Preface

The present monograph is devoted to the theoretical and numerical study of the interaction of electromagnetic fields with deformable media. The models considered are based on different combination of the Lamé and Maxwell equations. Several direct and associated with them inverse problems are studied. Then speaking about inverse problems, electromagnetic and elastic characteristics of a medium are the subject of reconstruction. Values of physical fields are connected through electromagnetoelastic interactions.

The authors consider the processes which are observed when seismic waves propagate in the Earth's crust. Variations of seismic and electromagnetic fields arising in this case are called *electromagnetoelastic waves*. The following types of electromagnetoelastic interactions are distinguished: interaction based on the *electrokinetic properties of a medium*, interaction based on the *piezoelectric properties of a medium* and interaction based on the *velocity effect*.

First, different statements of mathematical model of electromagnetoelastic interactions is described. Then the theoretical results of the numerical solution are discussed for various direct and inverse problems for the equations of electromagnetoelasticity.

Finally, the authors give some results of the numerical solutions of two inverse problems for the system of equations describing linear processes of interaction of electromagnetic and elastic waves.

Even if a complete review of the area is not exposed, rather wide range of problems and methods for their solution are discussed. This makes the monograph interesting and useful.

Contents

Introduction

Recently the interaction of electromagnetic fields with deformable media has been a subject of many theoretical and experimental investigations in the field of continuum mechanics and geophysics. For the description of sufficiently simpleinteractions, theories of magnetohydrodynamics [2, 26], electroelasticity [13, 33], and magnetoelasticity [23, 48] were developed. These theories are, basically, a combination (without introducing the new conceptions) of objects and phenomena considered in continuum mechanics and electrodynamics.

Investigation of more complex electromagnetoelastic interactions in a continuum medium requires consideration of more complex models. For a more profound acquaintance with the state-of-the-art on theory of electromagnetoelastic interactions the reader is referred to, e.g., [18, 29, 34].

The present investigation is aimed at studying some direct and associated with them inverse problems of electromagnetic and elastic characteristics of a medium reconstruction connected with electromagnetoelastic interactions. The models considered here are based on simple variants of a combination of the Lamé and the Maxwell equations.

Let us characterize in brief the basic types of electromagnetoelastic interactions. It is well known that when an electriconducting elastic body oscillates in electromagnetic field, variations of the electrical and magnetic fields are observed as a result of this motion. Similar processes are also observed when seismic waves propagate in the Earth's crust. Variations of seismic and electromagnetic fields arising in this case are called *electromagnetoelastic waves.* Such waves contain a certain information about electromagnetic and elastic parameters of a medium. In this case, as a rule, the following types of electromagnetoelastic interactions are distinguished:

- Interaction based on the **electrokinetic properties of a medium**. It is supposed that generation of electric signals under elastic waves propagation is connected precisely with manifestation of electrokinetic properties of a medium. This effect is used for development of electroseismic methods of "viewing" of the Earth's crust, electroacoustic investigations in boreholes, etc.
- Interaction based on the **piezoelectric properties of a medium**. This interaction is connected with propagation of elastic waves in crystal rocks when the elastic deformation of lattice of a material produces displacement of electrons and, as consequence, there arises an electrical field induced by such deformations.

- Interaction based on the **velocity effect**. Whereas, for example, the electrokinetic effect is connected with local interactions of elastic waves with a flow of the pore liquid, the velocity effect is based on slow movement of a medium in external electromagnetic field. In the case of geophysical or seismological problems, the velocity effect leads to so-called **seismomagnetic effect** based on the interaction of seismic waves with the Earth's magnetic field [31]. This interaction results in induced electromagnetic waves propagating with velocities commensurable with those of seismic waves.

Anisimov et al. [5] confirm simultaneous propagation of seismic waves with induced geomagnetic variations and point to a possibility to record such geomagnetic variation with confidence.

The first result on inverse problems of electromagnetoelasticity was obtained, apparently, by Burdakova and Yakhno [14]. Omitting here the geophysical aspect of such problems we would like to point out the paper by Alekseev [1] which is, in our opinion, of substantial mathematical and methodological importance. It turned out that, in essence, the inverse problems of electromagnetoelasticity as a part of so-called **combined inverse problems** provide a possibility of a more successful solution than the study of each of the inverse problems separately, taking into account the data obtained in order to get the general idea of the medium in study. He gave a mathematical definition of the combined inverse problem and showed that it is not equivalent to a simple set of individual problems.

A systematic study of theoretical questions which are connected with the uniqueness, existence, and stability of solutions of inverse problems (and associated with them direct ones) for electromagnetoelasticity system began in the nineties of the XX-th century. In this connection, we should note the works [7, 8, 9, 20, 22, 24, 27, 28, 30, 35, 36, 37, 41, 42, 50, 51].

As for applications of such problems the reader is referred, for example, to [18].

This monograph is not a complete exposition of this field of research. Its main objective is to give a conception of the range of problems and methods for their solution.

Chapter 1

Mathematical model of electromagnetoelastic interactions

The interaction of electromagnetic fields with deformable media is considered with point of view of linear elasticity connected with electrodynamic of elastic moving media by means of motion of particles in the electromagnetic field. We do not consider any effects of interactions, which could arise as a result of some kind of relations in constitutive equations besides velocity. We, basically, follow Dunkin and Eringen [16] when defining a mathematical model for electromagnetoelastic effect.

1.1. Electromagnetic theory

Let $\mathbb{R}^3$ be a three-dimensional Euclidean space of points $x = (x_1, x_2, x_3)$. The process of propagation of electromagnetic waves in $\mathbb{R}^3$ will be described by the following Maxwell system:

$$\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} = \operatorname{rot} \boldsymbol{H}, \qquad \frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{D} = \rho_e, \qquad \operatorname{div} \boldsymbol{B} = 0. \tag{1.2}$$

Here $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$, and $\boldsymbol{B}$ are the electromagnetic vectors, $\boldsymbol{J}$ is the current density, ρ_e is the charge density, and all quantities are expressed in the MKS units. When a medium is at rest, the electromagnetic constitutive equations of an isotropic medium are

$$\boldsymbol{D}^0 = \epsilon \boldsymbol{E}^0, \qquad \boldsymbol{B}^0 = \mu \boldsymbol{H}^0, \qquad \boldsymbol{J}^0 = \sigma \boldsymbol{E}^0, \tag{1.3}$$

where ϵ, μ are called the electric and the magnetic permeabilities, and σ is the electrical conductivity. The same equations are assumed to be valid at each point in the reference frame moving with the velocity of a material point, i.e., the *proper* frame, but they are expressed in terms of the field measured in the *laboratory* frame in which motion is observed. For small velocities the proper quantities are related to the laboratory ones by the equations (cf. [48])

$$\boldsymbol{E}^0 = \boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}, \qquad \boldsymbol{D}^0 = \boldsymbol{D} + c^{-2} \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H},$$

$$
\boldsymbol{H}^0 = \boldsymbol{H} - \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{D}, \qquad \boldsymbol{B}^0 = \boldsymbol{B} - c^{-2}\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{E},
$$
$$
\boldsymbol{J}^0 = \boldsymbol{J} - \rho_e \frac{\partial \boldsymbol{u}}{\partial t}, \qquad \rho_e^0 = \rho_e, \qquad c \equiv (\epsilon_0 \mu_0)^{-1/2},
$$

where ϵ_0, μ_0 are the dielectric and the magnetic permeabilities of the vacuum, and $\boldsymbol{u}$ is a displacement vector. Let us substitute these relations into constitutive equations (1.3). If the terms of order $|\frac{\partial \boldsymbol{u}}{\partial t}|^2/c^2$ and higher are dropped, the results of [48] are as follows:

$$
\boldsymbol{D} = \epsilon \boldsymbol{E} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}, \quad \boldsymbol{B} = \mu \boldsymbol{H} - \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{E}, \quad \alpha \equiv \epsilon\mu - \epsilon_0\mu_0, \tag{1.4}
$$
$$
\boldsymbol{J} = \rho_e \frac{\partial \boldsymbol{u}}{\partial t} + \sigma\Big(\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}\Big). \tag{1.5}
$$

For more details of electromagnetic theory, the reader is referred to many textbooks that treat this research field, e.g., [18, 21, 26], etc.

Thus, we have obtained a complete system for freely moving media. They are Maxwell's equations (1.1), (1.2) and the constitutive relations (1.4), (1.5). Equation (1.5) is a modification of Ohm's law, where appears a term reflecting the influence of particles moving in the magnetic field with a current density.

The electromagnetic matching conditions are obtained in the following manner. First rewrite equations (1.1), (1.2) in the equivalent form

$$
\begin{aligned}
&\operatorname{rot}\Big(\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}\Big) = -\frac{\partial \boldsymbol{B}}{\partial t} - \frac{\partial \boldsymbol{u}}{\partial t}\operatorname{div}\boldsymbol{B} + \operatorname{rot}\Big(\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}\Big),\\
&\operatorname{rot}\Big(\boldsymbol{H} - \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{D}\Big) = \frac{\partial \boldsymbol{D}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial t}\operatorname{div}\boldsymbol{D} - \operatorname{rot}\Big(\frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{D}\Big) + \boldsymbol{J} - \rho_e\frac{\partial \boldsymbol{u}}{\partial t},
\end{aligned} \tag{1.6}
$$
$$
\operatorname{div}\boldsymbol{B} = 0, \qquad \operatorname{div}\boldsymbol{D} = \rho_e. \tag{1.7}
$$

Then integral analogues of these equations can be obtained by integrating (1.6) over the surface S' composed of material particles and bounded by a curve C and (1.7) over a volume V of material particles bounded by the surface S. Note that C, S', S, and V move with the material. After applying Stokes' theorem on the left-hand sides of (1.6), we obtain

$$
\begin{aligned}
&\int_C \Big(\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}\Big)\cdot d\boldsymbol{c} = -\frac{d}{dt}\int_{S'} \boldsymbol{B}\cdot d\boldsymbol{s}',\\
&\int_C \Big(\boldsymbol{H} - \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{D}\Big)\cdot d\boldsymbol{c} = \frac{d}{dt}\int_{S'} \boldsymbol{D}\cdot d\boldsymbol{s}' + \int_{S'}\Big(\boldsymbol{J} - \rho_e\frac{\partial \boldsymbol{u}}{\partial t}\Big)\cdot d\boldsymbol{s}',
\end{aligned} \tag{1.8}
$$

Applying the Gauss–Ostrogradskii theorem to (1.7) yields

$$
\int_S \boldsymbol{B}\cdot d\boldsymbol{s} = 0, \qquad \int_S \boldsymbol{D}\cdot d\boldsymbol{s} = \int_V \rho_e\, dx, \tag{1.9}
$$

where we have also used the well-known relation

$$\frac{d}{dt}\int_{S'} \boldsymbol{F}\cdot d\boldsymbol{S}' = \int_{S'}\Big[\frac{\partial \boldsymbol{F}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial t}\operatorname{div}\boldsymbol{F} - \operatorname{rot}\Big(\frac{\partial \boldsymbol{u}}{\partial t}\times\boldsymbol{F}\Big)\Big]\, d\boldsymbol{S}'.$$

Select S' to be a small rectangular area oriented perpendicular to the discontinuity surface such that one side lies in the part of material with one material properties and other one lies in the part with another material properties (Figure 1.1).

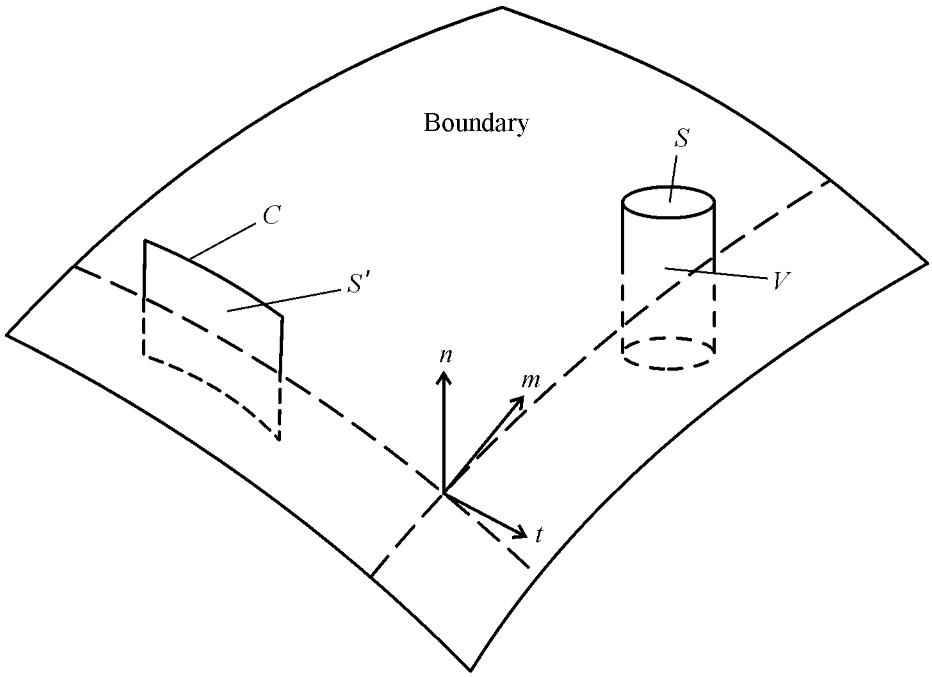

Figure 1.1

As the dimension of S', perpendicular to the boundary, tends to zero, equations (1.8) now look like

$$\Big[\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t}\times\boldsymbol{B}\Big]_t = 0, \qquad \Big[\boldsymbol{H} - \frac{\partial \boldsymbol{u}}{\partial t}\times\boldsymbol{D}\Big]_t = J_m^S - \rho_e^S\frac{\partial u_m}{\partial t}, \qquad (1.10)$$

where the symbol $[\boldsymbol{F}]_t$ means a jump of the tangential components of the vector $\boldsymbol{F}$ across the surface, where the coefficients of equations have breaks, and J_m^S, ρ_e^S represent the surface current and the charge, respectively. Here and in the sequel the subscripts t, m, n denote the vector components in the directions $\boldsymbol{t}, \boldsymbol{m}, \boldsymbol{n}$ which form the right-hand orthogonal triad (see Figure 1.1). Now, let us choose V to be a small cylindrical volume, whose axis is perpendicular to the discontinuity surface such that one of the circular ends lies in the part of the material with one material property and another one lies in the part with another material property (see Figure 1.1). As the height of the volume tends to zero, equations (1.9) take the form:

$$[\boldsymbol{B}]_n = 0, \qquad [\boldsymbol{D}]_n = \rho_e^S, \tag{1.11}$$

where $[\boldsymbol{F}]_n$ means a jump in the normal component of $\boldsymbol{F}$.

Equations (1.10), (1.11) constitute the complete electromagnetic matching conditions on a discontinuity surface.

