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Mixed spline approximation*

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The given paper studies the mixed spline approximation problem

$$A_1\hat{\sigma}_\alpha = z_1,$$
$$\alpha\|T\hat{\sigma}_\alpha\|^2 + \|A_2\hat{\sigma}_\alpha - z_2\|^2 \to \min.$$

Here the operator A_1 gives the interpolation conditions, smoothing is carried out using the operator A_2, and T is the energy operator. The necessary and sufficient conditions of the unique solvability for this problem are obtained. Incorrectness of D^m-spline approximation in $W_2^m(R^n)$ is proved.

1. Introduction

The mixed spline approximation problem combines the peculiarities of the problems of spline interpolation and smoothing that were studied by many mathematicians beginning from the works of Atteia [1, 2]. The monograph of Loran [3] should be mentioned specially. It gives the conditions of existence and uniqueness of the interpolating and smoothing splines in the general form.

The mixed formulation was first proposed in [4], although in practice the mixed problems were considered much earlier (however, without the use of the variational formulation). The main purpose of the given paper is to obtain precise conditions of unique solvability of the mixed problem. In particular, it is shown that if the kernel of the operator T is finite-dimensional, then the closeness of the image of the operator A for the unique solvability of the problems of spline approximation *is not required* (Theorem 5.4). At the same time, the closeness of the image of the operator T *is necessary* for the unique solvability *on the class* of operators A (Theorem 7.1).

The method used by the author to obtain the conditions of unique solvability of the mixed problem is in the replacement of this problem by

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the equivalent problem of the best approximation in the convex set (affine subspace), the unique solvability of which is well-known. In this case some norm is being constructed with the help of the operators of the mixed problem, and the requirement of the unique solvability turns out to be directly related to the equivalence conditions of this norm and the original norm of the space X.

2. Basic definitions

2.1. Let X be the real Banach space with the norm $\|\cdot\|_X$. If, moreover, X is the Hilbert space, then the scalar product in it will be denoted by $(\cdot,\cdot)_X$. If it is clear from the context what norm or scalar product is meant, then the subscript X will be omitted. The zero element of the space X will be denoted by the symbol 0.

The adjoint space of the linear continuous functionals over X is denoted by X^*, and the Banach space of the linear limited operators, acting from X into the Banach space Y is denoted by $L(X,Y)$.

2.2. Let $A \in L(X,Y)$ be some linear operator. $N(A)$ and $R(A)$ denote the kernel and the image of the operator A:

$$N(A) = \{x \in X : \ Ax = 0\}, \qquad R(A) = AX.$$

It is evident that $N(A)$ is the closed subspace.

Let $M \subset Y$ be some set. Then its preimage will be denoted in the following way:

$$A^{-1}(M) = \{x \in X : \ Ax \in M\}.$$

2.3. The norm generated by an operator. Consider the functional

$$\|x\|_A \stackrel{df}{=} \|Ax\|_Y,$$

which is, evidently, *a semi-norm* in X. If, moreover, $N(A) = \{0\}$, then the functional $\|\cdot\|_A$ will be a norm in X.

If X and Y are the Hilbert spaces, then the bilinear function

$$(u,v)_A \stackrel{df}{=} (Au, Av)_Y$$

will give *a scalar semi-product* in X, which will be a scalar product if $N(A) = \{0\}$.

2.4. Direct sum of the Banach spaces. Let X and Y be the Banach spaces. Consider the linear space

$$Z = \{x \oplus y : \; x \in X, \; y \in Y\}$$

with the operations of summation and multiplication:

$$\begin{aligned} x_1 \oplus y_1 + x_2 \oplus y_2 &= (x_1 + x_2) \oplus (y_1 + y_2), \\ \lambda(x \oplus y) &= \lambda x \oplus \lambda y. \end{aligned}$$

This space becomes the Banach space if we introduce a norm on it, for example, in the following way:

$$\|x \oplus y\|_Z = \left(\|x\|_X^p + \|y\|_Y^p\right)^{1/p}, \tag{2.1}$$

where $1 \le p \le \infty$ (at $p = \infty$, the sum of norms in (2.1) is replaced by the maximum). Let us denote this space by $X \oplus_p Y$ and call it *the direct sum of the Banach spaces X and Y with the p-norm.* We shall more often use the 2-norm, and therefore we shall write $X \oplus Y$ instead of $X \oplus_2 Y$ for simplicity.

It is evident that all the p-norms given by (2.1) are equivalent.

2.5. Direct sum of operators. Let X, Y, Z be the Banach spaces, and $A \in L(X,Y)$, $B \in L(X,Z)$ be some operators. Construct the operator $A \oplus_p B$ by the rule

$$A \oplus_p B\, x = Ax \oplus Bx.$$

Here the subscript p indicates that the operator acts into the Banach space $Y \oplus_p Z$.

It is evident that the operator $A \oplus_p B$ that will be called *the direct sum of operators*, is continuous. We shall write simply $A \oplus B$ instead of $A \oplus_2 B$ at $p = 2$.

2.6. Problems of spline approximation. Let X, Y, Z be the real Hilbert spaces and the operators $T \in L(X,Y)$ and $A \in L(X,Z)$ be given.

