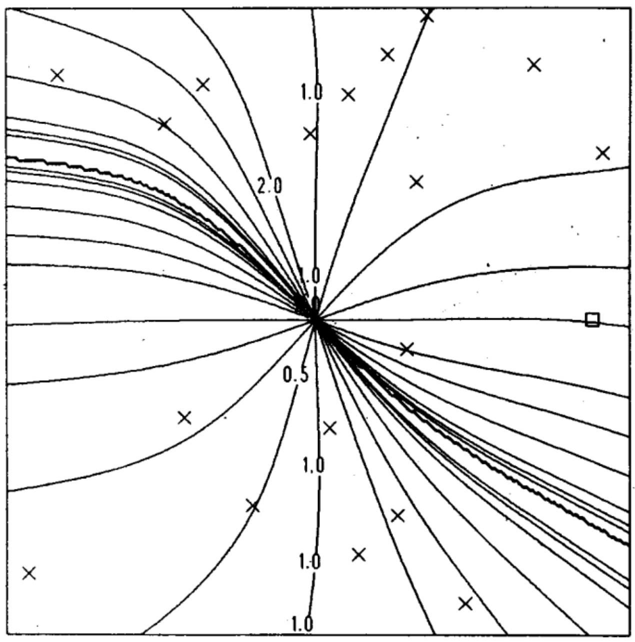

Figure 4

12. Numerical experiments

Figures 1, 2 show the graphics of the rational D^2-splines which interpolate the functions

$$\frac{e^x}{1.01 - \cos x} \quad \text{and} \quad \frac{1 - x^5}{(x+3)(x^2+1)(x-4)^2}$$

respectively. The graphics of the interpolated functions practically coincide with these ones and, therefore, they are not presented.

Figure 3 shows the isolines of the function $(y - \sin x)/(y + \sin x)$ on the square $[-2,2] \times [-2,2]$ and Figure 4 shows the rational D^2-spline for this function constructed on a scattered mesh with 20 nodes. The mesh nodes are marked by the "crosses" and the tie node is marked by the "box".

The computation was hold on the LIDA-3 [9] software library subpackages RATIO and GRATIO designed by the author.

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