1.2. Elastic theory

Consider now the equations of motion of a deformable medium. The mechanical equations will be derived by applying the conservation of momentum to the volume of a material, V, with the bounding surface S in the absence of a mechanical force, using an assumption that only a mechanical effect of the electromagnetic fields is the introduction of the Lorentz force

$$\boldsymbol{f}^e = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}.$$

Thus the equation of global conservation of momentum in the rectangular coordinates is the following

$$\int_S T \cdot \boldsymbol{n}\, ds + \int_V \boldsymbol{f}^e\, dx = \frac{d}{dt}\int_V \boldsymbol{g}^m\, dx, \tag{1.12}$$

where T is a stress tensor and $\boldsymbol{g}^m$ is momentum per unit volume. Using the Gauss–Ostrogradskii theorem for the surface integral and differentiation of the volume integral according to [17, Eq. 20.9],

$$\frac{d}{dt}\int_V \boldsymbol{g}^m\, dv = \int_V \Big(\frac{\partial \boldsymbol{g}^m}{\partial t} + \operatorname{Div}\Big(\boldsymbol{g}^m \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big)\Big)\, dx \tag{1.13}$$

we obtain

$$\int_V \Big(\operatorname{Div} T + \boldsymbol{f}^e - \operatorname{Div}\Big(\boldsymbol{g}^m \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big) - \frac{\partial \boldsymbol{g}^m}{\partial t}\Big)\, dx = 0,$$

where

$$\operatorname{Div} T = \Big(\sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij}\Big)_{i=1}^{3}.$$

If the mechanical momentum is locally conserved, then

$$\operatorname{Div} T + \boldsymbol{f}^e = \operatorname{Div}\Big(\boldsymbol{g}^m \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big) + \frac{\partial \boldsymbol{g}^m}{\partial t}. \tag{1.14}$$

In an elastic solid $\boldsymbol{g}^m = \rho \frac{\partial \boldsymbol{u}}{\partial t}$, where ρ is the material density, and the assumption of infinitesimal strains and rotations (1.14) reduces to

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \boldsymbol{f}^e. \tag{1.15}$$

The elastic matching conditions on stress are obtained by applying (1.15) to appropriate differential elements. Introduce Maxwell's stresses T^e and the electromagnetic momentum $\boldsymbol{g}^e$ according to Minkowski (cf. [44]),

$$T^e = \boldsymbol{E} \otimes \boldsymbol{D} + \boldsymbol{H} \otimes \boldsymbol{B} - \frac{1}{2}(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})I, \qquad \boldsymbol{g}^e = \boldsymbol{D} \times \boldsymbol{B},$$

where I is the unit matrix of order 3×3. Let us show that the Lorentz force $\boldsymbol{f}^e$ can be represented in the following form

$$\boldsymbol{f}^e = \operatorname{Div} T^e - \frac{\partial \boldsymbol{g}^e}{\partial t}. \tag{1.16}$$

It is easy to check the correctness of formula (1.16) taking into account equations (1.1), (1.2) and constitutive relations

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H}.$$

After simple transformations we come to the equation

$$\operatorname{Div} T = \rho_e \boldsymbol{E} + \operatorname{rot} \boldsymbol{E} \times \boldsymbol{D} + \operatorname{rot} \boldsymbol{H} \times \boldsymbol{B}.$$

Using Maxwell's equations (1.1), (1.2) we obtain

$$\operatorname{Div} T = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} + \frac{\partial \boldsymbol{D}}{\partial t} \times \boldsymbol{B} - \frac{\partial \boldsymbol{B}}{\partial t} \times \boldsymbol{D}$$

from which follows the next formula:

$$\operatorname{Div} T = \rho_e \boldsymbol{E} + \boldsymbol{J} \times VB + \frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) = \boldsymbol{f}^e + \frac{\partial \boldsymbol{g}^e}{\partial t}.$$

This formula proves the representation (1.16).

The volume integral containing $\boldsymbol{f}^e$ can be written down as

$$\int_V \boldsymbol{f}^e dx = \int_S \Big(T + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big) \cdot \boldsymbol{n}\, ds - \int_V \Big(\frac{\partial \boldsymbol{g}^e}{\partial t} + \operatorname{Div}\Big(\boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big)\Big)\, dx,$$

where $\operatorname{Div}(\boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t})$ was added and subtracted, and the Gauss–Ostrogradskii theorem was used to convert the volume integral to the surface one. Using this expression and (1.13) in (1.12) gives us

$$\int_S \Big(T + T^e + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big) \cdot \boldsymbol{n}\, ds = \frac{d}{dt}\int_V \Big(\boldsymbol{g}^m + \boldsymbol{g}^e\Big)\, dx, \tag{1.17}$$

which is the form appropriated for obtaining the matching conditions on surface fractions.

Now let S and V be the surface and the volume of a small cylindrical element, whose axis is perpendicular to the discontinuity surface such that one end of the cylinder lies in the part of a material with certain material properties and another one lies in the part with other material properties (see Figure 1.1). Applying (1.17) to this cylindrical region and allowing the axial dimension to approach zero, (1.17) becomes

$$\Big[T + T^e + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big] \cdot \boldsymbol{n} = 0. \tag{1.18}$$

In the case of a body surrounded by vacuum $T = 0$ outside the body and (1.18) reduces to

$$T \cdot \boldsymbol{n} = -\Big[T^e + \boldsymbol{g}^e \otimes \frac{\partial \boldsymbol{u}}{\partial t}\Big] \cdot \boldsymbol{n} \quad \text{on} \quad \Omega,$$

where Ω is the body surface.

The mechanical constitutive equations are taken to be the usual Hook's Law for an isotropic elastic medium, i.e.

$$T = \lambda \operatorname{tr} S \cdot I + 2\varkappa S, \tag{1.19}$$

where S is the strain tensor defined by the formula

$$S_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3.$$

In the above formulas, $\lambda, \varkappa$ are the Lamé coefficients. When using these relations it is assumed that the stresses and strains for this combined system in proper and laboratory frames are the same. Due to the fact that the system has been split to two parts, the mechanical part and the electromagnetic part, as expressed by the Minkowski energy-momentum tensor, this question needs further consideration. For the present purposes we simply assume that constitutive relations (1.19) for a purely elastic medium are unaffected by the electromagnetic fields. For very large fields or finite deformations the interaction terms will enter the constitutive relations thereby coupling together the elastic and the electromagnetic constitutive equations [18].

1.3. Summary of equations and matching conditions

Here we summarize the basic field equations and matching conditions for an electromagnetoelastic medium.

Field equations:

$$\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} = \operatorname{rot} \boldsymbol{H}, \qquad \operatorname{div} \boldsymbol{D} = \rho_e,$$
$$\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \qquad \operatorname{div} \boldsymbol{B} = 0, \tag{1.20}$$

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}. \tag{1.21}$$

Constitutive equations:

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}, \quad \boldsymbol{B} = \mu \boldsymbol{H} - \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{E}, \quad \alpha \equiv \epsilon\mu - \epsilon_0 \mu_0,$$
$$\boldsymbol{J} = \rho_e \frac{\partial \boldsymbol{u}}{\partial t} + \sigma\left(\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}\right), \tag{1.22}$$

$$T = \lambda \operatorname{tr} S \cdot I + 2\varkappa S, \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3. \tag{1.23}$$

Matching conditions:

$$\left[\boldsymbol{E} + \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{B}\right]_t = 0, \qquad \left[\boldsymbol{H} - \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{D}\right]_t = J_m^S - \rho_e^S \frac{\partial u_m}{\partial t},$$
$$[\boldsymbol{B}]_n = 0, \qquad [\boldsymbol{D}]_n = \rho_e^S, \tag{1.24}$$

$$\left[T + \boldsymbol{E} \otimes \boldsymbol{D} + \boldsymbol{H} \otimes \boldsymbol{B} - \frac{1}{2}\Big(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\Big) I + \Big(\boldsymbol{D} \times \boldsymbol{B}\Big) \otimes \frac{\partial \boldsymbol{u}}{\partial t}\right] \cdot \boldsymbol{n} = 0. \tag{1.25}$$

Chapter 2

Direct problems

In this chapter, we present some results of the solution of direct problems for the system of equations describing linear and nonlinear processes of the interaction of electromagnetic and elastic waves based on motion of particles.

2.1. The Cauchy problem for the electromagnetoelasticity equations for weakly conducting media

In this section, following the work [41], we present some results of solution of the Cauchy problem for a system of equations describing the linear interaction process of electromagnetic and elastic waves in a weakly conducting elastic medium. The linear interaction of electromagnetic field with an elastic isotropic medium based on motion of particles is described by the following equations:

$$\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} + \boldsymbol{j} = \operatorname{rot} \boldsymbol{H}, \qquad \frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \tag{2.1}$$

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \mu \boldsymbol{J} \times \boldsymbol{h}^0 + \boldsymbol{f}, \tag{2.2}$$

where the vectors $\boldsymbol{j}$ and $\boldsymbol{f}$ characterize the external source of currents and the external source of elastic oscillations. In this section, $\boldsymbol{j}, \boldsymbol{f}$ are supposed to be distributions with finite supports and

$$(\boldsymbol{j}, \boldsymbol{f}) \equiv 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}_-, \tag{2.3}$$

where $\mathbb{R}_- = \{t \in \mathbb{R} \mid t < 0\}$. The defining relations for the stress tensor T and components of the electric induction $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$ for an elastic isotropic inhomogeneous space are as follows:

$$\begin{gathered} \boldsymbol{D} = \varepsilon \boldsymbol{E} + \alpha \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{h}^0, \qquad \boldsymbol{J} = \sigma \boldsymbol{E} + \sigma\mu \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{h}^0, \qquad \boldsymbol{B} = \mu \boldsymbol{H}, \\ T = \lambda \cdot \operatorname{tr} S \cdot I + 2\varkappa S, \\ S: \quad S_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3. \end{gathered} \tag{2.4}$$

Here $\boldsymbol{h}^0$ is the magnetic intensity vector characterizing the external constant magnetic field. In this case we assume that μ is a positive constant and $\boldsymbol{h}^0$ is a constant nonzero vector. The material properties of a medium are described by the smooth bounded functions

$$\varepsilon, \rho, \lambda, \varkappa: \ \mathbb{R}^3 \to \mathbb{R}_+, \qquad \sigma: \ \mathbb{R}^3 \to \overline{\mathbb{R}}_+.$$

For system (2.1)–(2.4), we consider the Cauchy problem with the initial data

$$(\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{u},) \equiv 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_-. \tag{2.5}$$

We shall treat the solution of the Cauchy problem (2.1)–(2.5) as a generalized function defined over the space of infinitely differentiable compactly supported functions. Let us now make a general note about terminology. Usually, system (2.1) is supplemented with the two equations

$$\operatorname{div} \boldsymbol{B} = 0, \qquad \operatorname{div} \boldsymbol{D} = \rho_e. \tag{2.6}$$

According to the classical theory, it is system (2.1), (2.6) which is used as a system of Maxwell's equations. However, it is often used to regard system (2.1) as an independent object, ignoring equations (2.6). This treatment is based on the following reasons. The first equation in (2.6) is a direct corollary of (2.1); so it is fulfilled for any solution to problem (2.1)–(2.5). The second equation in (2.6) can be naturally considered as an independent equation for determining the charge density ρ_e, but this problem is beyond our interest here. At the same time, the electric strength vector $\boldsymbol{E}$ can be found from (2.1)–(2.5). Thus, equations (2.1) are the major and quite independent part of Maxwell's equations.

System (2.1)–(2.4) arises in result of linearization of a more complicated nonlinear system (1.20)–(1.23). These equations show that interaction of elastic medium with electromagnetic field is uniliteral interaction for nonconductive media ($\sigma = 0$): the displacement vector $\boldsymbol{u}$ is independent of electromagnetic field, but at the same time the vectors $\boldsymbol{E}, \boldsymbol{H}$ are dependent on the vector $\boldsymbol{u}$. The assumption about weak electrical conductivity of the medium ($\sigma \approx 0$) makes it possible to linearize the original equations (2.1)–(2.5) with respect to σ calculating the Frechèt derivative on $\sigma = 0$.

Let

$$\boldsymbol{H} = \boldsymbol{H}^0 + \boldsymbol{H}^1, \qquad \boldsymbol{E} = \boldsymbol{E}^0 + \boldsymbol{E}^1, \qquad \boldsymbol{u} = \boldsymbol{u}^0 + \boldsymbol{u}^1, \tag{2.7}$$

where $(\boldsymbol{H}^0, \boldsymbol{E}^0, \boldsymbol{u}^0)$ is the solution of the Cauchy problem (2.1)–(2.5) with $\sigma \equiv 0$:

$$\varepsilon \frac{\partial \boldsymbol{E}^0}{\partial t} + \alpha \frac{\partial^2 \boldsymbol{u}^0}{\partial t^2} \times \boldsymbol{h}^0 + \boldsymbol{j} = \operatorname{rot} \boldsymbol{H}^0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{2.8}$$

$$\mu \frac{\partial \boldsymbol{H}^0}{\partial t} + \operatorname{rot} \boldsymbol{E}^0 = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{2.9}$$

$$\rho\frac{\partial^2\boldsymbol{u}^0}{\partial t^2} - \operatorname{Div} T(\boldsymbol{u}^0) = \boldsymbol{f}, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}, \tag{2.10}$$

$$(\boldsymbol{H}^0, \boldsymbol{E}^0, \boldsymbol{u}^0) \equiv 0, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}_-, \tag{2.11}$$

Thus, the problem corresponding to linearization with respect to σ in the neighborhood of $\sigma\equiv 0$ has the following form:

$$\varepsilon\frac{\partial\boldsymbol{E}^1}{\partial t} + \alpha\frac{\partial^2\boldsymbol{u}^1}{\partial t^2}\times\boldsymbol{h}^0 + \sigma\boldsymbol{E}^0 + \mu\sigma\frac{\partial\boldsymbol{u}^0}{\partial t}\times\boldsymbol{h}^0 = \operatorname{rot}\boldsymbol{H}^1, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}, \tag{2.12}$$

$$\mu\,\frac{\partial\boldsymbol{H}^1}{\partial t} + \operatorname{rot}\boldsymbol{E}^1 = 0, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}, \tag{2.13}$$

$$\rho\frac{\partial^2\boldsymbol{u}^1}{\partial t^2} - \operatorname{Div} T(\boldsymbol{u}^1) - \sigma\mu\Big(\boldsymbol{E}^0 + \mu,\frac{\partial\boldsymbol{u}^0}{\partial t}\times\boldsymbol{h}^0\Big)\times\boldsymbol{h}^0 = 0, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}, \tag{2.14}$$

$$(\boldsymbol{H}^1, \boldsymbol{E}^1, \boldsymbol{u}^1) \equiv 0, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}_-, \tag{2.15}$$

where as $T(\boldsymbol{u}^k)$, $k=0,1$, we mark the value of the stress tensor on the functions $\boldsymbol{u}^k$, $k=0,1$, respectively.

Thus, the interaction of electromagnetic field with a weakly conductive elastic medium is described by equations (2.7)–(2.15). Obtained in such way system is the basic subject of our investigation in this section.

Problem statement and basic result

The basic question, which we are going to study here is the structure of the solution to the Cauchy problem (2.7)–(2.15). First of all we are interested in singular part of the Cauchy problem solution, and as well that regular part, which defined discontinuities on characteristic cones of the solution. The structure of the solution of the Cauchy problem (2.7)–(2.15) will be studied under the condition that the vector-functions $\boldsymbol{j}$ and $\boldsymbol{f}$ have the form

$$\boldsymbol{j} = \boldsymbol{j}^0\,\delta(x-x^0,t), \qquad \boldsymbol{f} = \boldsymbol{f}^0\,\delta(x-x^0,t),$$

where $\boldsymbol{j}^0$ and $\boldsymbol{f}^0$ are constant vectors and $x^0\in\mathbb{R}^3$ is a fixed point.

Let c_1, c_2, and c_3 be velocities of electromagnetic, longitudinal, and transverse elastic waves, respectively:

$$c_1 = \sqrt{\frac{1}{\varepsilon\mu}}, \qquad c_2 = \sqrt{\frac{\lambda+2\varkappa}{\rho}}, \qquad c_3 = \sqrt{\frac{\varkappa}{\rho}}. \tag{2.16}$$

For each of them we introduce the Riemannian metric with the length element $d\tau_k$ defined by the formula

$$d\tau_k = c_k(x)\,ds, \quad k = 1, 2, 3, \tag{2.17}$$

where ds is the length element in the Euclidean metric. Let us denote by $\Gamma_k(x^0, x)$ the geodesic of metric (2.17) connecting the points x^0 and x and by $\tau_k(x^0, x)$ its length. It is well-known that $\tau_k(x^0, x)$ as a function of the variable x satisfies the relations

$$|\operatorname{grad}_x \tau_k(x^0, x)| = c_k^{-2}(x), \quad \tau_k(x^0, x) = O(|x - x^0|) \quad \text{for} \quad x \to x^0. \tag{2.18}$$

In what follows we will assume that each of metrics (2.17) is simple, i.e., each pair of points x^0, x is connected by one and only one geodesic $\Gamma_k(x^0, x)$. In addition, we assume that $c_k(x)$, $k = 1, 2, 3$, satisfy the conditions

$$0 < m_3 \le c_3(x) < c_2(x) < c_1(x) \le M_1 < \infty, \tag{2.19}$$

where m_3 and M_1 are constants defined by the formulas

$$m_3 = \inf_{x \in \mathbb{R}^3} c_3, \qquad M_1 = \sup_{x \in \mathbb{R}^3} c_1.$$

Let $\theta_0(t)$ be the Heaviside function:

$$\theta_0(t) = 1 \quad \text{for} \quad t \ge 0, \qquad \theta_0(t) = 0 \quad \text{for} \quad t < 0.$$

Introduce the functions

$$\begin{gathered} \theta_n(t) = \frac{t^n}{n!}\theta_0(t), \qquad \theta_{-n}(t) = \frac{d^n}{dt^n}\theta_0(t), \quad n = 1, 2, 3, \ldots, \\ S_k = S_k(x, t, x^0) \equiv t - \tau_k(x^0, x), \quad k = 1, 2, 3. \end{gathered} \tag{2.20}$$

The differentiation in (2.20) is understood in the sense of the theory of distributions. The equalities $S_k = 0$, $k = 1, 2, 3$, define the characteristic cones corresponding to the velocities c_k, $k = 1, 2, 3$.