Let $z \in R(A)$. Then the solution to the problem

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|^2 \tag{2.2}$$

is called *the interpolating spline.*

Let $\alpha > 0$ be some parameter and $z \in Z$. Then the solution to the problem

$$\sigma_\alpha = \arg\min_{x \in X} \alpha\|Tx\|^2 + \|Ax - z\|^2 \tag{2.3}$$

is called *the smoothing spline.*

Let $Z = Z_1 \oplus Z_2$, $A = A_1 \oplus A_2$, where $A_1 \in L(X, Z_1)$ and $A_2 \in L(X, Z_2)$ are some operators. Let, also, $z_1 \in R(A_1)$, $z_2 \in Z_2$ be some elements and $\alpha > 0$ be a parameter. Then the solution to the problem

$$\hat{\sigma}_\alpha = \arg \min_{x \in A_1^{-1}(z_1)} \alpha \|Tx\|^2 + \|A_2 x - z_2\|^2 \tag{2.4}$$

is called *the mixed spline.*

The vector of the initial data z in the corresponding problem of spline approximation will be called *an acceptable vector* if it satisfies the conditions imposed in the definition of the problem. So, the set of acceptable vectors for the problem of spline interpolation is $R(A)$, this set for the smoothing problem is Z, and for the mixed problem it is $R(A_1) \oplus Z_2$.

3. The sufficient criterion for the unique solvability of the mixed problem

It should be noted that the interpolating and smoothing splines are special cases of the mixed spline: we obtain the interpolating spline if we set $Z_2 = \{0\}$ in the problem (2.4), and we get the smoothing spline if $Z_1 = \{0\}$. Therefore, it is sufficient to determine the unique solvability for the mixed problem (2.4).

3.1. Lemma. *If there exists the operator $B \in L(X, W)$ acting into some Hilbert space W such that*

(a) $N(A_1) \subset N(B)$,

(b) *the norms* $\|x\|_{T\oplus B\oplus A_2} \overset{df}{=} \left(\|Tx\|^2 + \|Bx\|^2 + \|A_2 x\|^2\right)^{1/2}$ *and* $\|x\|_X$ *are equivalent,*

then the solution to the mixed problem (2.4) *exists and is unique for any acceptable initial data.*

Proof. It is evident that $A_1^{-1}(z_1) = x_* + N(A_1)$, where $x_* \in A_1^{-1}(z_1)$ is an arbitrary element. Then, taking into account (a), we have

$$\forall x \in A_1^{-1}(z_1) \quad Bx = Bx_* + B(x - x_*) = Bx_*.$$

Consequently, the functional of the problem (2.4) can be corrected, with the addition of the value $\|Bx\|^2$ to it, which is equal to the constant on $A_1^{-1}(z_1)$:

$$\Phi_\alpha(x) \overset{df}{=} \alpha\|Tx\|^2 + \|Bx\|^2 + \|A_2 x - z_2\|^2 = \|x\|^2_{\sqrt{\alpha}T\oplus B\oplus A_2} - 2(z_2, A_2 x) + C_1.$$

It follows from the condition (b) that the norm $\|\cdot\|_{\sqrt{\alpha}T\oplus B\oplus A_2}$ is equivalent to the norm $\|\cdot\|_X$. Therefore, by Riesz's theorem, the functional $(z_2, A_2 x)$ can be represented in the form

$$(z_2, A_2 x) = (f, x)_{\sqrt{\alpha}T \oplus B \oplus A_2},$$

where $f \in X$ is some element. Then

$$\Phi_\alpha(x) = \|x - f\|^2_{\sqrt{\alpha}T \oplus B \oplus A_2} + C_2,$$

and the problem (2.4) reduces to the equivalent problem

$$\hat{\sigma}_\alpha = \arg \min_{x \in A_1^{-1}(z_1)} \|x - f\|^2_{\sqrt{\alpha}T \oplus B \oplus A_2},$$

which is in the minimization of the distance from the closed affine subspace $A_1^{-1}(z_1)$ to the element f. It is well-known (see, for example, [3]) that this problem has the unique solution. □

Let us next obtain the conditions providing the equivalence of the norm generated by the operator $T \oplus B \oplus A_2$, and the original norm of the space X. We shall study this question in a more general case of the Banach space.

4. Equivalent norms in the Banach spaces

Let us give some known theorems of functional analysis (see, for example, [3, 5, 6]) which we shall need later.

4.1. Theorem. *If the Banach space X is reflexive ($X^{**} = X$), then, from any sequence $x_n \in X$ bounded over the norm, a weak convergent subsequence can be selected.*

4.2. Theorem (on open mapping). *Let the operator $A \in L(X,Y)$ act on the space Y. Then the image of any open set in X by the mapping A is open in Y.*

Corollary. *Let $A \in L(X,Y)$ and $M \subset X$ be some set. If*

(a) *AM is closed in Y,*

then

(b) *$N(A) + M$ is closed in X.*

If, moreover, $R(A)$ is closed in Y, then (b) *implies* (a).

Proof. (a) ⇒ (b). It is evident that $N(A) + M = A^{-1}(AM)$ and the set $N(A) + M$ is closed as the preimage of the closed set AM.

(b) $\Rightarrow$ (a). As the subspace $R(A)$ is closed in Y, it can be considered as the Banach space with the topology induced from Y. Then the mapping $A : X \to R(A)$ is open and transforms the open set $U \stackrel{df}{=} X \setminus (N(A) + M)$ to the open set AU. As $N(A) + M = A^{-1}(AM)$, then $AU = R(A) \setminus AM$. Consequently, the set AM is closed as the complement to the open set AU.

□

4.3. Theorem. *Let the operator $A \in L(X,Y)$ act on the space Y. Then there exists the constant $C > 0$ such that, for any point $y \in Y$, the point $x \in A^{-1}(y)$ will be found such that $\|x\| \leq C\|y\|$.*

Remark. If operator A acts not on the whole space Y, then this theorem is valid at the condition that $R(A)$ is closed in Y and the point y is taken from $R(A)$.

