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Convergence of variational splines I

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Strong convergence of interpolating splines on the *imbedded meshes* is established without the assumption that the system of operators corresponding to the added interpolation conditions is *correct*. It is also shown that correctness of the system of operators is equivalent to the zero intersection of their kernels.

The necessary and sufficient conditions of convergence of the mixed splines on the subspaces to the mixed spline on the whole space are obtained, and simple sufficient conditions of their convergence are found.

1. Introduction

We use the term *the sequence of the interpolating splines on the imbedded meshes* when the sequence of the coresponding measurement operators A_i satisfies the condition $N(A_{i+1}) \subset N(A_i)$, $i \in \mathbb{N}$. In [1, 2] it was proved that convergence of such splines to the interpolating function takes place if and only if the system of operators corresponding to the added mesh nodes is *correct*. In [3] the other convergence condition was obtained:

$$\bigcap_{i=1}^{\infty} N(A_i) = \{0\}.$$

We show the equivalence of these two criteria and, moreover, we improve these results by finding the limit element of the spline sequence if the system of operators is not correct.

The convergence of interpolating and smoothing splines on subspaces was studied by Vasilenko [4, 5]. We extend the proof of convergence to the mixed splines on subspaces and improve the known results.

Let us give a brief summary of the paper. The rest part of this section presents notations which will be used. Section 2 is devoted to the convergence of the best approximations on the convex sets. Its results are used

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further in the proof of the convergence of interpolating splines on imbedded meshes (Section 3) and for mixed splines on subspaces (Section 4).

1.1. Let X be the real Hilbert space. The norm and the scalar product in it will be denoted by $\|\cdot\|_X$ and $(\cdot,\cdot)_X$, respectively. If it is clear from the context what norm or scalar product is meant, then the subscript will be omitted. The zero element of the space X will be denoted by the symbol 0. $L(X,Y)$ will stand for the Banach space of linear bounded operators acting from X into the Hilbert space Y.

1.2. Let $A \in L(X,Y)$ be some linear operator. $N(A)$ and $R(A)$ denote the kernel and image of operator A:

$$N(A) = \{x \in X : \ Ax = 0\}, \qquad R(A) = AX.$$

The preimage of the point $y \in Y$ will be denoted by

$$A^{-1}(y) = \{x \in X : \ Ax = y\}.$$

The functionals

$$\|x\|_A \stackrel{df}{=} \|Ax\|_Y, \qquad (u,v)_A \stackrel{df}{=} (Au, Av)_Y$$

give the semi-norm and the scalar semi-product on X, respectively. If $N(A) = \{0\}$, then $\|\cdot\|_A$ will be a norm, and $(\cdot,\cdot)_A$ will be a scalar product on X.

1.3. We use the notation

$$X \oplus Y = \{x \oplus y : \ x \in X, \ y \in Y\}$$

for the direct sum of the Hilbert spaces X and Y with the operations of summation and multiplication by scalar

$$\begin{aligned} x_1 \oplus y_1 + x_2 \oplus y_2 &= (x_1 + x_2) \oplus (y_1 + y_2), \\ \lambda(x \oplus y) &= \lambda x \oplus \lambda y \end{aligned}$$

and the norm

$$\|x \oplus y\|_{X\oplus Y} = \left(\|x\|_X^2 + \|y\|_Y^2\right)^{1/2}.$$

Let $A \in L(X,Y)$, $B \in L(X,Z)$ be some operators. $A \oplus B$ stands for the direct sum of the operators A and B given by the rule

$$A \oplus B\, x = Ax \oplus Bx.$$

The norm generated from this operator is determined in the following way:

$$\|x\|_{A\oplus B} = \left(\|Ax\|^2 + \|Bx\|^2\right)^{1/2}.$$

2. Convergence of best approximations on convex sets

Let X be Hilbert space, $M \subset X$ be the non-empty closed convex set, and $f \in X$. It is well-known that the problem

$$h = \arg\min_{x \in M} \|x - f\|_X \tag{2.1}$$

has the unique solution. Let us construct the operator $P_M : X \to X$ mapping each element $f \in X$ to $h \in X$ by the formula (2.1).