We now introduce new vector functions $\boldsymbol{V}$, $\boldsymbol{V}^0$, $\boldsymbol{V}^1$, defined by the formulas

$$\boldsymbol{V} = \boldsymbol{V}^0 + \boldsymbol{V}^1, \quad \boldsymbol{V}^k = (\boldsymbol{V}^{1k}, \boldsymbol{V}^{2k}, \boldsymbol{V}^{3k}) \equiv (\boldsymbol{H}^k, \boldsymbol{E}^k, \boldsymbol{u}^k), \quad k = 0, 1.$$

The following theorem holds.

Theorem 2.1. *Let there be a certain number $\delta_0 > 0$ such that the coefficients ε, σ, ρ, λ, and $\varkappa$ are constants in the domain*

$$D_0 = \left\{ x \in \mathbb{R}^3 \;\middle|\; |x - x^0| < \delta_0 \right\}$$

and belong to $C^m(\mathbb{R}^3)$, $m > 3(N + 10) + 10$, for a certain integer $N \ge 5$. Then the solution to the Cauchy problem (2.7)–(2.15) can be represented in the form

$$\boldsymbol{V}^{ik}(x,t) = \sum_{n=-2}^{N} \{\boldsymbol{\alpha}_n^{ik}(x)\,\theta_n(S_1) + \boldsymbol{\beta}_n^{ik}(x)\,\theta_n(S_2) + \boldsymbol{\gamma}_n^{ik}(x)\,\theta_n(S_3)\} + \boldsymbol{V}_N^{ik}(x,t), \quad i=1,2,3; \quad k=0,1, \tag{2.21}$$

where $\boldsymbol{V}_N^{ik}(x,t) \in C^N(\mathbb{R}^3 \times \mathbb{R})$ *and the coefficients* $\boldsymbol{\alpha}_n^{ik}(x)$, $\boldsymbol{\beta}_n^{ik}(x)$, *and* $\boldsymbol{\gamma}_n^{ik}(x)$ *have the following properties:*

a) $\boldsymbol{\alpha}_{-2}^{3k}(x) = \boldsymbol{\beta}_{-2}^{3k}(x) = \boldsymbol{\gamma}_{-2}^{3k}(x) \equiv 0$, $k=0,1$,

b) $\boldsymbol{\alpha}_n^{ik}(x)$, $\boldsymbol{\beta}_n^{ik}(x)$, *and* $\boldsymbol{\gamma}_n^{ik}(x)$ *are analytic as real-valued functions in the domain* $D_0 \setminus \{x^0\}$ *and smooth outside of* D_0 *(more precisely, these are functions of the class* $C^{m-2n-8}(\mathbb{R}^3 \setminus D_0)$*). Moreover, there exists a positive constant* $C>0$ *depending only on the values of the coefficients* ε, μ, σ, ρ, λ, *and* $\varkappa$ *in the domain* D_0 *and on the values of* $|\boldsymbol{j}^0|$, $|\boldsymbol{f}^0|$, *and* $|\boldsymbol{h}^0|$ *such that*

$$\big(|\boldsymbol{\alpha}_n^{i0}|, |\boldsymbol{\beta}_n^{i0}|, |\boldsymbol{\gamma}_n^{i0}|\big) \le C \cdot \begin{cases} |x-x^0|^{-(3+n)}, & i=1,2, \\ |x-x^0|^{-(2+n)}, & i=3; \end{cases} \tag{2.22}$$

$$\big(|\boldsymbol{\alpha}_n^{i1}|, |\boldsymbol{\beta}_n^{i1}|, |\boldsymbol{\gamma}_n^{i1}|\big) \le C \cdot \begin{cases} |x-x^0|^{-(2+n)}, & i=1,2, \\ |x-x^0|^{-(1+n)}, & i=3; \end{cases} \tag{2.23}$$

c) *in the domain* $\{(x,t) \mid S_3 > 0,\ x \in D_0\}$, *the functions*

$$\widetilde{\boldsymbol{V}}_N^{ik}(x,t) = \sum_{n=-2}^{N} \{\boldsymbol{\alpha}_n^{ik}(x)\,\theta_n(S_1) + \boldsymbol{\beta}_n^{ik}(x)\,\theta_n(S_2) + \boldsymbol{\gamma}_n^{ik}(x)\,\theta_n(S_3)\}$$

satisfy the estimates

$$\widetilde{\boldsymbol{V}}_N^{10} = \widetilde{\boldsymbol{V}}_N^{30} \equiv 0, \quad |\widetilde{\boldsymbol{V}}_N^{2k}| \le C\,|x-x^0|^{-3}, \quad k=0,1,$$
$$|\widetilde{\boldsymbol{V}}_N^{11}| \le C\,|x-x^0|^{-2}, \quad |\widetilde{\boldsymbol{V}}_N^{31}| \le C\,|x-x^0|^{-1}$$

with the same constant C *as in* (2.22) *and* (2.23).

Remark 2.1. For the correctness of representation (2.21) at some fixed point x^1 for values $t \le T$, for any $T > \tau_1(x^0,x^1)$, it is sufficient, by virtue of hyperbolic system (2.7)–(2.15), to suppose the condition of simplicity of the metrics (2.17), and the conditions of smoothness of the coefficients be valid in a finite domain of the space $\mathbb{R}^3$ bounded by the Riemannian ellipsoid

$$\tau_1(x^0,x) + \tau_1(x,x^1) = T.$$

Representation (2.21) is also correct for all the points satisfying the inequalities

$$\tau_1(x^0,x) \le t \le T - \tau_1(x,x^1).$$

Remark 2.2. Apparently, the above theorem is valid without hypothesis about homogeneity of medium in the domain D_0.

The above theorem is a basis for formulating inverse problems for weakly conducting media.

Sketch of the proof

The basic idea of the proof of the theorem is the following. We construct for a homogeneous medium the solution to problem (2.7)–(2.15) having the form of (2.21) with $N = 5, \boldsymbol{V}_N^{ik} \equiv 0$. The thus obtained solution is equal to the required one for $t < t_0$, where $t_0 = \delta_0/M_1$, by virtue of the hyperbolic system and an assumption about constancy of the coefficients in the theorem. By this reason we can consider the problem for $t > t_0$ only. Representing for this case the functions $\boldsymbol{V}^{ik}(x,t)$ in the form of (2.21) with $N \geq 5$, for $\alpha_n^{ik}, \beta_n^{ik}, \gamma_n^{ik}$ a system of algebraic and ordinary differential equations along geodesic metrics (2.17) is obtained. The solution to such equations is selected from the condition of their coincidence with the solution constructed in the domain D_0. As a result of such action we obtain for the functions $\boldsymbol{V}_N^{ik}(x,t)$ the Cauchy problem with zero initial data for $t < t_0$ and a smooth right-hand side. Using the method of energetic estimates enables us attain necessary smoothness of the functions $\boldsymbol{V}_N^{i}(x,t)$. The proof utilizes the methods developed in [39, 49] for investigation of the structure of fundamental solutions to the Cauchy problem for hyperbolic equations, as well as for the Lamè and Maxwell's systems.

For the complete proof the reader is referred to the original work [41].

2.2. Initial boundary-value problem for the electromagnetoelasticity equations with partially nonlinear interaction

In this section, following the original work Lorenzi and Priimenko [27], we present some results of solution of the first initial boundary-value problem for the electromagnetoelasticity system in the case when the nonlinear term describing the interaction of the electromagnetic and the elastic fields is presented in Maxwell's system only.

We will consider one possible statement of the problem which arises in the theory of electromagnetoelasticity under the following assumptions:

1. Ω_1, Ω_2, and Ω are three bounded connected open sets in $\mathbb{R}^3$ such that Ω_2 and Ω belong to the classes C^3 and C^2, respectively, and the following conditions are fulfilled: $\overline{\Omega}_2 \Subset \Omega$, $\Omega_1 = \Omega \setminus \overline{\Omega}_2$.

2. An oscillating inhomogeneous isotropic electrical-conducting elastic body B, which occupies the domain $\Omega_2 \subset \mathbb{R}^3$, is placed into the domain Ω where the process of propagation of electromagnetic waves occurs.
3. The electromagnetic field arises as a result of propagation of elastic oscillations. Moreover, we neglect the reverse effect of the electromagnetic field on the process of elastic waves propagation.
4. We neglect the transport currents in the domain Ω.
5. The motion of the medium occurs with velocities which are lower than those of electromagnetic waves in the elastic medium.

By virtue of the previous assumptions the constitutive relations (1.22) take the form

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \qquad \boldsymbol{B} = \mu \boldsymbol{H}, \qquad \boldsymbol{J} = \sigma \Big[\boldsymbol{E} + \mu \frac{\partial \widetilde{\boldsymbol{u}}}{\partial t} \times \boldsymbol{H} \Big],$$

where $\widetilde{\boldsymbol{u}}$ is an continuation of the function $\boldsymbol{u}$ by zero over the whole of the domain $(0,T) \times \Omega$, $T > 0$. Using the relations obtained, we can write down the Maxwell system (1.20) in the domain $(0,T) \times [\Omega_1 \cup \Omega_2]$ in the form

$$\begin{aligned} &\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + \sigma \boldsymbol{E} + \sigma\mu \frac{\partial \widetilde{\boldsymbol{u}}}{\partial t} \times \boldsymbol{H} = \operatorname{rot} \boldsymbol{H}, \\ &\mu \frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \qquad \operatorname{div} \mu \boldsymbol{H} = 0. \end{aligned} \tag{2.24}$$

According to assumptions 2 and 3, the propagation of elastic waves in the body B is governed by the ordinary system of the Lamé equations

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \boldsymbol{f}, \quad (t,x) \in (0,T) \times \Omega_2, \tag{2.25}$$

where $\rho : \Omega_2 \to \mathbb{R}_+$ and $\boldsymbol{f}, \boldsymbol{u} : (0,T) \times \Omega_2 \to \mathbb{R}^3$, and the stress tensor T is defined by formula (1.23) with $\lambda, \varkappa : \Omega_2 \to \mathbb{R}_+$. In this section, we assume the function $\boldsymbol{f}$ to have the representation $\boldsymbol{f}(t,x) = f(t)\,\boldsymbol{g}(t,x)$, where $\boldsymbol{g} : [0,T] \times \Omega_2 \to \mathbb{R}^3$ and $f : [0,T] \to \mathbb{R}$ are known functions.

Our main problem consists in determining the functions $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{u}$. To this end, we need to supplement differential equations (2.24) and (2.25) with appropriate initial and boundary conditions and with the gluing conditions for the solution of the problem on the surfaces, where the coefficients of differential equations have breaks.

Now we can formulate the direct problem.

Direct Problem 2.1. *Determine a set of the functions*

$$\boldsymbol{u} : [0,T] \times \Omega_2 \to \mathbb{R}^3, \quad \boldsymbol{E}, \boldsymbol{H} : [0,T] \times \Omega \to \mathbb{R}^3,$$

such that

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + f(t)\boldsymbol{g}(t,x), \quad (t,x) \in (0,T) \times \Omega_2, \tag{2.26}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \quad \frac{\partial \boldsymbol{u}}{\partial t}(0,x) = \boldsymbol{u}_1(x), \quad x \in \Omega_2, \tag{2.27}$$

$$\boldsymbol{u}(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega_2, \tag{2.28}$$

$$\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} + \sigma \boldsymbol{E} + \sigma\mu \frac{\partial \widetilde{\boldsymbol{u}}}{\partial t} \times \boldsymbol{H} = \operatorname{rot} \boldsymbol{H}, \quad (t,x) \in (0,T) \times [\Omega_1 \cup \Omega_2], \tag{2.29}$$

$$\mu \frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \quad \operatorname{div} \mu \boldsymbol{H} = 0, \quad (t,x) \in (0,T) \times [\Omega_1 \cup \Omega_2], \tag{2.30}$$

$$\boldsymbol{E}(0,x) = \boldsymbol{E}_0(x), \quad \boldsymbol{H}(0,x) = \boldsymbol{H}_0(x), \quad x \in \Omega, \tag{2.31}$$

$$\boldsymbol{n} \times \boldsymbol{E} = 0, \quad (t,x) \in (0,T) \times \partial\Omega, \tag{2.32}$$

$$[\boldsymbol{E} \times \boldsymbol{n}]_{\partial\Omega_2} = [\boldsymbol{H} \times \boldsymbol{n}]_{\partial\Omega_2} = 0, \quad (t,x) \in (0,T) \times \partial\Omega_2. \tag{2.33}$$

It is assumed that the functions $\varepsilon, \mu : \overline{\Omega} \to \mathbb{R}_+$, $\sigma : \overline{\Omega} \to \overline{\mathbb{R}}_+$, and $\boldsymbol{E}_0, \boldsymbol{H}_0 : \Omega \to \mathbb{R}^3$ are continuous in the domain $\overline{\Omega} \setminus \partial\Omega_2$ with possible jumps on the surface $\partial\Omega_2$. We also assume that the functions $f : [0,T] \to \mathbb{R}$, $\boldsymbol{g} : [0,T] \times \Omega_2 \to \mathbb{R}^3$, $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega_2 \to \mathbb{R}^3$ are given and have sufficient smoothness.

As is easy to see, the solution of Direct Problem 2.1 can be divided into two steps because we can separate the solution of problem (2.26)–(2.28) from the solution of problem (2.29)–(2.33) (the coupling term is presented in Maxwell's system only). For this reason we divide this process into two ones: the first – to solve the direct problem (2.26)–(2.28), and the second – to solve the direct problem (2.29)–(2.33).

Solution of direct problem (2.26)–(2.28)

It is worth noting that the term $\partial\widetilde{\boldsymbol{u}}/\partial t \times \boldsymbol{H}$ occurring in equation (2.26) creates certain difficulties in the course of solution of direct problem (2.26)–(2.28). In fact, we cannot confine ourselves to consideration of the weak solution $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{u})$ of the problem since in this case the product $\partial\widetilde{\boldsymbol{u}}/\partial t \times \boldsymbol{H}$ may fail to be an element of the space $L^2(\Omega_2; \mathbb{R}^3)$. In order that imposing too severe constraints on the function be avoided we should require at least, that both the multipliers $\partial\widetilde{\boldsymbol{u}}/\partial t$ and $\boldsymbol{H}$ be elements of the space $L^4(\Omega_2; \mathbb{R}^3)$. For this, we assume the density ρ, the Lamé coefficients λ and $\varkappa$, the free term $f\boldsymbol{g}$, and the initial data $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ of problem (2.26)–(2.28) to satisfy the following conditions:

$$\rho \in H^2(\Omega_2;\mathbb{R}), \qquad \varkappa, \lambda \in W^{2,\infty}(\Omega_2;\mathbb{R}), \tag{2.34}$$

$$\min\big(\rho(x), \lambda(x), \varkappa(x)\big) \geq \rho_0 > 0, \quad \forall x \in \overline{\Omega}_2\,, \tag{2.35}$$

$$f \in L^p((0,T);\mathbb{R}), \quad \boldsymbol{g} \in L^{p'}\big((0,T); H^2(\Omega_2;\mathbb{R}^3)\big) \,\cap\, H^1_0(\Omega_2;\mathbb{R}^3), \tag{2.36}$$

$$\boldsymbol{U}_0 \in H^3(\Omega_2;\mathbb{R}^3) \cap H^1_0(\Omega_2;\mathbb{R}^3), \qquad \operatorname{Div} T \in H^1_0(\Omega_2;\mathbb{R}^3), \tag{2.37}$$

$$\boldsymbol{U}_1 \in H^2(\Omega_2;\mathbb{R}^3) \cap H^1_0(\Omega_2;\mathbb{R}^3). \tag{2.38}$$

Here $1/p + 1/p' = 1$.