4.4. Theorem (the Banach theorem on the continuity of the inverse operator). *If the operator $A \in L(X,Y)$ acts on the space Y and $N(A) = \{0\}$, then operator A^{-1} is continuous.*

Corollary (inversion of Theorem 4.3). *Let X be reflexive, $A \in L(X,Y)$ and there exist the constant $C > 0$ such that, for any point $y \in R(A)$, the point $x \in A^{-1}(y)$ will be found such that $\|x\| \leq C\|y\|$. Then $R(A)$ is closed in Y.*

Proof. Let us consider the fundamental sequence $y_n \in R(A)$ and construct the sequence x_n so that $Ax_n = y_n$ and

$$\|x_n\| \leq C\|y_n\|. \tag{4.1}$$

As the sequence y_n converges, then, due to (4.1), x_n is the bounded sequence. Consequently, by Theorem 4.1, the subsequence $x_{n'}$ can be chosen from it that weakly converges to some point $x_* \in X$. Then $y_{n'} = Ax_{n'} \xrightarrow{c} Ax_*$. As the sequence y_n converges strongly, its limit coincides with Ax_*.

□

4.5. Lemma. *Let $A \in L(X,Y)$. Then the following statements are equivalent:*

(a) *the norms $\|\cdot\|_A$ and $\|\cdot\|_X$ are equivalent;*

(b) *$N(A) = \{0\}$ and $R(A)$ is closed in Y.*

Proof. (a) $\Rightarrow$ (b). As the functional $\|\cdot\|_A$ gives the norm in X, then $N(A) = \{0\}$. Consequently, operator A is the isomorphism between X

and $R(A)$, and the operator $A^{-1} : R(A) \to X$ exists. It follows from the equivalence of the norms $\|\cdot\|_A$ and $\|\cdot\|_X$ that the operator A^{-1} is bounded and, consequently, $R(A)$ is closed as the preimage of the closed set X.

(b) $\Rightarrow$ (a). As the subspace $R(A)$ is closed in Y, then it can be considered as the Banach space. Then, in accordance with Theorem 4.4, the operator $A^{-1} : R(A) \to X$ is continuous, i.e.,

$$\forall y \in R(A) \qquad \|A^{-1}y\|_X \le \|A^{-1}\| \cdot \|y\|_Y .$$

Denoting $x = A^{-1}y$ and taking into account that the operator A^{-1} acts on the space X, we obtain

$$\forall x \in X \qquad \|x\|_X \le \|A^{-1}\| \cdot \|Ax\|_Y = \|A^{-1}\| \cdot \|x\|_A .$$

On the other hand,

$$\|x\|_A = \|AX\|_Y \le \|A\| \cdot \|x\|_X .$$

Consequently, the norms $\|\cdot\|_A$ and $\|\cdot\|_X$ are equivalent. □

4.6. Theorem. *Let X, Y, Z be the Banach spaces, and $A \in L(X,Y)$, $B \in L(X,Z)$ be some operators. Then the following statements are valid at any $1 \le p \le \infty$:*

(a) *if $R(A \oplus_p B)$ is closed, then $N(A) + N(B)$ is closed;*

(b) *if $R(A)$ and $BN(A)$ are closed and X is reflexive, then $R(A \oplus_p B)$ is closed;*

(c) *if $R(A)$, $R(B)$ and $N(A) + N(B)$ are closed and X is reflexive, then $R(A \oplus_p B)$ is closed;*

(d) *if $R(A)$ is closed and $N(A)$ is finite-dimensional, then $R(A \oplus_p B)$ is closed.*

Proof. (a) As $N(A \oplus_p B) \subset N(A)$, then

$$N(A \oplus_p B) + N(A) = N(A),$$

i.e., the subspace $N(A \oplus_p B) + N(A)$ is closed. It follows from the corollary of Theorem 4.2 that the subspace $(A \oplus_p B)N(A) = \{0\} \oplus BN(A)$ is closed. Hence, the set $BN(A)$ is closed and, consequently, its preimage $B^{-1}(BN(A)) = N(A) + N(B)$ is closed.

(b) It follows from the corollary of Theorem 4.4 that, for the proof of the closeness of the subspace $R(A \oplus_p B)$, it will suffice to find, for any point $y \oplus z \in R(A \oplus_p B)$, the point $x \in (A \oplus_p B)^{-1}(y \oplus z)$ such that

$$\|x\|_X \le C\|y \oplus z\|_{Y\oplus_p Z}$$

with the constant $C > 0$ which does not depend on $y \oplus z$. As all the p-norms in $Y \oplus Z$ are equivalent, it can be considered that $p = 1$.

It follows from the closeness of $R(A)$ and Theorem 4.3 that, for any $y \in R(A)$, the point $u \in A^{-1}(y)$ will be found such that

$$\|u\| \le C_1\|y\| \tag{4.2}$$

with the constant $C_1 > 0$ which does not depend on y.

If $x \in (A\oplus_p B)^{-1}(y\oplus z)$ is any point, then $x - u \in N(A)$. Consequently, the point x can be sought in the form $u + v$, where the point $v \in N(A)$ satisfies the condition $Bu + Bv = z$.