2.1. Lemma [6]. *The following statements are equivalent:*

(a) $h = P_M f$;

(b) $h \in M$ *and* $\forall x \in M \ (h - f, h - x)_X \le 0$.

If M *is affine subspace in* X ($M = x_* + K$), *then* (b) *can be replaced by*

(b′) $h \in M$ *and* $\forall u \in K \ (h - f, u)_X = 0$.

The inequality in the condition (b) can also be written in the form:

$$\|x - h\|^2 \le \|x - f\|^2 - \|h - f\|^2. \tag{2.2}$$

Actually,

$$\begin{aligned} \|x - h\|^2 &= \|x - f\|^2 - 2(x - f, h - f) + \|h - f\|^2 \\ &= \|x - f\|^2 - 2(h - f - h + x, h - f) + \|h - f\|^2 \\ &= \|x - f\|^2 - \|h - f\|^2 + 2(h - x, h - f) \le \|x - f\|^2 - \|h - f\|^2. \end{aligned}$$

2.2. Lemma.

$$\forall f, g \in X \qquad \|P_M f - P_M g\| \le \|f - g\|. \tag{2.3}$$

Proof. We conclude from Lemma 2.1 that

$$\begin{aligned} (P_M f - f, P_M f - P_M g) &\le 0 \\ (P_M g - g, P_M g - P_M f) &\le 0. \end{aligned}$$

Subtracting the first inequality from the second one, we obtain

$$\begin{aligned} 0 &\ge (P_M g - g, P_M g - P_M f) - (P_M f - f, P_M f - P_M g) \\ &= (P_M g - P_M f + f - g, P_M g - P_M f) \\ &= \|P_M g - P_M f\|^2 - (g - f, P_M g - P_M f). \end{aligned}$$

Hence,

$$\|P_M g - P_M f\|^2 \le (g - f, P_M g - P_M f) \le \|g - f\| \cdot \|P_M g - P_M f\|. \quad (2.4)$$

After the reduction of the factor $\|P_M g - P_M f\|$ in the left-hand and right-hand sides of (2.4), we obtain (2.3). □

2.3. Theorem. *Let $M_i \subset X$ be non-empty closed convex sets, and $f_i \in X$ be some elements, $i \in \mathbb{N}$. Assume that $f_i \to f$ as $i \to \infty$ and the sets M_i satisfy one of the following conditions:*

(a) *$M_{i+1} \subset M_i$ and $\bigcap_{i \in \mathbb{N}} M_i = M$;*

(b) *$M_i \subset M$ and there exists the sequence $x_i \in M_i$ converging to $h \overset{df}{=} P_M f$.*

Then the sequence $h_i \overset{df}{=} P_{M_i} f_i$ converges to h as $i \to \infty$.

Proof. It is sufficient to prove the statement of the theorem for $f_i = f$. Really, taking into account Lemma 2.2, we have

$$\|h_i - h\| \le \|h_i - P_{M_i} f\| + \|P_{M_i} f - h\| \le \|f_i - f\| + \|P_{M_i} f - h\| \to 0$$

if $P_{M_i} f \to h$. Therefore, we shall consider further that $f_i = f$.

(a). Let us denote $\alpha_i = \|h_i - f\|^2$. As $M_{i+1} \subset M_i$, the sequence α_i monotonically increases and is, evidently, bounded above by the number $\alpha = \|h - f\|^2$. Consequently, the sequence α_i converges. After applying (2.2) for the set M_i and substituting the element h_j $(j > i)$ instead of x, we obtain the inequality

$$\|h_i - h_j\|^2 \le \alpha_j - \alpha_i,$$

from which it follows that the sequence h_i converges.

Let $h_* = \lim_{i \to \infty} h_i$. Fix i and take an arbitrary $j > i$. Since $M_j \subset M_i$, we have $h_j \in M_i$. Hence, $h_* \in M_i$ and, consequently, $h_* \in \bigcap M_i = M$. At the same time,

$$\|h_* - f\|^2 = \lim_{i \to \infty} \|h_i - f\|^2 = \lim_{i \to \infty} \alpha_i \le \alpha = \|h - f\|^2.$$

So, $h_* = h$.