Theorem 2.2. *Let ρ, λ, $\varkappa$, f, $\boldsymbol{g}$, $\boldsymbol{u}_0$, and $\boldsymbol{u}_1$ satisfy conditions* (2.34)–(2.38)*. Then there exists a unique solution $\boldsymbol{u}(f)$ to problem* (2.26)–(2.28)*. This solution satisfies the conditions*

$$\begin{aligned}\boldsymbol{u}(f) \in C\big([0,T]; H^3(\Omega_2;\mathbb{R}^3)\big) \cap H^1_0(\Omega_2;\mathbb{R}^3) \cap \\ C^1\big([0,T]; H^2(\Omega_2;\mathbb{R}^3)\big) \cap W^{2,1}\big((0,T); H^1(\Omega_2;\mathbb{R}^3)\big),\end{aligned} \tag{2.39}$$

$$\begin{aligned}&\Big(\|\boldsymbol{u}(f)(t)\|^2_{3,2} + \Big\|\frac{\partial}{\partial t}\boldsymbol{u}(f)(t)\Big\|^2_{2,2} + \Big\|\frac{\partial^2}{\partial t^2}\boldsymbol{u}(f)(t)\Big\|^2_{1,2}\Big)^{1/2} \\ &\quad \leq C_1\big(\rho_0^{-1}, \|\rho\|_{2,2}, \|\varkappa\|_{2,\infty}, \|\lambda\|_{2,\infty}\big) \times \\ &\qquad \Big[\big(\|\boldsymbol{u}_0\|^2_{3,2} + \|\boldsymbol{u}_1\|^2_{2,2}\big)^{1/2} + \|\boldsymbol{g}\|_{t,0,p',2,2} \cdot \|f\|_{t,0,p}\Big], \\ &\qquad\qquad \forall t \in [0,T], \quad \forall f \in L^p((0,T);\mathbb{R}),\end{aligned} \tag{2.40}$$

$$\begin{aligned}&\Big(\|\boldsymbol{u}(f_2)(t) - \boldsymbol{u}(f_1)(t)\|^2_{3,2} + \Big\|\frac{\partial}{\partial t}\boldsymbol{u}(f_2)(t) - \frac{\partial}{\partial t}\boldsymbol{u}(f_1)(t)\Big\|^2_{2,2} + \\ &\Big\|\frac{\partial^2}{\partial t^2}\boldsymbol{u}(f_2)(t) - \frac{\partial^2}{\partial t^2}\boldsymbol{u}(f_1)(t)\Big\|^2_{1,2}\Big)^{1/2} \\ &\quad \leq C_1\big(\rho_0^{-1}, \|\rho\|_{2,2}, \|\varkappa\|_{2,\infty}, \|\lambda\|_{2,\infty}\big)\|\boldsymbol{g}\|_{t,0,p',2,2} \cdot \|f_2 - f_1\|_{t,0,p}, \\ &\qquad\qquad \forall t \in [0,T], \quad \forall f_1, f_2 \in L^p((0,T);\mathbb{R}).\end{aligned} \tag{2.41}$$

Here $\|\cdot\|_{j,2}$, $\|\cdot\|_{t,0,q,j,2}$, and $\|\cdot\|_{t,0,p}$ are the norms in the spaces $H^j(\Omega_2;\mathbb{R}^3)$, $L^q((0,t); H^j(\Omega_2;\mathbb{R}^3))$, and $L^p((0,t);\mathbb{R})$, respectively, and C_1 is a nonnegative function continuous and nondecreasing in each of its arguments.

Solution of direct problem (2.29)–(2.33)

Now we are able to solve direct problem (2.29)–(2.33). We make the following assumptions about the coefficients and the initial data of the problem:

$$\begin{aligned}\varepsilon, \mu, \sigma \in W^{1,\infty}(\Omega_1;\mathbb{R}) \cap W^{1,\infty}(\Omega_2;\mathbb{R}), \\ \min\big\{\varepsilon(x), \mu(x)\big\} \geq \gamma^{-1} > 0, \quad \forall x \in \Omega_1 \cup \Omega_2;\end{aligned} \tag{2.42}$$

$$\begin{aligned}\boldsymbol{E}_0 \in H(\mathrm{rot},\Omega), \quad \boldsymbol{H}_0 \in H(\mathrm{rot},\Omega) \cap H^1(\Omega_1;\mathbb{R}^3) \cap H^1(\Omega_2;\mathbb{R}^3), \\ \mu\boldsymbol{H}_0 \in H(\mathrm{div};\Omega);\end{aligned} \tag{2.43}$$

$$\boldsymbol{n} \times \boldsymbol{E}_0 = 0, \quad x \in \partial\Omega; \qquad \operatorname{div} \mu \boldsymbol{H}_0 = 0, \quad x \in \Omega;$$
$$\boldsymbol{n} \cdot \boldsymbol{H}_0 = 0, \quad x \in \partial\Omega. \tag{2.44}$$

The following theorem is valid under these assumptions.

Theorem 2.3. *Let the vector-functions $\boldsymbol{g}$, $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{E}_0$, and $\boldsymbol{H}_0$ satisfy conditions* (2.36)–(2.38), (2.43), *and* (2.44). *Then for every function $f \in L^p((0,T);\mathbb{R})$ problem* (2.29)–(2.33) *has a unique solution $(\boldsymbol{E}, \boldsymbol{H}) = (\boldsymbol{E}(f), \boldsymbol{H}(f))$ satisfying the conditions*

$$\boldsymbol{E}(f) \in C\big([0,T]; H(\mathrm{rot};\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega;\mathbb{R}^3)\big), \tag{2.45}$$

$$\boldsymbol{H}(f) \in C\big([0,T]; H(\mathrm{rot};\Omega) \cap H^1(\Omega_2;\mathbb{R}^3)\big) \cap C^1\big([0,T]; L^2(\Omega;\mathbb{R}^3)\big), \tag{2.46}$$

$$\Big\|\frac{\partial}{\partial t}\boldsymbol{E}(f)(t)\Big\|_{0,2,\Omega} + \Big\|\frac{\partial}{\partial t}\boldsymbol{H}(f)(t)\Big\|_{0,2,\Omega} + \|\mathrm{rot}\, \boldsymbol{E}(f)(t)\|_{0,2,\Omega} +$$
$$\|\boldsymbol{E}(f)(t)\|_{0,2,\Omega} + \|\boldsymbol{H}(f)(t)\|_{1,2,\Omega_1} + \|\boldsymbol{H}(f)(t)\|_{1,2,\Omega_2}$$
$$\leq C_2(T) + TC_3(T, \|f\|_{T,0,p}), \quad \forall t \in (0,T), \tag{2.47}$$

where C_2 and C_3 are positive nondecreasing continuous functions depending also on the norms of the data of the problem. Moreover, for every pair of the functions $f_1, f_2 \in L^p((0,T);\mathbb{R})$ the following estimate holds:

$$\Big\|\frac{\partial}{\partial t}\boldsymbol{E}(f_2)(t) - \frac{\partial}{\partial t}\boldsymbol{E}(f_1)(t)\Big\|_{0,2,\Omega} + \Big\|\frac{\partial}{\partial t}\boldsymbol{H}(f_2)(t) - \frac{\partial}{\partial t}\boldsymbol{H}(f_1)(t)\Big\|_{0,2,\Omega} +$$
$$\|\mathrm{rot}\, \boldsymbol{E}(f_2)(t) - \mathrm{rot}\, \boldsymbol{E}(f_1)(t)\|_{0,2,\Omega} + \|\boldsymbol{E}(f_2)(t) - \boldsymbol{E}(f_1)(t)\|_{0,2,\Omega} +$$
$$\|\boldsymbol{H}(f_2)(t) - \boldsymbol{H}(f_1)(t)\|_{1,2,\Omega_1} + \|\boldsymbol{H}(f_2)(t) - \boldsymbol{H}(f_1)(t)\|_{1,2,\Omega_2}$$
$$\leq C_4(T, \|f_1\|_{T,0,p}) \int_0^t h(f_2)(t-s)\, \|f_2 - f_1\|_{s,0,p}\, ds, \tag{2.48}$$

where $h(f)(t) = \exp\big[t\,\big(\gamma\,\|\sigma\|_{0,\infty,\Omega} + C_5(T)\,\|f\|_{T,0,p} \cdot \|g\|_{T,0,p',2,2,\Omega_2}\big)\big]$ and C_4 and C_5 are positive nondecreasing continuous functions depending also on the norms of the data of the problem.

The proofs of Theorems 2.2 and 2.3 are rather bulky and therefore are omitted. For their complete proofs the reader is referred to the original paper [27].

2.3. Initial boundary-value problem for the electromagnetoelasticity equations with complete nonlinear interaction

In this section, we will present some results of solution of the first initial boundary-value problem for a system of electromagnetoelasticity, when nonlinear terms describing the interaction of electromagnetic and elastic fields are presented in both Maxwell's and the Lamè systems.

Basic equations

Consider the case of diffusion approximation of Maxwell's system. This means that in the field equations (1.20), (1.21) we neglect by displacement current $\frac{\partial \boldsymbol{D}}{\partial t}$ formally assuming $\epsilon = 0$, and set $\rho_e = 0$. Simultaneously we put in constitutive equations (1.22), (1.23) $\alpha = 0$ and $\rho_e = 0$, too. It is easy to show that in this case in the presence of external electromagnetic $\boldsymbol{j}$ and elastic $\boldsymbol{f}$ sources of oscillations we can form the following electromagnetoelasticity system

$$\begin{aligned}
&\sigma \boldsymbol{E} + \sigma\mu \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H} + \boldsymbol{j} = \operatorname{rot} \boldsymbol{H}, \\
&\mu \frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{rot} \boldsymbol{E} = 0, \quad \operatorname{div} \mu \boldsymbol{H} = 0, \\
&\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + \mu \operatorname{rot} \boldsymbol{H} \times \boldsymbol{H} + \boldsymbol{f}.
\end{aligned}$$

We make the following assumptions about the functions $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{u}, \boldsymbol{j}, \boldsymbol{f}$:

$$\begin{aligned}
\boldsymbol{E} = (0,1,0)E(z,t), \quad \boldsymbol{H} = (1,0,0)H(z,t), \quad \boldsymbol{u} = (0,0,1)u(z,t), \\
\boldsymbol{j} = (0,1,0)j(z,t), \qquad \boldsymbol{f} = (0,0,1)f(z,t),
\end{aligned} \tag{2.49}$$

where the variable z stands for the variable x_3. Under such assumptions for the case $\rho = \text{const}, \mu = \text{const}$ we can form the following non-dimensional model system (cf. [3]):

$$h_t = (rh_z)_z - (hu_t)_z - (rj)_z, \qquad u_{tt} = (\nu^2 u_z)_z - phh_z + f,$$

where h, u, j, f are dimensionless analogues of the functions, introduced by formulas (2.49), $r^{-1} = \mu L V_0 \sigma$ is the magnetic Reynolds number, $p = \mu H_0^2 \rho^{-1} V_0^{-2}$, $\nu = \sqrt{(\lambda + 2\varkappa)/\rho V_0^2}$ is dimensionless velocity of the elastic waves propagation; and L, V_0, H_0 are characteristic values of length, seismic velocity and magnetic field, respectively.

The problem statement

Now we are able to formulate the basic problem to be studied. Consider the equations

$$h_t = (rh_z)_z - (hu_t)_z - (rj)_z, \quad (z,t) \in Q_T, \tag{2.50}$$

$$u_{tt} = (\nu^2 u_z)_z - phh_z + f, \quad (z,t) \in Q_T, \tag{2.51}$$

where $Q_T = \Omega \times (0,T)$, $\Omega = (-l,l)$. The functions r, ν, f, j are supposed to be smooth functions with possible jumps in points z_m: $-l < z_1 < z_2 <$

$\ldots < z_m < l$, $r(z) \geq r_0 > 0$; $\nu(z) \geq \nu_0 > 0$, p is a positive number. The following initial boundary value problem is considered for equations (2.50), (2.51) with the initial conditions

$$h(z,0) = h_0(z), \quad u(z,0) = u_0(z), \quad u_t(z,0) = u_1(z), \qquad z \in \Omega, \quad (2.52)$$

and the boundary conditions

$$h(-l,t) = h(l,t) = 0, \quad u(-l,t) = u(l,t) = 0, \qquad t \in (0,T). \quad (2.53)$$

The initial boundary value problem (2.50)–(2.53) can be considered as a diffraction problem for parabolic-hyperbolic system (2.50), (2.51), i.e., as a problem in the cylinder Q_T consisting of several media. The following transmission conditions on the boundaries of such media were assumed: *continuity of the solution and its derivatives in co-normal directions to discontinuous surfaces*. In our problem, the discontinuous surfaces are the lines $z = z_i$, $i = 1, 2, \ldots, m$, in the cylinder Q_T. These conditions mean the physical absence of discontinuities of a medium and equilibrium of the effective forces on discontinuous surfaces. Mathematically we can form the following transmission conditions at the points of discontinuity of the coefficients:

$$\begin{gathered} [h(z,t)]_{z=z_i} = 0, \quad [r(z)(h_z(z,t) - j(z,t))]_{z=z_i} = 0, \\ [u(z,t)]_{z=z_i} = 0, \quad [\nu^2(z) u_z(z,t)]_{z=z_i} = 0, \quad i = 1, \ldots, m. \end{gathered} \quad (2.54)$$

For studying this problem we will make use of the fact that any diffraction problem can be considered as a generalized solution of an initial boundary-value problem with discontinuous coefficients [25, Chapter III, p. 224–232].

To introduce the generalized solution of the initial boundary-value problem (2.50)–(2.54) we need some functional spaces.

The Banach space $L_q(\Omega)$ consists of all measurable functions on Ω that are the qth-power ($q \geq 1$) summable on Ω provided with the norm $\|v\|_{q,\Omega} = (\int_\Omega |v(z)|^q dz)^{1/q}$. Measurability and summability are to be understood in the sense of Lebesgue.

The Banach space $L_{q,r}(Q_T)$, $q, r \geq 1$, consists of all measurable on Q_T functions with the finite norm $\|v\|_{q,r,Q_T} = (\int_0^T (\int_\Omega |v(z,t)|^q dz)^{\frac{r}{q}} dt)^{1/r}$. In the case $q = r$, the Banach space $L_{q,q}(Q_T)$ will be denoted by $L_q(Q_T)$, and the norm $\|v\|_{q,q,Q_T}$ – by $\|v\|_{q,Q_T}$.

The generalized derivatives are understood in the sense accepted in the theory of generalized functions [43, 47].

The Banach space $W_q^l(\Omega)$ of all the functions from $L_q(\Omega)$ has generalized derivatives up to order l (integers inclusively), that are the qth-power summable on Ω. The norm in $W_q^l(\Omega)$ is defined by the equality

$$\|v\|_{q,\Omega}^{(l)} = \sum_{j=0}^{l} \|D_z^s v\|_{q,\Omega}^{l},$$

$\overset{\circ}{W}{}_q^l(\Omega)$ is a subspace of $W_q^l(\Omega)$ in which the set of all functions that are infinitely differentiable and finite in Ω is dense.

The Banach space $W_q^{2l,l}(Q_T)$ for l integral $(q \geq 1)$ of all $L_q(Q_T)$-elements has generalized derivatives $D_t^r D_z^s$, where the numbers r, s satisfies the inequality $2r + s \leq 2l$. The norm in $W_q^{2l,l}(Q_T)$ is defined in the following way:

$$\|v\|_{q,Q_T}^{(2l)} = \sum_{j=0}^{l} \sum_{2r+s=j} \|D_t^r D_z^s v\|_{q,Q_T}.$$

The Hilbert space $W_2^{1,k}(Q_T)$, $k = 0, 1$, has a scalar product defined by $(u, v)_{W_2^{1,k}(Q_T)} = \int_{Q_T} (uv + u_z v_z + k u_t v_t) dz dt$.