As the subspace $BN(A)$ is closed, then, if we consider the restriction of operator B on $N(A)$ and apply Theorem 4.3, the point $v \in N(A)$ will be found such that $Bv = z - Bu$ and

$$\|v\| \le C_2\|z - Bu\|. \tag{4.3}$$

It is obvious that $x \overset{df}{=} u + v \in (A \oplus_p B)^{-1}(y \oplus z)$. Then, taking into account the inequalities (4.2) and (4.3), we obtain

$$\begin{aligned} \|x\| &\le \|u\| + \|v\| \le \|u\| + C_2\|z - Bu\| \\ &\le (1 + C_2\|B\|)\|u\| + C_2\|z\| \le (1 + C_2\|B\|)C_1\|y\| + C_2\|z\| \\ &\le \max\{(1 + C_2\|B\|)C_1, C_2\} \cdot \|y \oplus z\|_{X\oplus_1 Y}. \end{aligned}$$

(c) Applying the corollary of Theorem 4.2, we conclude from the closeness of $R(B)$ and $N(A) + N(B)$ that $BN(A)$ is closed. Then we use the statement (b).

(d) The given statement is proved using the explicit construction of the continuous operator that is "pseudo-inverse" for $A \oplus_p B$.

As the subspace $N(A)$ is finite-dimensional, then it is *complementable* [6, Lemma 4.21], i.e., the closed subspace $U \subset X$ will be found such that $N(A) \cap U = \{0\}$ and $N(A) + U = X$. It is evident that restriction $\hat{A}$ of operator A on U is one-to-one correspondence between U and $R(A)$. As $R(A)$ is closed, then, by Theorem 4.4, operator $\hat{A}^{-1}$ is continuous.

Furthermore, due to the finite dimensionality of $N(A)$, the set $BN(A)$ is closed and $N(A) \cap N(B)$ is finite-dimensional. Consequently, the closed subspace $V \subset N(A)$ will be found which is the complement to $N(A)\cap N(B)$, and the continuous operator $\hat{B}^{-1}$ can be constructed which is the inverse operator relative to the restriction of operator B on V.

Assume that $u = \hat{A}^{-1}y$ and $v = \hat{B}^{-1}(z - Bu)$. Then

$$x \stackrel{df}{=} u + v \in (A \oplus_p B)^{-1}(y \oplus z),$$

and, given in this way, the linear mapping C from $R(A \oplus_p B)$ on $U + V$ is continuous and has one-to-one correspondence. As V is finite-dimensional, then the subspace $U + V$ is closed and, consequently, $R(A \oplus_p B)$ is closed as the preimage of the closed set by the mapping C. □

4.7. Theorem. *Let $1 \le p \le \infty$. The following statements are valid:*

(a) *if the norms $\|\cdot\|_{A\oplus_p B}$ and $\|\cdot\|_X$ are equivalent, then $N(A) \cap N(B) = \{0\}$ and $N(A) + N(B)$ is closed;*

(b) *if, on the subspace $N(A)$, the norms $\|\cdot\|_B$ and $\|\cdot\|_X$ are equivalent, $R(A)$ is closed and X is reflexive, then the norms $\|\cdot\|_{A\oplus_p B}$ and $\|\cdot\|_X$ are equivalent;*

(c) *if $R(A)$, $R(B)$ and $N(A) + N(B)$ are closed, $N(A) \cap N(B) = \{0\}$ and X is reflexive, then the norms $\|\cdot\|_{A\oplus_p B}$ and $\|\cdot\|_X$ are equivalent;*

(d) *if $R(A)$ is closed, $N(A)$ is finite-dimensional and $N(A) \cap N(B) = \{0\}$, then the norms $\|\cdot\|_{A\oplus_p B}$ and $\|\cdot\|_X$ are equivalent.*

Proof. All these statements easily follow from Lemma 4.5, the identity $N(A \oplus_p B) = N(A) \cap N(B)$ and the corresponding statements of Theorem 4.6. □

5. The necessary and sufficient conditions for the unique solvability of the mixed problem

5.1. Lemma. *Conditions of Lemma 3.1 are equivalent to the following:*

> *On the subspace $N(A_1)$, the norms $\|\cdot\|_X$ and $\|\cdot\|_{T\oplus A_2}$ are equivalent.*

Proof. The norm $\|\cdot\|_{T\oplus B\oplus A_2}$ on the subspace $N(A_1)$ coincides with $\|\cdot\|_{T\oplus A_2}$. Therefore, the equivalence of the norms $\|\cdot\|_{T\oplus A_2}$ and $\|\cdot\|_X$ on $N(A_1)$ is evident. It remains to show the reverse, i.e., to construct some operator B which satisfies the conditions of Lemma 3.1.

As B, let us take an arbitrary operator with the closed image and the kernel $N(A_1)$ (for example, an orthoprojector on $N(A_1)^\perp$) and, applying the statement (b) of Theorem 4.7 to the operators B and $T \oplus A_2$, we obtain the required condition (reflexivity condition of space X is fulfilled automatically, because it is the Hilbert space). □

5.2. Lemma. *The condition*

(a) $N(T) \cap N(A_1) \cap N(A_2) = \{0\}$

is required for the uniqueness of the solution to the problem (2.4). *If, moreover,* $R(T)$ *is closed, then the condition*

(b) $(T \oplus A_2)N(A_1)$ *is closed in* $Y \oplus Z_2$

is required for the existence of the solution to the problem (2.4).

Remark. In accordance with Lemma 4.5, the conditions (a), (b) of the given lemma provide the equivalence of the norms $\|\cdot\|_{T\oplus A_2}$ and $\|\cdot\|_X$ on $N(A_1)$.

Proof. (a) If this condition is not fulfilled, then the non-zero element $u \in N(T) \cap N(A_1) \cap N(A_2)$ will be found. Then, if the solution $\hat{\sigma}_\alpha$ of the problem (2.4) exists at some initial data, $\hat{\sigma}_\alpha + u$ will also be its solution.