(b). Denote $d = \|h - f\|$ and $\varepsilon_i = \|x_i - h\|$. Then

$$\|h_i - f\| \le \|x_i - f\| \le \|x_i - h\| + \|h - f\| = d + \varepsilon_i.$$

After applying (2.2) for the set M and substituting the element h_i instead of x, we obtain

$$\|h_i - h\|^2 \le \|h_i - f\|^2 - \|h - f\|^2 \le (d + \varepsilon_i)^2 - d^2 = \varepsilon_i(2d + \varepsilon_i).$$

Under condition of the theorem, $\varepsilon_i \to 0$ as $i \to \infty$. Consequently, $h_i \to h$. □

3. Convergence of interpolating splines on imbedded meshes

Let X, Y, Z_i be the Hilbert spaces, and $T \in L(X,Y)$, $A_i \in L(X,Z_i)$ be some operators, $i \in \mathbb{N}$. Assume that $R(T)$ is closed and the following spline interpolation problems

$$\sigma_i = \arg \min_{x \in A_i^{-1}(z_i)} \|Tx\|^2, \quad i \in \mathbb{N}, \tag{3.1}$$

are uniquely solvable for any $z_i \in R(A_i)$. This means, in accordance with [6], that the sets $N(T) + N(A_i)$ are closed and $N(T) \cap N(A_i) = \{0\}$. Introduce the operators of spline interpolation $S(A_i, T, X)$ which maps every element $f \in X$ to the interpolating splines σ_i for $z_i = A_i f$.

3.1. Definition. Let U, V be the Banach spaces. The sequence of operators $B_i \in L(U,V)$ is called *strongly converging to* $B \in L(U,V)$, if

$$\forall x \in U \qquad B_i x \to Bx \quad \text{as } i \to \infty.$$

3.2. Theorem. *Let $R(T)$ be closed and the problems* (3.1) *be uniquely solvable for any acceptable initial data. If $N(A_{i+1}) \subset N(A_i)$, $i \in \mathbb{N}$, and $A_* \in L(X, Z_*)$ is some operator with the kernel*

$$N(A_*) = \bigcap_{i \in \mathbb{N}} N(A_i), \tag{3.2}$$

then the sequence of operators $S_i \overset{df}{=} S(A_i, T, X)$ strongly converges to the operator $S_* \overset{df}{=} S(A_*, T, X)$.

Proof. Let $x_* \in X$ be some element. By definition, we have

$$S_i x_* = \arg \min_{x \in A_i^{-1}(A_i x_*)} \|Tx\|^2. \tag{3.3}$$

Under conditions of the theorem, $N(T) + N(A_1)$ is closed and $N(T) \cap N(A_1) = \{0\}$. Consequently [7], such operator $\tilde{A} \in L(X, \tilde{Z})$ will be found that $N(A_1) \subset N(\tilde{A})$ and the norm $\|\cdot\|_{T \oplus \tilde{A}}$ is equivalent to the norm $\|\cdot\|_X$. Taking into account the evident identity

$$A_i^{-1}(A_i x_*) = x_* + N(A_i)$$

and the inclusions $N(A_i) \subset N(A_1) \subset N(\tilde{A})$, we rewrite the problem (3.3) in the equivalent form

$$S_i x_* = \arg \min_{x \in x_* + N(A_i)} \|x\|^2_{T \oplus \tilde{A}}.$$

Hence, by Theorem 2.3, the sequence $S_i x_*$ strongly converges to the solution of the problem

$$\sigma_* = \arg \min_{x \in x_* + \bigcap N(A_i)} \|x\|^2_{T \oplus \tilde{A}}.$$

However, from (3.2) we have $\sigma_* = S_* x_*$, i.e., $S_i x_* \to S_* x_*$ as $i \to \infty$. □

Corollary. *Let $R(T)$ be closed, the problems* (3.1) *be uniquely solvable for any acceptable initial data, and $N(A_{i+1}) \subset N(A_i)$, $i \in \mathbb{N}$. Then the following statements are equivalent:*

(a) *the sequence of operators S_i strongly converges to the identical operator I, i.e., $\forall x \in X \;\; S_i x \to x$;*

(b) $\bigcap_{i \in \mathbb{N}} N(A_i) = \{0\}$.