The Banach space $V_2(Q_T)$ of all $W_2^{1,0}(Q_T)$-elements has the finite norm

$$|v|_{Q_T} = \max_{0 \leq t \leq T} \|v\|_{2,\Omega} + \|v_z\|_{2,Q_T}$$

where

$$\|v_z\|_{2,Q_T} = \left(\int_{Q_T} v_z^2 dz dt \right)^{1/2}.$$

The Banach space $V_2^{1,0}(Q_T)$ is obtained from $W_2^{1,1}(Q_T)$ by closing in $V_2(Q_T)$-norm. $V_2^{1,1/2}(Q_T)$ is a subset of those $V_2^{1,0}(Q_T)$-elements, for which

$$\int_0^{T-\tau} \int_\Omega \tau^{-1} (v(z, t+\tau) - v(z,t))^2 \, dz \, dt \to 0 \,\text{as}\, \tau \to 0.$$

Zero over $W_2^{1,0}(Q_T), W_2^{1,1}(Q_T),\ V_2(Q_T),\ V_2^{1,0}(Q_T),\ V_2^{1,1/2}(Q_T)$ means that only those elements of these spaces are taken, which vanish on $S_T = \partial\Omega \times (0, T)$.

The space $C^{\alpha,\alpha/2}(Q_T)$ is a set of all the functions continuous in $\overline{Q}_T$ with the Hölder indices α by z and $\alpha/2$ by t [25, Chapter I, p. 2–10].

Now we can define the solution of problem (2.50)–(2.53).

Definition 2.1. The functions $h(z,t) \in \overset{\circ}{V}_2(Q_T), u(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ are called the generalized solution of the initial boundary value problem (2.50)–(2.54) if they satisfy the integral equalities

$$-\int_{Q_T} h\eta_t \, dz \, dt + \int_{Q_T} r h_z \eta_z \, dz \, dt - \int_{Q_T} h u_t \eta_z \, dz \, dt$$
$$= \int_{Q_T} r j \eta_z \, dz \, dt + \int_\Omega h_0(z) \eta(z, 0) \, dz, \qquad (2.55)$$

$$-\int_{Q_T} u_t\zeta_t\,dz\,dt + \int_{Q_T} \nu^2 u_z\zeta_z\,dz\,dt + \int_{Q_T} phh_z\zeta\,dz\,dt$$
$$= \int_{Q_T} f\zeta\,dz\,dt + \int_\Omega u_1(z)\zeta(z,0)\,dz, \qquad (2.56)$$

$u(z,0) = u_0(z)$, $z \in \Omega$, for any $\eta(z,t), \zeta(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ such that $\eta(z,T) = \zeta(z,T) = 0$.

The generalized solution of (2.50)–(2.54) can be done in another equivalent form.

Definition 2.2. The functions $h(z,t) \in \overset{\circ}{V}_2(Q_T)$, $u(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ are called the generalized solution of problem (2.50)–(2.54) if almost for all $t_1 \in [0,T]$ they satisfy the equalities

$$-\int_{Q_{t_1}} h\eta_t\,dz\,dt + \int_{Q_{t_1}} rh_z\eta_z\,dz\,dt - \int_{Q_{t_1}} hu_t\eta_z\,dz\,dt$$
$$= \int_{Q_{t_1}} rj\eta_z\,dz\,dt + \int_\Omega h_0(z)\eta(z,0)\,dz - \int_\Omega h(z,t_1)\eta(z,t_1)\,dz, \quad (2.57)$$
$$-\int_{Q_{t_1}} u_t\zeta_t\,dz\,dt + \int_{Q_{t_1}} \nu^2 u_z\zeta_z\,dz\,dt + \int_{Q_{t_1}} phh_z\zeta\,dz\,dt$$
$$= \int_{Q_{t_1}} f\zeta\,dz\,dt + \int_\Omega u_1\zeta(z,0)\,dz\,dt - \int_\Omega u_t(z,t_1)\zeta(z,t_1)\,dz, \qquad (2.58)$$

$u(z,0) = u_0(z)$, $z \in \Omega$, where $Q_{t_1} = \Omega \times (0,t_1)$, $\eta(z,t), \zeta(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$.

The equivalence of Definitions 2.1 and 2.2 was proved in [35]. The generalized solution is considered in the sense of the distributions theory and for this reason we should understand the fulfillment of the transmission conditions (2.54) in the sense of integral equalities (2.55), (2.56). For more details the reader is referred to [35].

Existence theorem

Let us prove the following existence theorem about the solvability of problem (2.50)–(2.54). For this purpose we assume that the functions r, ν, the free members f, j, the constant p and initial data h_0, u_0, u_1 in problem (2.50)–(2.54) enjoy the properties

a) the functions r, ν, f, j are supposed to be smooth functions with possible jumps in the points z_m: $-l < z_1 < z_2 < \ldots < z_m < l$, $r(z) \geq r_0 > 0$; $\nu(z) \geq \nu_0 > 0$ and p is a positive number;

b) $h_0 \in C^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$, $h_0(\pm l) = 0$, and $u_0 \in \overset{\circ}{W}{}_2^1(\Omega)$ and $u_1 \in L_2(\Omega)$.

Theorem 2.4. *In the conditions, formulated above, problem* (2.50)–(2.54) *has the solution*

$$h(z,t) \in \overset{\circ}{V}_2(Q_T), \quad u(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T).$$

Proof. Theorem 2.4 will be proved using the Bubnov–Galerkin method. Let us check that the functions h, u satisfy equalities (2.57), (2.58). Let us consider in $\overset{\circ}{W}{}_2^{1,1}(Q_T)$ a fundamental sequence of the functions $\psi_k(z)$, orthogonal in $L_2(\Omega)$, such that $(\psi_k, \psi_l) = \delta_{kl}$. The approximate solution will be constructed in the following form

$$h^N(z,t) = \sum_{k=1}^{N} a_k^N(t)\psi_k(z), \quad u^N(z,t) = \sum_{k=1}^{N} b_k^N(t)\psi_k(z),$$

where

$$a_k^N = (h^N, \psi_k), \quad b_k^N = (u^N, \psi_k), \quad k = 1, \ldots, N.$$

The functions a_k^N, b_k^N are determined from the conditions

$$\begin{aligned} &\frac{d}{dt}(h^N, \psi_k) = ((rh_z^N)_z - (u_t^N h^N)_z - (rj)_z, \psi_k), \\ &(h^N(z,0), \psi_k) = h_{0k}, \end{aligned} \tag{2.59}$$

$$\begin{aligned} &\frac{d^2}{dt^2}(u^N, \psi_k) = ((\nu^2 u_z^N)_z - ph^N h_z^N + f, \psi_k), \\ &(u^N(z,0), \psi_k) = u_{0k}, \quad \frac{d}{dt}(u^N(z,0), \psi_k) = u_{1k}. \end{aligned} \tag{2.60}$$

System (2.59), (2.60) is a nonlinear one of ordinary differential equations. Its solution exists on the interval $[0, \tau)$ and $\max_k(|a_k^N(t)|, |b_k^N(t)|) \to \infty$ when $t \to \tau$. We will prove $|a_k^N(t)|$, $|b_t^N(t)|$, $k = 1, \ldots, N$, to be bounded functions for $t \in [0, T]$, and for this reason system (2.59), (2.60) has a solution on the interval $[0, T]$ with any positive T.

Let us do the following transformations of equations (2.59), (2.60):

- multiply equalities (2.59) by $pa_k^N(t)$ and sum up the results for all $k = 1, \ldots, N$;
- multiply equalities (2.60) by $b_k^N(t)$ and sum up the results for all $k = 1, \ldots, N$;
- integrate the equalities obtained with respect to the variable t over the interval $(0, t_1), t_1 \leq T$, and sum up the final results.

The above steps give us the equality

$$\frac{1}{2}p\|h^N(z,t)\|^2_{2,\Omega}\Big|_{t=0}^{t=t_1} + \frac{1}{2}\|u_t^N(z,t)\|^2_{2,\Omega}\Big|_{t=0}^{t=t_1} + \frac{1}{2}\|\nu u_t^N(z,t)\|^2_{2,\Omega}\Big|_{t=0}^{t=t_1} + \|\sqrt{pr}h_z\|^2_{2,Q_{t_1}} = -\int_{Q_{t_1}} p(rj)_z h^N\,dz\,dt + \int_{Q_{t_1}} fu^N\,dz\,dt. \quad (2.61)$$

Note that

$$\|h^N(z,0)\|^2_{2,\Omega} = \sum_{k=1}^N a_k^2(0) \le \|h_0\|^2_{2,\Omega},$$

$$\|u_t^N(z,0)\|^2_{2,\Omega} = \sum_{k=1}^N b_k'^2(0) \le \|u_1\|^2_{2,\Omega},$$

$$\|\nu u_z^N(z,0)\|^2_{2,\Omega} \le \sum_{k=1}^N \nu_0^2 b_k^2(0) \le \nu_0^2\|u_0\|^2_{2,\Omega},$$

$$-\frac{1}{2}\int_{Q_{t_1}} pr[(h_z^N)^2 + 2h_z^N j + j^2]\,dz\,dt \le 0.$$

For this reason the following inequality is valid

$$\frac{1}{2}p\|h^N(z,t_1)\|^2_{2,\Omega} + \frac{1}{2}\|u_t^N(z,t_1)\|^2_{2,\Omega} + \frac{1}{2}\|\nu u_z^N(z,t_1)\|^2_{2,\Omega} + \frac{1}{2}\|\sqrt{pr}h_z^N\|^2_{2,Q_{t_1}} \le \mu_2 + |\int_{Q_{t_1}} fu_t^N\,dz\,dt|,$$

where

$$\mu_2 = \frac{1}{2}p\|h_0\|^2_{2,\Omega} + \frac{1}{2}\|u_1\|^2_{2,\Omega} + \frac{1}{2}\|\nu u_{0,z}\|^2_{2,\Omega} + \frac{1}{2}\int_{Q_{t_1}} prj^2\,dz\,dt.$$

In particular, we have

$$\frac{1}{2}\|u_t^N(z,t_1)\|^2_{2,\Omega} \le \mu_2 + \left|\int_{Q_{t_1}} fu_t^N\,dz\,dt\right|.$$

Let us integrate the latter inequality over the interval $[0,T]$

$$\frac{1}{2}\|u_t^N\|^2_{2,Q_T} \le \mu_2 T + \int_0^T \left|\int_{Q_{t_1}} fu_t^N\,dz\,dt\right| dt_1. \quad (2.62)$$

Note that

$$\left|\int_{Q_{t_1}} fu_t^N dzdt\right| \le \delta^2\int_{Q_T}(u_t^N)^2 dzdt + \frac{1}{2\delta^2}\int_{Q_T} f^2\,dz\,dt. \quad (2.63)$$

Setting $\delta^2 = 1/4T$ we have

$$\int_0^T \left|\int_{Q_{t_1}} fu_t^N\,dz\,dt\right| dt_1 \le \frac{1}{4}\int_{Q_T}(u_t^N)^2\,dz\,dt + 2T^2\int_{Q_T} f^2\,dz\,dt. \quad (2.64)$$

Then inequalities (2.62), (2.64) yield

$$\frac{1}{4}\|u_t^N\|^2_{2,Q_T} \le \mu_2 T + 2T\int_{Q_T} f^2\,dz\,dt$$

Formula (2.63) with $\delta^2 = 1/4T$ gives us

$$\begin{aligned}\left|\int_{Q_{t_1}} f u_t^N\,dz\,dt\right| &\le \frac{1}{4T}\int_{Q_T}(u_t^N)^2\,dz\,dt + 2T\int_{Q_T} f^2\,dz\,dt\\ &\le \mu_2 + 4T\int_{Q_T} f^2\,dz\,dt.\end{aligned}$$

Hence, from (2.61) the following inequality is obtained:

$$\begin{aligned}\frac{p}{2}\|h^N(z,t_1)\|^2_{2,\Omega} + \frac{1}{2}\|u_t^N(z,t_1)\|^2_{2,\Omega} + \frac{1}{2}\|\nu u_z^N(z,t_1)\|^2_{2,\Omega} + \frac{1}{2}\|\sqrt{pr}h_z^N\|^2_{2,Q_{t_1}}\\ \le 2\mu_2 + 4T\int_{Q_T} f^2\,dz\,dt \equiv \mu_1, \qquad (2.65)\end{aligned}$$

with the constant μ_1 independent of N. From (2.65) follows that all functions

$$a_k^N(t) = (h^N(z,t),\psi_k(z)),$$
$$b_k^N(t) = (u^N(z,t),\psi_k(z)), \quad b_k'^N(t) = (u_t^N(z,t),\psi_k(z))$$

are uniformly bounded on the interval $(0,t_1)$, $t_1 \le T$.

Let us show that they are equicontinuous functions on the interval $[0,T]$ for a fixed number k and any $N \ge k$. In fact, from (2.59) we can obtain

$$a_k^N(t+\Delta t) - a_k^N(t) = \int_{Q_{t,t+\Delta t}} [(rh_z^N)_z - (u_t^N h^N)_z - (rj)_z]\psi_k(z)\,dz\,dt,$$

where $Q_{t,t+\Delta t} = \Omega\times(t,t+\Delta t)$. To estimate the right-hand side one should use the following inequalities

$$\int_{Q_{t,t+\Delta t}} |v_1v_2v_3|\,dz\,dt \le \|v_1\|_{q_1,r_1,Q_{t,t+\Delta t}}\|v_2\|_{q_2,r_2,Q_{t,t+\Delta t}}\|v_3\|_{q_3,r_3,Q_{t,t+\Delta t}}, \quad (2.66)$$

$$q_i, r_i \in [1,\infty), \quad i=1,2,3, \quad \frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3} = 1, \quad \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} = 1;$$

$$\|v\|_{q,r,Q_{t,t+\Delta t}} \le \beta\frac{2}{r}\|v_z\|_{2,Q_{t,t+\Delta t}} + \beta\Big(1-\frac{2}{r}\Big)\,\text{vrai}\max_{0\le t\le T}|v|_{2,\Omega}, \qquad (2.67)$$

$$r\in[4,\infty), \quad q\in[2,\infty), \quad \frac{1}{r}+\frac{1}{2q} = \frac{1}{4};$$

$$\int_{Q_{t,t+\Delta t}} |rh_z^N\psi_{k,z}|\,dz\,dt \le C\|\psi_{k,z}\|_{2,Q_{t,t+\Delta t}}\|h_z^N\|_{2,Q_{t,t+\Delta t}}, \qquad (2.68)$$

$$\int_{Q_{t,t+\Delta t}} |u_t^N h^N \psi_{k,z}|\,dz\,dt \le \|\psi_{k,z}\|_{2,Q_{t,t+\Delta t}} \|u_t^N\|^2_{1,2,Q_{t,t+\Delta t}} \|h^N\|_{\infty,4,Q_{t,t+\Delta t}}. \tag{2.69}$$

Using the fact that integrals (2.68), (2.69) tend to zero when $\Delta t \to 0$, we obtain

$$|a_k^N(t+\Delta t) - a_k^N(t)| \le \epsilon(\Delta t)\|\psi_{k,z}\|_{2,\Omega}$$

with $\epsilon(\Delta t)$ independent of N and tending to zero when $\Delta t \to 0$, i.e., the equicontinuity of a_k^N, $N = k, k+1, \dots$, with respect to the variable t.

The uniform continuity of the functions b_k^N follows from the boundedness of their derivatives with respect to the variable t. By the usual diagonal process we can select a subsequence N_m, $m = 1, 2, \dots$, such that the functions $a_k^{N_m}$, $a_k^{N_m}$ be uniformly converging on $[0, T]$ to some continuous functions $a_k(t)$, $b_k(t)$. The functions a_k, b_k define the two functions

$$h = \sum_{k=1}^{\infty} a_k \psi_k, \qquad u = \sum_{k=1}^{\infty} b_k \psi_k.$$

The convergence of the functions $h_k^{N_m}$ to the function h is a weak $L_2(\Omega)$-convergence and uniform on $[0, T]$. In fact, for any $\psi(z) \in L_2(\Omega)$ we have

$$(h^{N_m} - h, \psi) = \sum_{k=1}^{s} (\psi, \psi_k)(h^{N_m} - h, \psi_k) + \left(\sum_{k=s+1}^{\infty} (h^{N_m} - h, (\psi, \psi_k)\psi_k \right),$$

with

$$\left| \left(h^{N_m} - h, \sum_{k=s+1}^{\infty} (\psi, \psi_k)\psi_k \right) \right| \le C' \left(\sum_{k=s+1}^{\infty} (\psi, \psi_k) \right)^{1/2} \equiv C'R(s),$$

where C' is independent of s. Let ϵ be a small positive number. We can choose a number s so small that $C'R(s) < \epsilon/2$, and a number N so big that

$$\left| \sum_{k=1}^{s} (\psi, \psi_k)(h^{N_m} - h, \psi_k) \right| < \epsilon/2.$$

As a result we obtain $|(h^{N_m} - h, \psi)| < \epsilon$ for all $t \in [0, T]$. Thus it has been shown that the sequence h^{N_m} has a weak converge in $L_2(\Omega)$ uniformly by $t \in [0, T]$.