(b) Let us reduce the problem (2.4) to the problem on the subspace $N(A_1)$ which is equivalent to (2.4). To do this, let us make the substitution $x = x_* + u$, where $x_* \in A_1^{-1}(z_1)$ is an arbitrary element. As a result we obtain the problem

$$\hat{u} = \arg \min_{u \in N(A_1)} \alpha\|Tu - f\|^2 + \|A_2 u - g\|^2, \tag{5.1}$$

where

$$\hat{u} = \hat{\sigma}_\alpha - x_*, \quad f = -Tx_*, \quad g = z_2 - A_2 x_*.$$

As the problem (2.4), in accordance with the condition of this lemma, has the solution at any acceptable initial data, the element x_*, at any z_1, "passes" through the whole space X. Consequently, the element f passes the whole set $R(T)$, and g passes the whole space Z_2 (z_2 changes independently of z_1). Thus, the problem (5.1) must have the solution at any $f \oplus g \in R(T) \oplus Z_2$.

Assume that the set $(T \oplus A_2)N(A_1)$ is not closed, i.e., the element

$$f_* \oplus g_* \in \overline{(T \oplus A_2)N(A_1)}$$

exists which is not contained in $(T \oplus A_2)N(A_1)$. As $R(T)$ is closed, then the closure of the set $(T \oplus A_2)N(A_1)$ is contained in $R(T) \oplus Z_2$. Consequently, the problem (5.1) must have the solution at $f \oplus g = f_* \oplus g_*$. However, this is not fulfilled, as $f_* \oplus g_* \notin (T \oplus A_2)N(A_1)$ and, at the same time, the element $u \in N(A_1)$ will be found such that the element $Tu \oplus A_2 u$ is as near to $f_* \oplus g_*$ as possible. □

After combining Lemmas 3.1, 5.1 and 5.2, we obtain the following theorem:

5.3. Theorem. *Let the subspace $R(T)$ be closed in Y. Then, for the unique solvability of the mixed problem (2.4) at any acceptable initial data, it is necessary and sufficient for the norm $\|\cdot\|_X$ on the subspace $N(A_1)$ to be equivalent to the norm $\|\cdot\|_{T\oplus A_2}$.*

If $N(T)$ is finite-dimensional, then the conditions of the unique solvability of the problem (2.4) are somewhat simplified.

5.4. Theorem. *Let $R(T)$ be closed and $N(T)$ be finite-dimensional. Then, for the unique solvability of the problem (2.4) at any acceptable initial data, it is necessary and sufficient that*

$$N(T) \cap N(A) = \{0\}. \tag{5.2}$$

The necessity of the condition (5.2) follows from Lemma 5.2, and its sufficiency follows from the statement (d) of Theorem 4.7 applied to the operators T and A, and also from Lemma 3.1 at $B = A_1$.

6. The case of infinite dimensional kernel of operator T

If $N(T)$ is infinite dimensional, then the condition of the equivalence of the norms in Theorem 5.3 can be replaced by other conditions that can be easily verified. To do this we shall need the concept of linear independence of the operators A_1 and A_2.

6.1. Definition. The family of operators $A_i \in L(X, Z_i)$, $i = 1, \dots, N$, is *linearly independent* if

$$R(A_1 \oplus \dots \oplus A_N) = R(A_1) \oplus \dots \oplus R(A_N).$$

The criterion of linear independence for a pair of operators can be formulated in different ways.

6.2. Lemma. *The following statements are equivalent:*

(a) *operators A_1 and A_2 are linearly independent;*

(b) $A_2 N(A_1) = R(A_2)$ (*or* $A_1 N(A_2) = R(A_1)$);

(c) $N(A_1) + N(A_2) = X$.

Proof. (a) $\Rightarrow$ (b). Let us take the element $0 \oplus z_2 \in R(A_1) \oplus R(A_2)$. As, by the condition, it belongs to $R(A_1 \oplus A_2)$, then the element $x \in X$ will be found such that

$$A_1 x = 0, \qquad A_2 x = z_2. \tag{6.1}$$

It follows from the first equation in (6.1) that $x \in N(A_1)$. In other words, the conditions (6.1) imply that, for any $z_2 \in R(A_2)$, $x \in N(A_1)$ will be found such that $A_2 x = z_2$. Hence, $A_2 N(A_1) = R(A_2)$. (Similarly, $A_1 N(A_2) = R(A_1)$.)

(b) $\Rightarrow$ (c). Let us consider the preimages of the sets in the statement (b):

$$X = A_2^{-1}(R(A_2)) = A_2^{-1}(A_2 N(A_1)) = N(A_1) + N(A_2).$$

(c) $\Rightarrow$ (a). Let $z_i \in R(A_i)$, $i = 1, 2$, be arbitrary elements. Let us choose some elements $x_i \in A_i^{-1}(z_i)$ and represent them, in accordance with (c), in the form $x_i = u_i + v_i$, where $u_i \in N(A_1)$ and $v_i \in N(A_2)$. Then

$$A_1 v_1 = A_1 x_1 = z_1, \quad A_2 u_2 = A_2 x_2 = z_2.$$

Assuming that $x = v_1 + u_2$, we obtain

$$Ax = A_1 x \oplus A_2 x = A_1 v_1 \oplus A_2 u_2 = z_1 \oplus z_2,$$

i.e., $z_1 \oplus z_2 \in R(A_1 \oplus A_2)$. □

6.3. Theorem. *Let $R(T)$ and $R(A_2)$ be closed and operators A_1 and A_2 be linearly independent. Then the conditions*

$$N(T) \cap N(A) = \{0\} \textit{ and } N(T) + N(A) \textit{ is closed}$$

are the necessary and sufficient conditions for the unique solvability of the problem (2.4) *at any acceptable initial data.*

Proof. *Necessity.* By Theorem 5.3, the norms $\|\cdot\|_{T\oplus A_2}$ and $\|\cdot\|_X$ are equivalent on $N(A_1)$. Further, by the statement (b) of Theorem 4.7, the norm $\|\cdot\|_{B\oplus(T\oplus A_2)}$ is equivalent to the norm $\|\cdot\|_X$ on X for any operator B with the closed image and the kernel $N(A_1)$. Finally, from the statement (a) of the same theorem we have that

$$N(T) \cap N(B \oplus A_2) = \{0\}, \quad N(T) + N(B \oplus A_2) \text{ is closed.}$$

Taking into account that $N(B) = N(A_1)$, we obtain what was required.