In [4, 5] this theorem is formulated in terms of correct system of operators.

3.3. Definition. Let $\mathcal{A} = \{A_i \in L(X, Z_i),\;\; i \in \mathbb{N}\}$ be a family of operators acting into some Hilbert spaces Z_i. We shall consider that the sequence $\{x_n \in X\}$ *converges to* $x \in X$ *by the system of operators* $\mathcal{A}$ ($x_n \xrightarrow{\mathcal{A}} x$), if

$$\forall A_i \in \mathcal{A} \qquad \lim_{n \to \infty} \|A_i(x_n - x)\| = 0.$$

3.4. Definition. The system $\mathcal{A}$ is said to be *correct* if the convergence $x_n \xrightarrow{\mathcal{A}} x$ implies weak convergence of x_n to x on some set K which is dense in X. Symbolically,

$$\left[x_n \xrightarrow{\mathcal{A}} x\right] \quad \Rightarrow \quad \left[\exists K \subset X : \; \overline{K} = X \;\; \& \;\; \forall k \in K \; (k, x_n) \to (k, x)\right].$$

3.5. Theorem. *The following statements are equivalent:*

(a) *the system $\mathcal{A}$ is correct;*

(b) $\bigcap_{i \in \mathbb{N}} N(A_i) = \{0\}$.

Proof. (a) ⇒ (b). Let $u \in \bigcap N(A_i)$, i.e.,

$$\forall i \in \mathbb{N} \qquad u \in N(A_i). \tag{3.4}$$

Consider the sequence $\{x_n \overset{df}{=} u\}$ and take any number $\alpha \in R$. Then, from (3.4), we obtain

$$\|A_i(x_n - \alpha u)\| = |1 - \alpha| \cdot \|A_i u\| = 0.$$

Hence, $x_n \overset{\mathcal{A}}{\to} \alpha u$. Since the system $\mathcal{A}$ is correct, we have

$$\forall k \in K \qquad (k, x_n) \to (k, \alpha u) = \alpha(k, u).$$

Due to the arbitrary choice of α we conclude that $(k, u) = 0$, i.e., $u \in K^\perp$. However, as K is dense in X, we have $K^\perp = \{0\}$ and $u = 0$.

(b) $\Rightarrow$ (a). Assume that

$$K = \bigcup_{i \in \mathbb{N}} R(A_i^*)$$

and prove that K is the required set, i.e., that $\overline{K} = X$ and

$$\left[x_n \overset{\mathcal{A}}{\to} x\right] \quad \Rightarrow \quad \left[\forall k \in K \ \ (k, x_n) \to (k, x)\right]. \tag{3.5}$$

It is evident that K is a subspace in X. Consequently, $\overline{K} = (K^\perp)^\perp$, i.e.,

$$\overline{K} = X \quad \Longleftrightarrow \quad K^\perp = \{0\}.$$

Using the identity $R(A_i^*)^\perp = N(A_i)$, we obtain

$$K^\perp = \left(\bigcup_{i \in \mathbb{N}} R(A_i^*)\right)^\perp = \bigcap_{i \in \mathbb{N}} N(A_i) = \{0\},$$

i.e., K is dense in X.

Now, let $x_n \overset{\mathcal{A}}{\to} x$, i.e., for any $i \in \mathbb{N}$, the sequence $\{A_i x_n\}_{n \in \mathbb{N}}$ strongly converges to $A_i x$. So, $A_i x_n \overset{w}{\to} A_i x$, i.e.,

$$\forall z \in Z_i \qquad (z, A_i x_n) \to (z, A_i x)$$

or

$$\forall z \in Z_i \qquad (A_i^* z, x_n) \to (A_i^* z, x).$$

Hence, if the vector z "passes through" the whole space Z_i, then the vector $k \overset{df}{=} A_i^* z$ passes through the whole $R(A_i^*)$. So, (3.5) is proved. □