Note that u^{N_m} are the bounded functions in $L^\infty(0, T; \overset{\circ}{W}{}_2^1(\Omega))$ and $u_t^{N_m}$ are the bounded ones in $L^\infty(0, T; L_2(\Omega))$. For this reason $u^{N_m} \to u$ $*$-weakly in $L^\infty(0, T; \overset{\circ}{W}{}_2^1(\Omega))$ but $u_t^{N_m} \to u_t$ $*$-weakly in $L^\infty(0, T; L_2(\Omega))$. The functions u^{N_m} belong to $\overset{\circ}{W}{}_2^{1,1}(Q_T)$ and as $\overset{\circ}{W}{}_2^{1,1}(Q_T) \hookrightarrow L_2(Q_T)$ we can say that

$u^{N_m} \to u$ strongly in $L_2(Q_T)$ almost everywhere. It follows from estimate (2.65) that one can extract from the sequence h^{N_m} a subsequence converging to h weakly in $L_2(Q_T)$ together with $h_z^{N_m}$. Without loss of generality we can assume that the sequences h^{N_m}, u^{N_m} tend to h, u in the above mentioned sense. By well-known properties of weak convergence inequality (2.65) is valid for the limiting functions h, u, too.

Let us now check that the functions h, u satisfy equalities (2.55), (2.56). First, we will show that the function h satisfies inequality (2.55). Multiply equality (2.59) with a smooth function $\alpha_k(t)$ equalled to zero for $t = T$, sum up the equalities obtained with all k from 1 up to $N' \le N$, and integrate the final result with respect to the variable t over the interval $[0, T]$. After integration by parts we obtain

$$\int_0^T (h^N, \Phi_t^{N'})\,dt = \int_0^T \Big[(rh_z^N, \Phi_z^{N'}) - (u_t^N h^N, \Phi_z^{N'}) - (rj, \Phi_z^{N'})\Big]\,dt, \quad (2.70)$$

where $\Phi^{N'}(z,t) = \sum_{k=1}^{N'} \alpha_k(t)\psi_k(z)$. In particular, it is possible in equality (2.70) to pass to a limit with the selected above subsequences h^{N_m}, u^{N_m} with fixed $\Phi^{N'}$ and to obtain equality (2.70) with h, u instead of h^N, u^N. The possibility of passing to the limit in $(u_t^N h^N, \Phi_z^{N'})$ follows from (2.65)–(2.69) and a weak convergence proved earlier. Taking advantage of Lemma 4.12 from [25, Chapter II, p. 89] we obtain the set of the functions $\Phi^{N'}$ is dense one in the space of all functions Φ used in the definition of generalized solution. By this reason, h satisfies equality (2.55) and belongs to $\overset{\circ}{V}_2(Q_T)$. In addition the following inequality is valid

$$\max_{Q_T} |h(z,t)| \le C, \qquad h(z,t) \in C^{\alpha,\alpha/2}(\overline{Q}_T).$$

Really, consider equality (2.55) separately from (2.56) and apply Theorems 7.1 and 10.1 from [25, Chapter III, p. 181, 204]. For application of these theorems it is necessary to fulfil the condition

$$u_t^2 \in L_{1,2/(1-2\kappa)}(Q_T), \qquad \kappa \in \Big(0, \frac{1}{2}\Big),$$

which follows from estimate (2.65). The set of the functions $\Phi^{N'}$ is dense one in $\overset{\circ}{W}{}_2^{1,1}(Q_T)$, i.e., $h(z,t)$ satisfies equality (2.55) and is the generalized solution from $\overset{0}{V}_2\,(Q_T)$.

Repeating all the transformations which were made with equality (2.59) with the function h for equality (2.60), we arrive at:

$$\int_0^T (u_t^N, \Phi_t^{N'})\,dt = \int_0^T \Big[(\nu^2 u_z^N, \Phi_z^{N'}) - (ph^N h_z^N, \Phi^{N'}) + (f, \Phi^{N'})\Big]\,dt + \int_\Omega u_1^N \Phi^{N'} dz, \qquad u^N(z,0) = u_0^N(z). \quad (2.71)$$

Considering the functions $\Phi^{N'}$ as fixed ones pass in the latter equality to the limit with respect to the above-selected sequence. For this reason we obtain equality (2.71) with the functions h, u instead of h^N, u^N

$$\int_0^T (u_t, \Phi_t^{N'})\, dt = \int_0^T \Big[(\nu^2 u_z, \Phi_z^{N'}) - (phh_z, \Phi^{N'}) + (f, \Phi^{N'})\Big]\, dt$$
$$\int_\Omega u_1 \Phi^{N'} dz, \qquad u(z,0) = u_0(z).$$

Note that since $\max_{Q_T} |h| \le C$, then $\int_0^T (phh_z, \Phi^{N'})\, dt$ is a bounded integral for any $\Phi^{N'} \in \overset{0}{W}{}_2^{1,1}(Q_T)$ and, as a set of the functions $\Phi^{N'}$ is dense in the space considered in the definition of a generalized solution (see Lemma 4.12 from [25, Chapter II, p. 89]), we obtain the function $u(z,t)$ satisfying equality (2.56) and being the generalized solution from $\overset{\circ}{W}{}_2^{1,1}(Q_T)$.

Thus, the existence theorem about solvability of problem (2.50)–(2.54) has been proved. □

Remark 2.3. Applying Lemma 4.1 from [25, Chapter III, p. 158] and Theorem 2.4 it is easy to show that any generalized solution $h(z,t)$ of problem (2.50)–(2.54) from $\overset{\circ}{V}_2(Q_T)$ belongs to $\overset{\circ}{V}{}_2^{1,1/2}(Q_T)$, too.

Uniqueness theorem

The proof of the uniqueness theorem is based on *a priori* estimate.

Lemma 2.1. *Let $h(z,t) \in \overset{\circ}{V}{}_2^{1,1/2}(Q_T), u(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ be the generalized solution of problem (2.50)–(2.54). Then the following inequality is valid for almost all $t_1 \in [0,T]$*

$$\frac{1}{2}\int_\Omega \{ph^2(z,t) + u_t^2(z,t) + \nu^2(z)u_z^2(z,t)\}\, dz\Big|_{t=0}^{t=t_1} + \frac{1}{2}\int_{Q_{t_1}} prh_z^2\, dz\, dt$$
$$\le \left(\frac{1}{2} + t_1\right)\int_{Q_{t_1}} prj^2\, dz\, dt + t_1^2 \int_{Q_{t_1}} f^2\, dz\, dt, \tag{2.72}$$

where $Q_{t_1} = \Omega \times (0, t_1)$.

Proof. Let h, u be the generalized solution of problem (2.50)–(2.54), i.e., (see formulas (2.55), (2.56))

$$-\int_{Q_T} h\eta_t\, dz\, dt + \int_{Q_T} rh_z\eta_z\, dz\, dt - \int_{Q_T} hu_t\eta_z\, dz\, dt$$
$$= \int_{Q_T} rj\eta_z\, dz\, dt + \int_\Omega h_0(z)\eta(z,0)dz,$$

$$-\int_{Q_T} u_t\zeta_t\,dz\,dt + \int_{Q_T} \nu^2 u_z\zeta_z\,dz\,dt + \int_{Q_T} phh_z\zeta\,dz\,dt$$
$$= \int_{Q_T} f\zeta\,dz\,dt + \int_\Omega u_1(z)\zeta(z,0)\,dz.$$

Consider the test functions $\hat\eta_{\bar k}(z,t)$ and $\hat\zeta_{\bar k}(z,t)$ defined by the formulas

$$\hat\eta_{\bar k} = \frac{1}{k}\int_{t-k}^{t} \hat\eta(z,\tau)\,d\tau, \qquad \hat\zeta_{\bar k} = \frac{1}{k}\int_{t-k}^{t} \hat\zeta(z,\tau)\,d\tau,$$

where $\hat\eta(z,t), \hat\zeta(z,t) \in W_2^{1,1}(Q_{-k,T})$, and $\hat\eta(z,t) = \hat\zeta(z,t) = 0$ for $t \in [-k,0]\cup [T-k,T]$, $Q_{-k,T} = \Omega\times(-k,T)$. Then we have

$$-\int_{Q_T} h\hat\eta_{\bar k,t}\,dz\,dt = -\int_{Q_T} h_k\hat\eta_t\,dz\,dt = \int_{Q_T} h_{k,t}\hat\eta\,dz\,dt.$$

To determine the latter equalities there was used the relation

$$\int_0^T h\hat\eta_{\bar k}\,dt = \int_0^{T-h} h_k\hat\eta\,dt,$$

which is valid for any piecewise summable on $[-k,T]$ functions $h, \hat\eta$ such that one of them is equal to zero for $t \in [-k,0]\cup[T-k,T]$ and $h_k = \frac{1}{k}\int_t^{t+k} h(z,\tau)d\tau$. In a similar way we obtain

$$-\int_{Q_T} u_t\hat\zeta_{\bar k,t}\,dz\,dt = \int_{Q_T} u_{k,tt}\hat\zeta\,dz\,dt.$$

In all other terms we can carry over the average $(\cdot)_{\bar k}$ from the functions $\hat\eta, \hat\zeta$ on others ones. Taking into account the permutability of such an average with differentiation with respect to the variable z we obtain the following equalities

$$\int_{Q_{T-k}} h_{k,t}\hat\eta\,dz\,dt + \int_{Q_{T-k}} (rh_z - hu_t - rj)_k\hat\eta_z\,dz\,dt = 0, \tag{2.73}$$

$$\int_{Q_{T-k}} u_{k,tt}\hat\zeta\,dz\,dt + \int_{Q_{T-k}} \nu^2 u_{k,z}\hat\zeta_z\,dz\,dt + \int_{Q_{T-k}} p(hh_z)_k\hat\zeta\,dz\,dt$$
$$= \int_{Q_{T-k}} f\hat\zeta\,dz\,dt. \tag{2.74}$$

Note that these equalities are valid for the functions $\hat\eta$, $\hat\zeta$ from a set of functions being wider than the above-considered one; namely, for any functions $\hat\eta$, $\hat\zeta$ which are equal to zero for $t \geq t_1$ and coincide with functions $\eta \in \overset{\circ}{V}{}_2^{1,0}(Q_{t_1})$, $\zeta \in \overset{\circ}{W}{}_2^{1,1}(Q_{t_1})$ for $t \in [0,t_1]$, where $0 \leq t_1 \leq T-k$. This fact was proved in [35].

Let us take $\eta = h_k$, $\zeta = u_{k,t}$ for $t \in [0,t_1]$ and rewrite in (2.73), (2.74) the corresponding terms in the following form

$$\int_{Q_{t_1}} h_{k,t} h_k \, dz \, dt = \frac{1}{2} \int_{\Omega} h_k^2 \, dz \Big|_{t=0}^{t=t_1},$$
$$\int_{Q_{t_1}} u_{k,tt} u_{k,t} \, dz \, dt = \frac{1}{2} \int_{\Omega} u_{k,t}^2 \, dz \Big|_{t=0}^{t=t_1},$$
$$\int_{Q_{t_1}} \nu^2 u_{k,z} u_{k,zt} \, dz \, dt = \frac{1}{2} \int_{\Omega} \nu^2 u_{k,z}^2 \, dz \Big|_{t=0}^{t=t_1}.$$

Using the formulas obtained and letting $k \to 0$, from (2.73), (2.74) we can obtain the equality

$$\frac{1}{2} \int_{\Omega} (ph^2 + u_t^2 + \nu^2 u_z^2) \, dz \Big|_{t=0}^{t=t_1} + \int_{Q_{t_1}} prh_z^2 \, dz \, dt$$
$$= \int_{Q_{t_1}} prjh_z \, dz \, dt + \int_{Q_{t_1}} f u_t \, dz \, dt.$$

Note that

$$\int_{Q_{t_1}} pr \left(\frac{1}{2} h_z^2 - jh_z + \frac{1}{2} j^2 \right) dz \, dt = \frac{1}{2} \int_{Q_{t_1}} pr(h_z - j)^2 \, dz \, dt \geq 0.$$

For this reason the following estimate is valid

$$\frac{1}{2} \int_{\Omega} (ph^2 + u_t^2 + \nu^2 u_z^2) \, dz \Big|_{t=0}^{t=t_1} + \frac{1}{2} \int_{Q_{t_1}} prh_z^2 \, dz \, dt$$
$$\leq \frac{1}{2} \int_{Q_{t_1}} prj^2 \, dz \, dt + \left| \int_{Q_{t_1}} f u_t \, dz \, dt \right|. \tag{2.75}$$

Integrating inequality (2.75) over $[0, t_1]$ we obtain the inequality:

$$\int_{Q_{t_1}} u_t^2 \, dz \, dt \leq \frac{1}{2} \int_{Q_{t_1}} prj^2 \, dz \, dt + t_1 \epsilon \int_{Q_{t_1}} u_t^2 \, dz \, dt + \frac{t_1}{2\epsilon} \int_{Q_{t_1}} f^2 \, dz \, dt.$$

Substituting the latter inequality with $t_1 \epsilon = 1/2$ in (2.75) gives us

$$\frac{1}{2} \int_{\Omega} (ph^2 + u_t^2 + \nu^2 u_z^2) \, dz \Big|_{t=0}^{t=t_1} + \frac{1}{2} \int_{Q_{t_1}} prh_z^2 \, dz \, dt$$
$$\leq \frac{1}{2} \int_{Q_{t_1}} prj^2 \, dz \, dt + t_1 \int_{Q_{t_1}} prj^2 \, dz \, dt + t_1^2 \int_{Q_{t_1}} f^2 \, dz \, dt$$
$$= \left(\frac{1}{2} + t_1 \right) \int_{Q_{t_1}} prj^2 \, dz \, dt + t_1^2 \int_{Q_{t_1}} f^2 \, dz \, dt. \qquad \square$$

Using estimate (2.72) we can show similar to [35] that

$$\max_{Q_T} |h(z,t)| \le C_0, \qquad h(z,t) \in C^{\alpha,\,\alpha/2}(\overline{Q}_T),$$
$$\|u(z,t)\|_{\overset{\circ}{W}{}_2^{1,1}(Q_T)} \le C_1. \tag{2.76}$$

Let us now prove the uniqueness theorem about solvability of problem (2.50)–(2.54).

Theorem 2.5. *Problem* (2.50)–(2.54) *cannot have more than one generalized solution.*

Proof. Let $h_n(z,t)$, $u_n(z,t)$, $n = 1,2$, be two solutions of problem (2.50)–(2.54). It follows from the above-mentioned arguments that both these solutions satisfy estimates (2.72), (2.76).