Sufficiency. Let $B \in L(X, W)$ be some operator with the closed image and the kernel $N(A_1)$. As operators A_1 and A_2 are linearly independent, then B and A_2 are also linearly independent (this follows from the statement (c) of Lemma 6.2). Taking into account the closeness of the images of operators B and A_2, we obtain that the subspace $R(B \oplus A_2)$ is closed. From the statement (c) of Theorem 4.7, the norms $\|\cdot\|_{T\oplus(B\oplus A_2)}$ and $\|\cdot\|_X$ are equivalent on X. Finally, applying Lemma 3.1, we get the unique solvability of the problem (2.4). □

Remark. In the special cases for the problems of spline interpolation and spline smoothing, one of the operators in the representation of the operator $A = A_1 \oplus A_2$ is a zero operator. It is evident that here the operators A_1 and A_2 are linearly independent.

7. Criterion for the closeness of the image of operator T

The requirement of closeness of the image of operator T which is present in the conditions of Theorems 5.3, 5.4, 6.3 is, generally speaking, not necessary for the unique solvability of the spline approximation problems. However, if we consider these problems on the *class* of operators A, this condition becomes necessary.

7.1. Theorem. *If $R(T)$ is not closed, then the operator $A \in L(X, Z)$ acting into some Hilbert space Z and satisfying the conditions*

$$N(T) \cap N(A) = \{0\}, \quad R(A) \text{ and } N(T) + N(A) \text{ are closed,} \tag{7.1}$$

will be found such that the spline interpolation problem (2.2) *does not have solution at some $z \in R(A)$.*

Proof. As $R(T)$ is not closed, then some point $y_* \in \overline{R(T)}$ will be found which does not belong to $R(T)$. It is obvious that $y_* \neq 0$. Let us take the linear functional $\varphi(y) \overset{df}{=} (y, y_*)_Y$ and consider the spline interpolation problem in the space Y:

$$\varphi(g) = \lambda, \quad \|g\|_Y \to \min, \tag{7.2}$$

where $\lambda = \varphi(y_*)$. It is easy to verify that its solution is unique and coincides with y_*.

Let us consider the problem (7.2) on the subspace $R(T)$. The interpolation condition in (7.2) is not contradictory on $R(T)$, as the set $R(T)$ is linear and y_* is its limit point. However, this problem does not have any solution, as $y_* \notin R(T)$ and the sequence $y_n \in R(T)$ will be found converging to y_*.

Let $B \in L(X, W)$ be the operator acting into some Hilbert space W and satisfying the conditions

$$N(T) \cap N(B) = \{0\}, \quad N(T) + N(B) = X, \quad R(B) \text{ is closed.} \tag{7.3}$$

(For example, the orthoprojector on $N(T)$ can be taken as B.) Assume that

$$\psi(x) = (Tx, y_*)_Y, \quad A = B \oplus \psi, \quad z = 0 \oplus \lambda.$$

It easily follows from (7.3) that operator A satisfies the conditions (7.1). It remains to show that the problem (2.2) does not have solution at the given A and z.

Let us assume the opposite. Let σ be the solution to the problem (2.2). By the interpolation conditions

$$B\sigma = 0, \quad \psi(\sigma) = \varphi(T\sigma) = \lambda.$$

Hence, $\sigma \in N(B)$ and $T\sigma$ satisfies the interpolation condition of the problem (7.2). It can be easily obtained from (7.3) that operator T is one-to-one correspondence between $N(B)$ and $R(T)$. Therefore we conclude that the vector $T\sigma$ is the solution to the problem (7.2) on $R(T)$. However, as it was shown above, this problem does not have solution. □

Thus, the closeness of $R(T)$ is necessary for the existence of the interpolating spline on the class of operators A satisfying the conditions (7.1). The following question arises: how can we verify that $R(T)$ is closed? Let us give one criterion of closeness which is based on the concept of the semi-Hilbert space.

7.2. Definition. The vector space X which has the semi-norm $|\cdot|$ with the kernel P is called *the semi-Hilbert space* if the factor space X/P is the Hilbert space over the norm $\|x + P\|_* \stackrel{df}{=} |x|$.