3.6. Let us consider the operators

$$B_i = A_1 \oplus \ldots \oplus A_i.$$

Assume that $R(T)$ is closed and the problems of spline interpolation

$$\sigma_i = \arg \min_{x \in B_i^{-1}(z_i)} \|Tx\|^2, \quad i \in \mathbb{N},$$

are uniquely solvable for any $z_i \in R(B_i)$. Then the theorem of convergence in terms of correct system of operators is formulated in the following way:

Theorem. *In order for the sequence of operators of the spline interpolation $S(B_i, T, X)$ to converge strongly to I, it is necessary and sufficient that the system $\mathcal{A}$ be correct.*

The proof of this theorem can be easily obtained from Theorem 3.5, the corollary of Theorem 3.2 and the obvious identity

$$\bigcap_{i \in \mathbb{N}} N(B_i) = \bigcap_{i \in \mathbb{N}} N(A_i).$$

4. Convergence of splines on subspaces

4.1. Let X, Y, Z, V be the Hilbert spaces, and $T \in L(X,Y)$, $A \in L(X,Z)$, $B \in L(X,V)$ be some operators. We consider the mixed problem of spline approximation [5, 7]

$$\hat{\sigma} = \arg \min_{x \in A^{-1}(z)} \|Tx\|^2 + \|Bx - v\|^2 \tag{4.1}$$

with the smoothing parameter equal to 1 (for simplicity).

Let us assume that $R(T)$ is closed and, on the subspace $N(A)$, the norms $\|\cdot\|_X$ and $\|\cdot\|_{T\oplus B}$ are equivalent. In accordance with [7], these conditions provide the unique solvability of the problem (4.1) for any $z \in R(A)$ and $v \in V$.

4.2. Let $\{E_\tau\}_{\tau>0}$ be a family of subspaces in X. The spline

$$\hat{\sigma}_\tau = \arg \min_{x \in A^{-1}(z) \cap E_\tau} \|Tx\|^2 + \|Bx - v\|^2 \tag{4.2}$$

is called the *mixed spline on the subspace E_τ*. Here it is assumed that $A^{-1}(z) \cap E_\tau \neq \emptyset$, i.e., the interpolation conditions $Ax = z$ are not contradictory on E_τ.

The unique solvability of the problem (4.2) takes place at the same conditions as for the problem (4.1).

4.3. Let us take the operator $\tilde{A} \in L(X, \tilde{Z})$ acting into some Hilbert space $\tilde{Z}$ and satisfying the following conditions: $N(\tilde{A}) \supset N(A)$, $R(\tilde{A})$ is closed, and $N(\tilde{A}) \cap N(T \oplus B) = \{0\}$. Then, by [7, Theorem 4.7], the norm

$$\|\cdot\|_* \stackrel{df}{=} \|\cdot\|_{\tilde{A}\oplus T\oplus B}$$

is equivalent to the norm $\|\cdot\|_X$ on X. Doing as in [7, Lemma 3.1], we replace the problem (4.1) by the equivalent problem

$$\hat{\sigma} = \arg \min_{x\in A^{-1}(z)} \|x - f\|_*^2, \tag{4.3}$$

where $f = B^* v$, and B^* is the operator adjoint to B with respect to the scalar product $(\cdot,\cdot)_*$ corresponding to the norm $\|\cdot\|_*$.

Similarly, the problem (4.2) is replaced by the equivalent problem

$$\hat{\sigma}_\tau = \arg \min_{x\in A^{-1}(z)\cap E_\tau} \|x - f\|_*^2. \tag{4.4}$$

4.4. Theorem. *The following statements are equivalent:*

(a) $\hat{\sigma}_\tau \to \hat{\sigma}$ *as* $\tau \to 0$*;*

(b) *the sets* $A^{-1}(z)\cap E_\tau$ *are asymptoticly dense at the point* $\hat{\sigma}$ *(such elements* $x_\tau \in A^{-1}(z)\cap E_\tau$ *will be found, that* $x_\tau \to \hat{\sigma}$ *as* $\tau \to 0$*).*

The proof of this theorem easily follows from Theorem 2.3 applied to the problems (4.3) and (4.4).