Introduce two functions $v(z,t)$, $w(z,t)$ by the formulas

$$v(z,t) = (h_2(z,t) - h_1(z,t))e^{-\lambda t}, \quad w(z,t) = (u_2(z,t) - u_1(z,t))e^{-\lambda t},$$

where λ is a positive number. Then the functions v, w will be the generalized solution of the problem

$$v_t + \lambda v = (r v_z)_z - (h_2 w_t + \lambda h_2 w + u_{1,t} v)_z, \tag{2.77}$$
$$w_{tt} + 2\lambda w_t + \lambda^2 w = (\nu^2 w_z)_z - p(h_2 v_z + h_{1,z} v), \tag{2.78}$$
$$v(\pm l, t) = w(\pm l, t) = 0, \tag{2.79}$$
$$v(z,0) = w(z,0) = w_t(z,0) = 0, \tag{2.80}$$

provided with transmission conditions (2.54) with a natural substitution of the functions v, w in the place of h, u. Thus the functions v, w satisfy the integral equalities

$$\int_{Q_T} \{-v\eta_t + \lambda v\eta + r v_z \eta_z - h_2 w_t \eta_z - \lambda h_2 w \eta_z - u_{1,t} v \eta_z\}\, dz\, dt = 0,$$
$$\int_{Q_T} \{-w_t\zeta_t + 2\lambda w_t \zeta + \lambda^2 w\zeta + \nu^2 w_z \zeta_z + p h_2 v_z \zeta + p h_{1,z} v\zeta\}\, dz\, dt = 0.$$

Reasoning as before we obtain

$$\int_{Q_{T-k}} \{v_{k,t}\hat\eta + \lambda v_k \hat\eta + (r v_z - h_2 w_t + \lambda h_2 w - u_{1,t} v)_k \hat\eta\}\, dz\, dt = 0,$$
$$\int_{Q_{T-k}} \{(w_{k,tt} + 2\lambda w_{k,t} + \lambda^2 w_k)\hat\zeta + (\nu^2 w_z)_k \hat\zeta_z\}\, dz\, dt + \tag{2.81}$$
$$\int_{Q_{T-k}} (p h_2 v_z + p h_{1,z} v)_k \hat\zeta\, dz\, dt = 0.$$

Assume

$$\hat{\eta}(z,t) = \begin{cases} \eta(z,t), & \text{if} \quad t \in (0,t_1], \\ 0, & \text{if} \quad t \notin (0,t_1], \end{cases}$$

$$\hat{\zeta}(z,t) = \begin{cases} \zeta(z,t), & \text{if} \quad t \in (0,t_1], \\ 0, & \text{if} \quad t \notin (0,t_1], \end{cases}$$

where $\eta = v_k$, $\zeta = w_{k,t}$, and $t_1 \in (0, T-k]$. Using these functions we obtain the following relations

$$\int_{Q_{T-k}} v_{k,t} v_k \, dz\, dt = \frac{1}{2} \int_\Omega v_k^2 \, dz \Big|_{t=0}^{t=t_1},$$
$$\int_{Q_{T-k}} w_{k,tt} w_{k,t} \, dz\, dt = \frac{1}{2} \int_\Omega w_{k,t}^2 \, dz \Big|_{t=0}^{t=t_1},$$
$$\int_{Q_{T-k}} w_{k,t} w_k \, dz\, dt = \frac{1}{2} \int_\Omega w_k^2 \, dz \Big|_{t=0}^{t=t_1}.$$

Let us multiply the first equality in formulas (2.81) by p and sum up with the second one. Using the latter formulas and the initial data (2.80) the result in limiting case $k \to 0$ can be transformed to the following form:

$$\frac{1}{2} \int_\Omega \{ p v^2(z,t_1) + w_t^2(z,t_1) + \nu^2(z) w_z^2(z,t_1) + \lambda^2 w^2(z,t_1) \} \, dz + \\ \int_{Q_{t_1}} \{ 2\lambda w_t^2 + \lambda p v^2 + p r v_z^2 - p u_{1,t} v_z v - \lambda p h_2 w v_z + p h_{1,z} v w_t \} \, dz\, dt = 0.$$

Using the Cauchy inequality gives us

$$\int_{Q_{t_1}} \{ 2\lambda w_t^2 + \lambda p v^2 + p r v_z^2 - p u_{1,t} v_z v - \lambda p h_2 w v_z + p h_{1,z} v w_t \} \, dz\, dt \\ \geq 2\lambda \|w_t\|^2_{2,Q_{t_1}} + p r_0 \|v_z\|^2_{2,Q_{t_1}} + \lambda p \|v\|^2_{2,Q_{t_1}} - p C_1 \max_{Q_{t_1}} |v| \, \|v_z\|_{2,Q_{t_1}} - \\ p \max_{Q_{t_1}} |v| \, \|h_{1,z}\|_{2,Q_{t_1}} \|w_t\|_{2,Q_{t_1}} - \lambda p C_1 \|w\|_{2,Q_{t_1}} \|v_z\|_{2,Q_{t_1}}.$$

Thus, for any $\lambda \geq 0$ and almost all $t_1 \in [0,T]$ the following inequality holds:

$$\frac{1}{2} \|v(z,t_1)\|^2_{2,\Omega} + \frac{1}{2} \|w_t(z,t_1)\|^2_{2,\Omega} + \frac{\lambda^2}{2} \|w(z,t_1)\|^2_{2,\Omega} + \\ 2\lambda \|w_t(z,t_1)\|^2_{2,\Omega} + \lambda p \|v\|^2_{2,Q_{t_1}} + p r_0 \|v_z\|^2_{2,Q_{t_1}} \\ \leq p C_1 \max_{Q_{t_1}} |v| \, \|v_z\|^2_{2,Q_{t_1}} + \lambda p \|v_z\|^2_{2,Q_{t_1}} \|w\|_{2,Q_{t_1}} + \\ p C_1 \max_{Q_{t_1}} |v| \, \|w_t\|^2_{2,Q_{t_1}}, \tag{2.82}$$

which is impossible for sufficiently large values of λ if $\|w(z,t_1)\|^2_{2,\Omega} \neq 0$. This means the uniqueness of the solution to problem (2.50)–(2.54).

If $w(z,t_1) = 0$ for almost all $t_1 \in [0,T]$, then from (2.77), (2.79), (2.80) we have

$$v_t + \lambda v = (rv_z)_z - (u_{1,t}v)_z, \quad v(z,0) = 0, \quad v(\pm l,t) = 0.$$

Applying the uniqueness theorem for a parabolic equation gives us $v(z,t) \equiv 0$. □

2.4. The Cauchy problem for the electromagnetoelasticity equations with complete nonlinear interaction

In this section, following the original work [35] we will give some results of solution of the Cauchy problem for electromagnetoelasticity equations when the nonlinear terms describing interaction of electromagnetic and elastic fields are presented in both Maxwell's and the Lamè systems. We note that some results of this section are novel. For example, there is a new proof of the existence theorem for any value of the parameter p in contrast to smallness of this parameter in the previous version of the existence theorem. Another proof of the uniqueness theorem is presented, too.

The problem statement

Consider the equations

$$h_t = (rh_z)_z-(hu_t)_z-(rj)_z, \quad u_{tt} = (\nu^2 u_z)_z-phh_z+f, \quad (z,t) \in \mathbb{R}_T, \quad (2.83)$$

where $\mathbb{R}_T = \mathbb{R} \times (0,T)$. The functions r, ν, f, j are supposed to be smooth functions with possible jumps in points z_m: $-\infty < z_1 < z_2 < \cdots < z_m < +\infty$, $r(z) \geq r_0 > 0$; $\nu(z) \geq \nu_0 > 0$, p is a positive number.

The following Cauchy problem will be treated for equations (2.83) with the initial data

$$h(z,0) = h_0(z), \quad u(z,0) = u_0(z), \quad u_t(z,0) = u_1(z), \quad z \in \mathbb{R}. \quad (2.84)$$

The Cauchy problem (2.83), (2.84) can be considered to be a diffraction problem for parabolic-hyperbolic system (2.83), (2.84), i.e., a problem in $\mathbb{R}_T$ consisting of several media. The following transmission conditions are supposed to be fulfilled on the boundaries of such media:

$$
\begin{aligned}
&[h(z,t)]_{z=z_i}=0, \quad [r(z)(h_z(z,t)-j(z,t))]_{z=z_i}=0,\\
&[u(z,t)]_{z=z_i}=0, \quad [\nu^2(z)u_z(z,t)]_{z=z_i}=0, \quad i=1,\dots,m.
\end{aligned}
\tag{2.85}
$$

Similar to the case of the previous Section 2.3 we can define the generalized solution to the Cauchy problem (2.83)–(2.85) in the following form.

Definition 2.3. The functions $h(z,t)\in V_2^{1,1/2}(R_T)$, $u(z,t)\in W_2^{1,1}(R_T)$, where $R_T=\mathbb{R}\times(0,T)$, are called the generalized solution of the Cauchy problem (2.83)–(2.85) if they satisfy the integral equalities

$$
\begin{aligned}
-\int_{R_T} h\eta_t\,dz\,dt &+ \int_{R_T} rh_z\eta_z\,dz\,dt - \int_{R_T} hu_t\eta_z\,dz\,dt\\
&= \int_{R_T} rj\eta_z\,dz\,dt + \int_{\mathbb{R}} h_0(z)\eta(z,0)\,dz,\\
-\int_{R_T} u_t\zeta_t\,dz\,dt &+ \int_{R_T} \nu^2u_z\zeta_z\,dz\,dt + \int_{R_T} phh_z\zeta\,dz\,dt\\
&= \int_{R_T} f\zeta\,dz\,dt + \int_{\mathbb{R}} u_1(z)\zeta(z,0)\,dz,
\end{aligned}
$$

$u(z,0)=u_0(z)$, $z\in\mathbb{R}$, for any $\eta(z,t),\zeta(z,t)\in W_2^{1,1}(R_T)$ such that $\eta(z,T)=\zeta(z,T)=0$.

Main results

Let us formulate the existence theorem about solvability of the Cauchy problem. For this purpose we assume that the functions r, ν, the free members f, j, the constant p, and initial data h_0, u_0, u_1 in the Cauchy problem (2.83)–(2.85) satisfy the properties

a) the functions r, ν, f, j are supposed to be smooth functions with a possible jump in the points z_m: $-\infty<z_1<z_2<\ldots<z_m<+\infty$, $r(z)\geq r_0>0$; $\nu(z)\geq\nu_0>0$, and p is a positive number;

b) $h_0\in C^\alpha(\mathbb{R})\cap L_2(\mathbb{R})$, $\alpha\in(0,1)$, and $u_0\in W_2^1(\mathbb{R})$ and $u_1\in L_2(\mathbb{R})$.

Theorem 2.6. *In the above-mentioned conditions the Cauchy problem* (2.83)–(2.85) *has the solution* $h(z,t)\in V_2^{1,1/2}(R_T)$, $u(z,t)\in W_2^{1,1}(R_T)$.

The proof of Theorem 2.4 mentioned in Section 2.3 for the bounded domain $\Omega=(-l,l)$ is based on estimates independent of l and, for this reason, readily transfers to the case $\mathbb{R}$.

It should be noted that the solution of the Cauchy problem (2.83)–(2.85) can be constructed as a limit for $l\to+\infty$ of the initial boundary value problem (2.50)–(2.54).

There is the following uniqueness theorem about solvability of the Cauchy problem.

Theorem 2.7. *The Cauchy problem* (2.83)–(2.85) *cannot have more than one solution.*

Demonstration of Theorem 2.7 is similar to the proof of Theorem 2.5 of the previous section.

Chapter 3

Inverse problems

In this chapter, we present some results of solution to inverse problems for a system of equations describing linear and nonlinear processes of interaction of electromagnetic and elastic waves based on motion of particles.

3.1. One-dimensional inverse problem

In this section, we present some results of solution to inverse problems for the equations of electromagnetoelasticity based on motion of particles. In our exposition, we follow the work [24].

Let $\mathbb{R}^3$ and $\mathbb{R}^3_\pm$ be a three-dimensional Euclidean space of the points $x = (x_1, x_2, x_3)$ and the half-spaces $\{x \in \mathbb{R}^3 \mid \pm x_3 > 0\}$, respectively. We assume the half-space $\mathbb{R}^3_-$ to correspond to the Earth's atmosphere with the constant electromagnetic parameters $\varepsilon_0 > 0$, $\mu_0 > 0$, and $\sigma_0 = 0$. Let us describe the propagation of electromagnetic waves in $\mathbb{R}^3_-$ with the help of the following Maxwell system:

$$\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{rot} \boldsymbol{H}, \qquad \mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{rot} \boldsymbol{E}, \qquad \operatorname{div} \boldsymbol{H} = 0. \tag{3.1}$$

At the same time, in the half-space $\mathbb{R}^3_+$ corresponding to the Earth's crust, we observe the interaction of electromagnetic and elastic waves described by the following system of electromagnetoelasticity:

$$\boldsymbol{J} = \operatorname{rot} \boldsymbol{H}, \quad \frac{\partial \boldsymbol{B}}{\partial t} = -\operatorname{rot} \boldsymbol{E}, \quad \operatorname{div} \boldsymbol{B} = 0, \quad \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T, \tag{3.2}$$

where

$$\operatorname{Div} T = \Big(\sum_{j=1}^{3} \frac{\partial}{\partial x_j} T_{ij} \Big)_{i=1}^{3}.$$

For the stress tensor T, the vectors of electric and magnetic inductions $\boldsymbol{D}$ and $\boldsymbol{B}$, and the vector of the electric current density $\boldsymbol{J}$, we have the following defining relations:

$$T = \lambda \operatorname{tr} S \cdot I + 2\varkappa S, \quad \boldsymbol{B} = \mu \boldsymbol{H}, \quad \boldsymbol{J} = \sigma\Big(\boldsymbol{E} + \mu \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{H}^0\Big), \tag{3.3}$$

where S is the strain tensor defined by the formula

$$S_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3,$$

and I is the unit 3×3-matrix. In the above formulas, $\rho, \lambda, \varkappa : \mathbb{R}^3_+ \to \mathbb{R}_+$ are the density of the inhomogeneous medium (the Earth's crust) and the Lamé coefficients, respectively; $\varepsilon, \mu, \sigma : \mathbb{R}^3_+ \to \mathbb{R}_+$ are the dielectric and magnetic permeabilities and the electrical conductivity of the crust, respectively; and $\boldsymbol{H}^0$ is a constant vector characterizing the Earth's magnetic field. Equations (3.1)–(3.3) were derived from the general model of equations of electromagnetoelasticity (1.20)–(1.25) for the case of an isotropic inhomogeneous earth by means of linearization in the neighborhood of the constant solution $(\boldsymbol{H}^0, \boldsymbol{E}^0, \boldsymbol{u}^0) = (\boldsymbol{H}^0, 0, 0)$ and discarding nonlinearities of orders higher than one. We also assume that the propagation of electromagnetic oscillations in the Earth's crust $\mathbb{R}^3_+$ is described by the quasi-stationary approximation of the Maxwell equations. Knopoff has shown [23] that in this case it is natural to neglect the reverse effect of the electromagnetic field on the elastic waves propagation. We assume that the following matching conditions are fulfilled on the Earth's surface $\Gamma = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$:

$$[\mu H_k] = [E_k] = 0, \quad k = 1, 2, \tag{3.4}$$

where the symbol $[\,\cdot\,]$ denotes a jump of the function across the surface, where the coefficients of the problem have breaks. As a direct problem for the system (3.1)–(3.4), we consider the Cauchy problem

$$(\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{u})\big|_{t<0} \equiv 0 \tag{3.5}$$

in the case, when the electromagnetic oscillations arise under the action of a vertical elastic source concentrated on the Earth's surface Γ:

$$T_{k3}(\boldsymbol{u})|_\Gamma = \delta_{k3}\delta(t, x_1, x_2), \quad k = 1, 2, 3. \tag{3.6}$$

Here $\delta(\cdot)$ is the delta-function concentrated at the point $(t, x_1, x_2) = (0, 0, 0)$ and δ_{k3} is the Kronecker symbol. Since the propagation of electromagnetic waves in the Earth's crust $\mathbb{R}^3_+$ is described by the quasi-stationary approximation of the Maxwell equations, then, to single out the unique solution it is natural to assume that the following radiation conditions at infinity are fulfilled:

$$|\boldsymbol{H}| \to 0, \quad |\boldsymbol{E}| \to 0 \quad \text{for} \quad |x| \to \infty. \tag{3.7}$$

Thus, the direct problem consists in finding the vectors $(\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{u})$ satisfying relations (3.1)–(3.7) provided the properties of the medium described by the coefficients ε_0, μ_0, ε, μ, σ, ρ, λ, and $\varkappa$, are known. The inverse problem is understood as problem of finding the electromagnetic and elastic parameters from equations (3.1)–(3.7) if we know some additional information about the behavior of certain components of the vectors $(\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{u})$ on the Earth's surface Γ. As an illustration to the proposed approach we will consider one of the simplest variants of the above inverse problem.