7.3. Lemma. *Let X be vector space and $T : X \to Y$ be the linear operator acting into the Hilbert space Y. Then the following statements are equivalent:*

(a) *the set $R(T)$ is closed in Y;*

(b) *$(X, \|\cdot\|_T)$ is the semi-Hilbert space.*

Proof. It is clear that the kernel of the semi-norm $\|\cdot\|_T$ coincides with $N(T)$. Further, let $\{\tilde{x}_n \overset{df}{=} x_n + N(T)\}$ be an arbitrary sequence in $X/N(T)$. We denote $y_n = Tx_n$. Then

$$\|\tilde{x}_n - \tilde{x}_m\|_T = \|y_n - y_m\|_Y.$$

Consequently, the sequence $\{\tilde{x}_n\}$ is fundamental in $X/N(T)$ if and only if the sequence $\{y_n\}$ is fundamental in Y. Thereby the conditions of closeness $R(T)$ and $X/N(T)$ are equivalent. □

8. D^m-splines in a bounded domain

8.1. Let $\Omega \subset R^n$ be a bounded domain, and $X = W_2^m(\Omega)$ be the Sobolev space with the norm

$$\|x\|_{W_2^m} = \Big(\sum_{|\alpha|\le m} \int_\Omega |D^\alpha x|^2 d\Omega \Big)^{1/2},$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n) \text{ is the multiindex}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n,$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \cdot \ldots \cdot \partial t_n^{\alpha_n}} \text{ is the operator of the partial derivative.}$$

Let us consider the problem of construction of the interpolating D^m-spline [4] using its values on the scattered (possibly infinite) mesh of nodes $\omega \subset \Omega$

$$\sigma(t) = z_t, \quad t \in \omega,$$

$$\|D^m\sigma\|^2_{L_2(\Omega)} \overset{df}{=} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha\sigma)^2 d\Omega \to \min \tag{8.1}$$

and specify the conditions at which this problem is stated correctly.

8.2. Conditions of closeness for the image of operator D^m. The operator of the generalized gradient of the m-th order

$$D^m x = \Big\{ \Big(\frac{m!}{\alpha!}\Big)^{1/2} D^\alpha x, \quad |\alpha| = m \Big\}$$

acting from $W_2^m(\Omega)$ to the direct sum of $(n + m - 1)!/(n - 1)!/m!$ spaces $L_2(\Omega)$ serves as the energy operator in the problem (8.1). The kernel of this operator is finite-dimensional and consists of the polynomials of the order $m-1$. The coefficients of the operator D^m were selected in such a way as to provide the invariance of the semi-norm $\|\cdot\|_{D^m}$ relative to the rotations of the Cartesian coordinate system.

In accordance with Lemma 7.3, the image of the operator D^m is closed if and only if the space $W_2^m(\Omega)$ is closed over the semi-norm $\|\cdot\|_{D^m}$. In other words, this means that the space $W_2^m(\Omega)$ must coincide, as a set, with the semi-Hilbert space $D^{-m}L_2(\Omega)$ (it is often denoted by $L_2^m(\Omega)$) consisting of functions with the bounded semi-norm $\|\cdot\|_{D^m}$.

The spaces $W_2^m(\Omega)$ and $D^{-m}L_2(\Omega)$ coincide (for the bounded domain) in the following cases [7]:

(a) Ω is a *star domain with respect to a ball*, i.e., such a ball lying in the domain will be found that any ray starting from its any point intersects the boundary of the domain exactly in one point;

(b) the boundary of the domain is *Lipschitz boundary*, i.e., its any point has such a neighbourhood U that the set $U \cap \Omega$ in some Cartesian coordinate system is given by the inequality $x_n < f(x_1, \ldots, x_{n-1})$ with some function f satisfying the Lipschitz condition;

(c) the domain *satisfies the cone condition*, i.e., a cone will be found having the constant height and opening, the top of which can touch any point of the domain so that the cone will be inside of the domain;

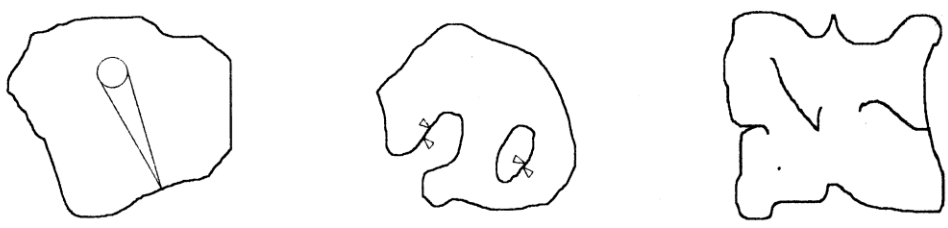
(d) Ω is a *star domain with respect to the finite number of balls*, i.e., it is a union of the finite number of star domains with respect to a ball.

If the domain is bounded (as in our case), then there are the following relations between these criteria:

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Leftrightarrow \text{(d)}, \tag{8.2}$$

i.e., the criterion (a) is stronger than (b), (b) is stronger than (c), (c) and (d) are equivalent. Differences between these criteria are seen in the figure. So, if a domain satisfies the criterion (b), then the cone can touch its any boundary point both from the inside and outside. At the same time, the criterion (c) provides the cone touching only from the inside of the domain.

The criteria (c) and (d) are interesting from the practical point of view, because they make it possible to consider the domain with cuts, punctured points and zero angles directed inward. In this case the space $W_2^m(\Omega)$ contains functions which have discontinuities in the punctured points and along the cuts. This fact serves to justify the algorithm for the construction of the discontinuous D^m-spline that was suggested in [8].



(a) (b) (c)

8.3. Remark. The definition of D^m-spline can be extended to a wider class of domains. We shall consider that the domain Ω has a *continuous boundary*, if every point $t \in \partial\Omega$ has such neighbourhood U that the set $\Omega \cap U$ in some Cartesian coordinate system is represented by the inequality $t_n < f(t_1, \ldots, t_{n-1})$, where f is a continuous function. If the bounded domain Ω is a combination of the finite number of domains with the continuous boundary, then the spaces $W_2^m(\Omega)$ and $D^{-m}L_2(\Omega)$ coincide [7, Remark to Lemma 1.1.11], which implies the closeness of the image of the operator D^m.