Remark. If the subspaces E_τ are asymptoticly dense in X ($E_\tau \to X$), then this condition provides convergence of the smoothing splines ($A = 0$) on the subspaces. In the interpolating ($B = 0$) and mixed cases this condition does not guarantee that the sets $A^{-1}(z)\cap E_\tau$ will be asymptoticly dense at the point $\hat{\sigma}$.

Further we shall find out when the condition $E_\tau \to X$ implies the convergence $\hat{\sigma}_\tau$ to $\hat{\sigma}$.

Let P_τ be the orthoprojector onto E_τ in the norm $\|\cdot\|_*$, i.e., $P_\tau^2 = P_\tau$, $\|P_\tau\|_* = 1$ and $R(P_\tau) = E_\tau$.

4.5. Lemma. *The problem* (4.4) *is equivalent to the problem*

$$\hat{\sigma}_\tau = \arg \min_{x\in (AP_\tau)^{-1}(z)} \|x - P_\tau f\|_*^2. \tag{4.5}$$

Proof. Let $f = f_1 + f_2$, where $f_1 = P_\tau f \in E_\tau$, $f_2 = (I - P_\tau)f \in E_\tau^\perp$. Then

$$\forall x \in E_\tau \qquad \|x - f\|_*^2 = \|x - f_1\|_*^2 + \|f_2\|_*^2.$$

Consequently, the solution will not change if f in (4.4) will be replaced by f_1. Further,

$$\hat{\sigma}_\tau = \arg \min_{x \in A^{-1}(z) \cap E_\tau} \|x - f_1\|_*^2 \geq \arg \min_{AP_\tau x = z} \|x - f_1\|_*^2 \stackrel{df}{=} \sigma_*.$$

If it will be shown that $\sigma_* \in E_\tau$, then it will follow that $\hat{\sigma}_\tau = \sigma_*$.

Assume the opposite. Let $\sigma_* = \sigma_1 + \sigma_2$, $\sigma_1 \in E_\tau$, $\sigma_2 \in E_\tau^\perp$ and $\sigma_2 \neq 0$. It is evident that $P_\tau \sigma_1 = P_\tau \sigma_*$. Hence, $AP_\tau \sigma_1 = AP_\tau \sigma_* = z$. However, it follows from the inequality

$$\|\sigma_* - f_1\|_*^2 = \|\sigma_1 - f_1\|_*^2 + \|\sigma_2\|_*^2 > \|\sigma_1 - f_1\|_*^2,$$

that σ_* cannot be the solution to the problem (4.5). □

4.6. It can be considered without the loss of generality that $R(A) = Z$ (since the norm of the space Z is not used in the definition of the mixed spline, we can always change it so that $R(A)$ will be the Hilbert space). It is known [5], that the solution to the problem (4.3) can be found from the system of operator equations

$$\hat{\sigma} + A^*\lambda = f, \qquad A\hat{\sigma} = z, \tag{4.6}$$

where A^* is the operator adjoint to A with respect to $(\cdot,\cdot)_*$. After obtaining $\hat{\sigma}$ from the first equation of (4.6) and substituting it into the second one, we have

$$Af - AA^*\lambda = z. \tag{4.7}$$

As $R(A) = Z$, the operator AA^* is invertible. Obtaining λ from (4.7) and substituting it into the first equation of (4.6), we get

$$\hat{\sigma} = A^*(AA^*)^{-1}z + \left[I - A^*(AA^*)^{-1}\right]f.$$

If $R(AP_\tau) = R(A)$, then, by Lemma 4.5, we also obtain

$$\hat{\sigma}_\tau = P_\tau A^*(AP_\tau A^*)^{-1}z + \left[I - P_\tau A^*(AP_\tau A^*)^{-1}\right]P_\tau f.$$

4.7. As $z \in R(A)$, such $x_* \in X$ will be found that $Ax_* = z$. Then, denoting

$$S = A^*(AA^*)^{-1}A, \qquad S_\tau = P_\tau A^*(AP_\tau A^*)^{-1}A,$$

we get

$$\hat{\sigma} = Sx_* + (I - S)f, \qquad \hat{\sigma}_\tau = S_\tau x_* + (I - S_\tau)P_\tau f. \tag{4.8}$$

Note that the operator S is the orthoprojector in the norm $\|\cdot\|_*$ with the kernel $N(A)$. It is the operator of spline interpolation for the problem

$$Sx_* = \arg \min_{Ax=Ax_*} \|Tx\|^2 + \|Bx\|^2.$$

It is clear that the operator S_τ is a projector, but it is not an orthoprojector in this norm.

4.8. Lemma. *Let $E_\tau \to X$ as $\tau \to 0$, and there exists $\tau_0 > 0$ such that for any $\tau \le \tau_0$ the norm of the operator*

$$M_\tau \overset{df}{=} A^*(AP_\tau A^*)^{-1}A$$

is bounded by a constant independent of τ. Then $\hat{\sigma}_\tau \to \hat{\sigma}$ as $\tau \to 0$.

Proof. 1. Since $\hat{\sigma}_\tau \in A^{-1}(z)$ and $A^{-1}(z)$ is the affine subspace, we conclude from (4.3) and Lemma 2.1, that

$$(\hat{\sigma} - f, \hat{\sigma} - \hat{\sigma}_\tau)_* = 0.$$

Hence,

$$\|\hat{\sigma} - \hat{\sigma}_\tau\|_*^2 = (\hat{\sigma}_\tau - f, \hat{\sigma}_\tau - \hat{\sigma})_*.$$

Taking into account that $\hat{\sigma}_\tau \in E_\tau$ and decomposing f and $\hat{\sigma}$ into the sums of orthogonal components from E_τ and $E_\tau^\perp$, we get

$$\|\hat{\sigma} - \hat{\sigma}_\tau\|_*^2 = (\hat{\sigma}_\tau - P_\tau f, \hat{\sigma}_\tau - P_\tau\hat{\sigma})_* + (P_\tau f - f, P_\tau\hat{\sigma} - \hat{\sigma})_*. \tag{4.9}$$

As $I - P_\tau$ is an orthoprojector, the second term in the right- hand side of (4.9) is transformed into $((I - P_\tau)f, \hat{\sigma})_*$ and it tends to zero as $\tau \to 0$.

2. It is the only problem now to estimate the first term in (4.9). Denote $x_\tau = x_* - P_\tau f$ and use (4.8), then we obtain

$$\begin{aligned} \hat{\sigma}_\tau - P_\tau f &= S_\tau x_* + P_\tau f - S_\tau P_\tau f - P_\tau f = S_\tau x_\tau, \\ \hat{\sigma}_\tau - P_\tau \hat{\sigma} &= S_\tau x_* + P_\tau f - S_\tau P_\tau f - P_\tau S x_* - P_\tau f + P_\tau S f \\ &= S_\tau x_\tau - P_\tau S x_* + P_\tau S f = S_\tau x_\tau - P_\tau S x_\tau + P_\tau S(I - P_\tau)f. \end{aligned}$$

Hence,

$$\begin{aligned} &(\hat{\sigma}_\tau - P_\tau f, \hat{\sigma}_\tau - P_\tau\hat{\sigma})_* \\ &= \|S_\tau x_\tau\|_*^2 - (S_\tau x_\tau, P_\tau S x_\tau)_* + (S_\tau x_\tau, P_\tau S(I - P_\tau)f)_* \\ &= \|S_\tau x_\tau\|_*^2 - \|S x_\tau\|_*^2 + (S x_\tau, (I - P_\tau)f)_*. \end{aligned} \tag{4.10}$$

Here we use the following properties of the operators S, S_τ and P_τ:

$$S = S^2 = S^* = SS_\tau, \quad P_\tau S_\tau = S_\tau, \quad P_\tau = P_\tau^*.$$

3. Since $\|x_\tau\|_* \le \|x_*\|_* + \|f\|_*$, the last term in (4.10) tends to zero as $\tau \to 0$. By using the simple transformations, the difference of the first two components in (4.10) reduces to

$$\|S_\tau x_\tau\|_*^2 - \|Sx_\tau\|_*^2 = ((M_\tau - S)x_\tau, x_\tau)_*.$$

It it easy to see that

$$M_\tau - S = M_\tau(I - P_\tau)S. \tag{4.11}$$

So,

$$\begin{aligned}((M_\tau - S)x_\tau, x_\tau)_* &= (M_\tau(I - P_\tau)Sx_\tau, x_\tau)_* \\ &\le \|M_\tau\|_* \cdot \|(I - P_\tau)Sx_\tau\|_* \cdot \|x_\tau\|_*. \end{aligned} \tag{4.12}$$

The last multiple in the estimate (4.12) is, evidently, bounded. The first one is bounded under the lemma condition.