8.4. Correctness of the problem (8.1). For the correct statement of the problem (8.1), along with the closeness of $R(D^m)$, the continuity of the operator $Ax = x\big|_\omega$ for the projection of the function $x \in W_2^m(\Omega)$ on the mesh ω is required. This will take place in any domain Ω (not necessarily satisfying the cone condition) if $m > n/2$. (In this case only *the local* properties of the functions from the space $W_2^m(\Omega)$ are important. As the mesh nodes are inside of the domain Ω, then the functions of the space $W_2^m(\Omega)$ will be continuous in the sufficiently small neighbourhood of these nodes.)

Mesh nodes can also be at the boundary of the domain, if in their neighbourhood there is Lipschitz boundary (this provides a possibility for the local extension of the function beyond the domain preserving the order of smoothness).

And, finally, for the unique solvability of the problem (8.1) (and, also, smoothing and mixed problems), it is necessary and sufficient (Theorem 5.4) that the kernels of the operators A and D^m intersect at the zero function (the closeness of the image of operator A is not required). This is true if $(n + m - 1)!/n!/(m - 1)!$ nodes will be found among the nodes of mesh ω where the problem of construction of the polynomial of the order $m - 1$ has the unique solution. Such set of nodes is usually called *the Lagrangian set* or simply L-set.

9. D^m-splines in R^n

9.1. Incorrectness of the problem of D^m-approximation in $W_2^m(R^n)$. In order to construct D^m-spline in a bounded domain, we need to know the *reproducing kernel* of the space $W_2^m(\Omega)$ connected with the semi-norm $\|\cdot\|_{D^m}$ [4] (it is also called the Green function of the polyharmonic operator $(-\Delta)^m$). In the multivariate case, the problem of construction of the reproducing kernel has most probably no analytical solution.

The situation changes for the better if $\Omega = R^n$. However, the space, where the D^m-approximation is fulfilled, should be correctly chosen. The fact is that the space $W_2^m(R^n)$ is not good for the construction of D^m-spline in R^n, as the image of the operator D^m is not closed. (It is easy to verify that $N(D^m) = \{0\}$. Therefore, if $R(D^m)$ is closed, then the norm $\|\cdot\|_{D^m}$, by Lemma 4.5, must be equivalent to the original norm. However, this is not the case, as the function $f(t) = C$ does not belong to $W_2^m(R^n)$, and, at the same time, $\|f\|_{D^m} = 0$.)

The fact of absence of the solution in $W_2^m(R^n)$ can be proved directly.

9.2. Lemma. *Let $m > n/2$. Then the problem of spline interpolation*

$$\sigma(0) = 1, \quad \|D^m\sigma\| \to \min \tag{9.1}$$

does not have solution in the space $W_2^m(R^n)$.

Proof. Let us assume that the solution $\sigma \in W_2^m(R^n)$ of the problem (9.1) exists. As $N(D^m) = \{0\}$, then $\|D^m\sigma\| > 0$. Let us take an arbitrary number $\varepsilon \in (0,1)$ and consider the function $\sigma_\varepsilon(x) \overset{df}{=} \sigma(\varepsilon x)$. It is easy to verify that the function σ_ε satisfies the interpolation condition and belongs to $W_2^m(R^n)$. At the same time,

$$\|D^m\sigma_\varepsilon\| = \varepsilon^{m-n/2}\|D^m\sigma\| < \|D^m\sigma\|. \qquad \square$$

9.3. The problem of spline approximation in R^n becomes correct if we consider the space $D^{-m}L_2(R^n)$ consisting of functions, m-th partial derivatives of which belong to $L_2(R^n)$. The norm in this space can be introduced, for example, in the following way:

$$\|x\|_{D^{-m}L_2} = \left(\int_\Omega x^2(t)d\mu + \|D^m x\|_{L_2}^2\right)^{1/2},$$

where μ is some measure in R^n, and the set $\Omega \subset R^n$ is bounded and contains the L-set for the operator D^m (for example, a unit ball can be taken as Ω).

This problem was studied in the works of Duchon [9–11] considering a more general space $D^{-m}\tilde{H}^r(R^n)$ consisting of functions, m-th partial derivatives of which belong to the Hilbert space $\tilde{H}^r(R^n)$ with the norm

$$\|x\|_{\tilde{H}^r} = \Big(\int |\mathcal{F}x(\tau)|^2 \cdot |\tau|^{2r} d\tau\Big)^{1/2}.$$

Here $\mathcal{F}x$ is the Fourier transform of the function x. The real parameter r must be chosen so that the parameter

$$\gamma \overset{df}{=} m - n/2 + r$$

can belong to the interval $(0, m)$. Here the condition $\gamma > 0$ provides the continuity of the point evaluation functionals (continuity of the operator for the projection on the mesh), and $\gamma < m$ provides the closeness of the image of the operator D^m acting from $D^{-m}\tilde{H}^r(R^n)$ into $\tilde{H}^r(R^n)$.

The reproducing kernel of the semi-Hilbert space $(D^{-m}\tilde{H}^r(R^n), \|\cdot\|_{D^m})$ is equal to

$$G_\gamma(s,t) = (-1)^{[\gamma]+1} \begin{cases} |s-t|^{2\gamma} \ln |s-t|, & \gamma \text{ is integer}, \\ |s-t|^{2\gamma} & \text{otherwise}, \end{cases}$$

with a positive normalizing factor. Here $[\gamma]$ is the entire part of γ, and

$$|s-t| = \Big(\sum_{i=1}^{n}(s_i - t_i)^2\Big)^{1/2}.$$

The sign of the function G_γ (obtained in [12]) is of considerable importance for the problem of spline smoothing, but the normalizing factor is, generally speaking, not essential.

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