4. Let us estimate the middle multiple in (4.12). Denote $\tilde{x} = x_* - f$. Then $x_\tau = \tilde{x} + (I - P_\tau)f$. Hence,

$$\|(I - P_\tau)Sx_\tau\|_* \le \|(I - P_\tau)S\tilde{x}\|_* + \|(I - P_\tau)S(I - P_\tau)f\|_*.$$

The first term in this estimate tends to zero, because $S\tilde{x}$ is independent of τ. Finally, the last term is estimated in the following way:

$$\begin{aligned}\|(I - P_\tau)S(I - P_\tau)f\|_* &\le \|I - P_\tau\|_* \cdot \|S\|_* \cdot \|(I - P_\tau)f\|_* \\ &= 1 \cdot 1 \cdot \|(I - P_\tau)f\|_* \to 0\end{aligned}$$

as $\tau \to 0$. □

4.9. It remains to find out under what conditions the norm of the operator M_τ is bounded by a constant independent of τ.

If $\dim R(A) < \infty$, then the boundedness of the norm of the operator M_τ is proved in the same way as in [5, Theorem 4.1]. Namely, first we prove the element-wise convergence of the matrix $AP_\tau A^*$ to AA^*. Hence, we conclude that the elements of the matrix $(AP_\tau A^*)^{-1}$ converges to the elements of $(AA^*)^{-1}$. And this implies the boundedness of $\|M_\tau\|_*$.

Let us consider the case $\dim R(A) = \infty$.

4.10. Lemma. *If* $\|(I - P_\tau)S\| \overset{df}{=} C_\tau < 1$, *then*

$$M_\tau = S(I - N_\tau)^{-1} = (I - N_\tau)^{-1}S = S(I - N_\tau)^{-1}S, \tag{4.13}$$

where $N_\tau = S(I - P_\tau)S$. *In this case*

$$\|M_\tau\|_* \leq (1 - C_\tau^2)^{-1}. \tag{4.14}$$

Proof. It follows from (4.11) that

$$M_\tau = S + M_\tau(I - P_\tau)S.$$

Hence, with the help of the recurrent substitution and taking into account the identity $S = S^2$, we formally obtain that

$$M_\tau = S \cdot \sum_{k=0}^{\infty} N_\tau^k. \tag{4.15}$$

As S and P_τ are ortoprojectors, the operator N_τ is self-adjoint and

$$(N_\tau x, x)_* = \|(I - P_\tau)Sx\|_*^2 \leq \|(I - P_\tau)S\|^2 \cdot \|x\|_*^2 \leq C_\tau^2\|x\|_*^2,$$

i.e.,

$$\|N_\tau\| \leq C_\tau^2 < 1. \tag{4.16}$$

Consequently, the series in (4.15) is absolutely convergent.

Taking into account that

$$\sum_{k=0}^{\infty} N_\tau^k = (I - N_\tau)^{-1},$$

we obtain the first equality in (4.13). The other equalities follow from the permutability of the operators S and N_τ and from the formula $S^2 = S$. The estimate (4.14) easily follows from (4.13) and (4.16). □

Combining results given above we obtain

4.11. Theorem. *Let* $E_\tau \to X$ *as* $\tau \to 0$, $R(AP_\tau) = R(A)$ *and one of the following conditions is fulfilled:*

(a) $\dim R(A) < \infty$;

(b) *there exists such* $\tau_0 > 0$, *that* $\|(I - P_\tau)S\|_* \leq C < 1$ *for* $\tau \leq \tau_0$.

Then $\hat{\sigma}_\tau \to \hat{\sigma}$ *as* $\tau \to 0$.

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