Statement of a simplified problem

Let us focus our attention on a simple version of equations (3.1)–(3.7). In this version, however, the main properties of more general models are kept. Assume that all the coefficients in the problem depend only on one variable x_3; in the sequel, denoted by z. For this case, consider the system

$$\frac{\partial^2 u}{\partial t^2} = v^2(z)\,\frac{\partial^2 u}{\partial z^2}, \quad (z,t) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.8}$$

$$\frac{\partial e}{\partial t} = c^2(z)\,\frac{\partial^2 e}{\partial z^2} + \mu h^0\,\frac{\partial^2 u}{\partial t^2}, \tag{3.9}$$

$$\frac{\partial u}{\partial z}\Big|_{z=0} = b\delta(t), \tag{3.10}$$

$$\Big(\frac{\partial}{\partial z} - c_0\Big(\frac{c_0}{c}\Big)^2\frac{\partial}{\partial t}\Big)\,e\Big|_{z=0} = 0, \tag{3.11}$$

$$\lim_{z\to\infty} e = 0, \qquad (e,u)\big|_{t<0} \equiv 0. \tag{3.12}$$

Here $u = \Re F_{x_1x_2}(u_3)\big|_{\nu_1=\nu_2=0}$, $e = \Re F_{x_1x_2}(E_1)\big|_{\nu_1=\nu_2=0}$ are the values of the generalized Fourier transforms with respect to the variables x_1, x_2 of the functions u_3 and E_1, respectively, at $\nu_1 = \nu_2 = 0$; ν_1, ν_2 are the variables dual to x_1, x_2; $v(z) = \sqrt{(\lambda + 2\varkappa)/\rho}$ is the velocity of longitudinal elastic waves; $c(z) = \sqrt{\sigma\mu}$ is a characteristic of the process of diffusion of electromagnetic waves in the Earth's crust; $c_0 = (\varepsilon_0\mu_0)^{-1/2}$ is the velocity of electromagnetic waves in the Earth's atmosphere; $b = (\lambda(0)+2\varkappa(0))^{-1}$; and h^0 is the number characterizing the constant magnetic field of the Earth.

System (3.8)–(3.12) is obtained by applying the Fourier transform to system (3.1)–(3.7). Since the coefficients of the problem are known functions in $\mathbb{R}^3_-$, the upper half-space was excluded from the consideration. For this, we replaced its effect by the corresponding boundary condition (3.11).

Let us introduce the following class of functions:

Definition 3.1. Let us say that the functions $c(z)$ and $v(z)$ belong to the class $\mathfrak{M}$ if

a) there exist positive numbers c_m, v_m, z_n, and z'_n, $m = 1, 2, \ldots, k+1$, $n = 1, 2, \ldots, k$, such that

$$c(z) = \begin{cases} c_m, & z \in (z_{m-1}, z_m), \quad m = 1, 2, \ldots, k, \\ c_{k+1}, & z > z_k, \end{cases}$$

$$v(z) = \begin{cases} v_m, & z \in (z'_{m-1}, z'_m), \quad m = 1, 2, \ldots, k, \\ v_{k+1}, & z > z'_k, \end{cases}$$

where $z'_0 = z_0 = 0$;

b) there exist positive constants τ_v and τ_c such that

$$\tau_v = \frac{z_1' - z_0'}{v_1} = \frac{z_2' - z_1'}{v_2} = \ldots = \frac{z_k' - z_{k-1}'}{v_k},$$
$$\tau_c = \frac{z_1 - z_0}{c_1} = \frac{z_2 - z_1}{c_2} = \ldots = \frac{z_k - z_{k-1}}{c_k}.$$

We can now formulate the inverse problem.

Inverse Problem 3.1. *The functions $u_0(t)$ and $e_0(t)$ and the constants τ_v and τ_c are known. It is required to find the functions $c(z), v(z) \in \mathfrak{M}$, i.e., the set $\{c_m, v_m, z_n, z_n'; \ m = 1, \ldots, k+1; \ n = 1, \ldots, k\}$, such that*

$$u|_{z=0} = u_0(t), \qquad e|_{z=0} = e_0(t),$$

where u, e is the solution to problem (3.8)–(3.12).

Inverse problem for a system of ordinary differential equations

To illustrate a possible approach to the solution of Inverse Problem 3.1 we consider the following system. In equations (3.8)–(3.12), we formally replace the derivative $\partial e/\partial t$ by the derivative $\partial^2 e/\partial t^2$, i.e., instead of equation (3.9) we consider the equation

$$\frac{\partial^2 e}{\partial t^2} = c^2(z)\, \frac{\partial^2 e}{\partial z^2} + \mu h^0\, \frac{\partial^2 u}{\partial t^2}.$$

To construct an algorithm for solving the so-modified problem we use the generalized Fourier transform with respect to the variable t. For convenience, we represent our problem in the form of the two subproblems:

$$\frac{d^2 u}{dz^2} + \omega^2 v^{-2} u = 0, \quad z \in \mathbb{R}_+, \tag{3.13}$$

$$\left.\frac{du}{dz}\right|_{z=0} = h_u(\omega), \qquad \frac{du}{dz} - \mathrm{i}\omega v^{-1} u \xrightarrow{z \to \infty} 0, \tag{3.14}$$

$$[u]\big|_{z=z_m'} = \left[v^2 \frac{du}{dz}\right]\Big|_{z=z_m'} = 0, \quad m = 1, 2, \ldots, k, \tag{3.15}$$

$$\frac{d^2 e}{dz^2} + \omega^2 c^{-2} e = \omega^2 c^{-2} \mu h^0 u, \quad z \in \mathbb{R}_+, \tag{3.16}$$

$$\left.\frac{de}{dz}\right|_{z=0} = h_e(\omega), \qquad \frac{de}{dz} - \mathrm{i}\omega c^{-1} e \xrightarrow{z \to \infty} 0, \tag{3.17}$$

$$[e]\big|_{z=z_m} = \left[c^2 \frac{de}{dz}\right]\Big|_{z=z_m} = 0, \quad m = 1, 2, \ldots, k. \tag{3.18}$$

Here $h_u(\omega)$ is the Fourier transform with respect to the variable t of the corresponding boundary condition on the function u in equations (3.8)–(3.12) and the function $h_e(\omega)$ will be defined later. Note that problems (3.13)–(3.15) and (3.16)–(3.18) can be successively solved.

Inverse Problem 3.2. *It is required to find a function $v(z) \in \mathfrak{M}$ such that*

$$u(0,\omega) = \Phi_u(\omega), \tag{3.19}$$

where $\Phi_u(\omega)$ is a given function and $u(z,\omega)$ is the solution to problem (3.11).

If we know the solution to Inverse Problem 3.2, we can also determine the function $c(z)$ because in this case the right-hand side of equation (3.16) will be a known function. Thus, we can consider the following inverse problem.

Inverse Problem 3.3. *We know the functions $\Phi_e(\omega)$ and $u(z,\omega)$. It is required to find a function $c(z) \in \mathfrak{M}$ such that*

$$e(0,\omega) = \Phi_e(\omega), \tag{3.20}$$

where $e(z,\omega)$ is the solution to problem (3.16)–(3.18).

Remark 3.1. Since the function $e(0,\omega)$ is assumed to be known, then we can also calculate the value of the derivative $\partial e/\partial t$ for $z = 0$. Thus, under the assumption that we know the constant c_1, the function $h_e(\omega)$ in equations (3.16)–(3.18) is the Fourier transform with respect to the variable t of the formula

$$c_0\Big(\frac{c_0}{c_1}\Big)^2 \frac{\partial e(0,t)}{\partial t}$$

and thus can be calculated.

Summarizing the above considerations we can say that the original Inverse Problem 3.1 was decomposed into two Inverse Problems 3.2 and 3.3. Using the results of solution to the corresponding inverse problems, we can propose the following recursive algorithm for the solution of Inverse Problem 3.2:

a) determine the numbers

$$\varphi_m = \int_{\omega_0}^{\omega_0+\pi/\tau_v} h_u^{-1}(\omega)\,\Phi_u(\omega)\exp(-2\mathrm{i}m\omega\tau_v)\,d\omega, \tag{3.21}$$

$$h_n^m = h_{m-1}^m h_{m-n-1}^{m-1} + h_n^{m-1}, \quad n = 0,1,\dots,m-2, \quad m = 2,3,\dots,k,$$

$$h_{m-1}^m = -\gamma_{m-1}, \qquad h_0^1 = h_m^m = 0, \qquad \gamma_1 = \varphi_1/2\varphi_0,$$

$$\gamma_m = (2\varphi_0)^{-1}\Big(\varphi_m + \sum_{n=1}^{m-1} h_n^m \varphi_{m-n}\Big)\Big/\prod_{n=1}^{m-1}(1-\gamma_n^2), \quad m = 2, 3, \ldots, k,$$

where ω_0 is an arbitrary positive number;

b) determine the numbers

$$v_1 = \frac{\pi}{\tau_v \varphi_0}, \qquad v_{m+1} = \frac{1+\gamma_m}{1-\gamma_m}\, v_m, \quad m = 1, 2, \ldots, k. \tag{3.22}$$

Using formulas (3.20), find the numbers z'_m, $m = 1, 2, \ldots, k$. This completes the solution of Inverse Problem 3.2.

To construct a solution of Inverse Problem 3.3 let us consider the sets

$$(z_{m-1}, z_m) \cap (z'_{n-1}, z'_n), \quad m = 1, 2, \ldots, k, \quad n = 1, 2, \ldots, k.$$

Since the function $u_n(z, \omega)$, $n = 1, 2, \ldots, k$, on the right-hand side of the differential equation (3.16) has the form

$$u_n(z, \omega) = u_n^1 + u_n^2 = B_1^n \exp\Big(\frac{\mathrm{i}\omega z}{v_n}\Big) + B_2^n \exp\Big(-\frac{\mathrm{i}\omega z}{v_n}\Big),$$

then we can use the following representation for the solution of problem (3.16)–(3.18):

$$e_m(z, \omega) = e_m^0 + e_m^1 + e_m^2, \tag{3.23}$$

where

$$e_m^0 = A_1^m \exp\Big(\frac{\mathrm{i}\omega z}{c_m}\Big) + A_2^m \exp\Big(-\frac{\mathrm{i}\omega z}{c_m}\Big)$$

is the general solution to the homogeneous differential equation (3.16) and e_m^j, $j = 1, 2$ are its particular solutions in the cases, when the function u on the right-hand side is replaced by u_m^j, $j = 1, 2$, respectively. These particular solutions can be represented in the form

$$e_m^j = r_m^j \exp\Big\{(-1)^{j+1}\frac{\mathrm{i}\omega z}{v_n}\Big\}, \quad j = 1, 2, \tag{3.24}$$

where

$$r_m^j = -\frac{\omega^2 v_n^2 h^0 \mu B_j^m}{\omega^2(v_n^2 - c_m^2)} = -\mu h^0 B_j^m \Big(1 - \Big(\frac{c_m}{v_n}\Big)^2\Big)^{-1}, \quad j = 1, 2. \tag{3.25}$$

It is known that for most of physical materials the ratio c_m/v_n is less than 10^{-1}, therefore, we can neglect the term $(c_m/v_n)^2$ in (3.25) and set

$$r_m^j = -\mu h^0 B_j^m, \quad j = 1, 2. \tag{3.26}$$

Taking into account the fact that we can represent the boundary condition in (3.17) in the form of recursive relationships with B_j^m, analogous formulas can be derived from relations (3.16)–(3.18), (3.21)–(3.24), and (3.26). Thus, the algorithm of solution of Inverse Problem 3.2 can also be used for solution of Inverse Problem 3.3. There is only one difference in the use of the additional information (3.20). We have

$$e_1(0,\omega) = e_1^0(0,\omega) + e_1^1(0,\omega) + e_1^2(0,\omega) = A_1^1 + A_2^1 - \mu h^0(B_1^1 + B_2^1) = \Phi_e(\omega),$$

$$\frac{de_1}{dz}\bigg|_{z=0} = \mathrm{i}\omega c_1^{-1}(A_1^1 - A_2^1) - \mathrm{i}\omega v_1^{-1}\mu h^0(B_1^1 - B_2^1) = h_e(\omega).$$

Since

$$B_1^1 + B_2^1 = \Phi_u(\omega), \qquad \mathrm{i}\omega v_1^{-1}(B_1^1 - B_2^1) = h_u(\omega),$$

then, the algorithm of solution of Inverse Problem 3.3 can be formulated as follows:

a) find the numbers

$$\varphi_m' = \int_{\omega_1}^{\omega_1+\pi/\tau_c} (h')^{-1}(\omega)\,\Phi'(\omega)\exp(-2\mathrm{i}m\omega\tau_c)d\omega, \tag{3.27}$$

where ω_1 is some positive number and

$$h'(\omega) = h_e(\omega) + \mu h^0 h_u(\omega), \qquad \Phi'(\omega) = \Phi_e(\omega) + \mu h^0 \Phi_u(\omega);$$

b) find the numbers

$$c_1 = \frac{\pi}{\tau_c\varphi_0'}, \qquad c_{m+1} = c_m\frac{1+\gamma_m}{1-\gamma_m}, \quad m = 1, 2, \ldots, k, \tag{3.28}$$

where γ_m, $m = 1, 2, \ldots, k$, are determined by formulas (3.21) with φ_m replaced by φ_m'.

The numbers z_m are found by formulas (3.20). Thus, Inverse Problem 3.1 is solved by formulas (3.21), (3.22), (3.27), and (3.28).

Remark 3.2. In fact, we have formulated an algorithm for solution of Inverse Problems 3.2 and 3.3 of reconstructing the parameters of a medium from additional information of the form (3.21) about the solution of the corresponding direct problems (3.13)–(3.15) and (3.16)–(3.18). Using the relationship between the solutions to differential equations of the parabolic and hyperbolic types, we can formulate a similar algorithm for solving the original Inverse Problem 3.1. Since the corresponding formulas are very bulky, we dwelt on demonstrating this approach in a simpler case, when we have formally changed the type of the differential equation for $e(z,t)$ from parabolic to hyperbolic.

3.2. Inverse problems for the electromagnetoelasticity equations for weakly conducting media

In this section, following the work [42], we present some results of solution of inverse problems for a system of equations describing the linear process of interaction of electromagnetic and elastic waves in a weakly conducting elastic medium. The results of the Cauchy problem solution represented in Section 2.1 will be sufficiently used.

Formulation of an inverse problem and main results

We now consider the case, when the coefficients of equations (2.12)–(2.15) are known constants outside a finite domain D and are unknown functions of x in D. We also assume that the magnetic permeability μ is constant everywhere in $\mathbb{R}^3$. For simplicity we assume that all the coefficients are functions of the class $C^m(\mathbb{R}^3)$ with a sufficiently large m and are continuous together with all their derivatives of order up to m on the boundary of the domain D. Let S be some closed sufficiently smooth surface enclosing the domain D. Assume that the function $\boldsymbol{V}^1 = (\boldsymbol{H}^1, \boldsymbol{E}^1, \boldsymbol{u}^1)$ (the solution to the Cauchy problem (2.12)–(2.15)) is known on the set $S \times [0, T]$, where T is a sufficiently large positive number, and x^0 is an arbitrary point of the surface S, i.e.,

$$\boldsymbol{V}^1 = \boldsymbol{F}(x, t, x^0), \quad (x, t, x^0) \in S \times [0, T] \times S \equiv G.$$

Remark 3.3. In what follows it is convenient for us to mark the dependence of functions on the variable x_0 because in the inverse problems under consideration the source position is a variable parameter which will be used.

Inverse Problem 3.4. *Find the coefficients $\varepsilon, \sigma, \lambda, \rho$, and $\varkappa$ in the domain D if the function $\boldsymbol{F}(x, t, x^0)$ is given.*

In the sequel, we will assume that $T \geq d/m_3$, where d is diameter of the domain D and m_3 is the constant from formula (2.19). Using representation (2.21) from the function $\boldsymbol{F}(x, t, x^0)$ we can uniquely determine the following quantities:

a) $\tau_k(x^0, x)$, $(x^0, x) \in S \times S$, $k = 1, 2, 3$;

b) $\boldsymbol{\alpha}_n^{i1}(x^0, x)$, $\boldsymbol{\beta}_n^{i1}(x^0, x)$, $\boldsymbol{\gamma}_n^{i1}(x^0, x)$, $(x^0, x) \in S \times S$, $i = 1, 2, 3$, $n \geq -2$.

Inverse Problem 3.4 is reduced to the successive solution of the inverse kinematic problem of reconstructing the velocities $c_k(x)$, $k = 1, 2, 3$, inside the domain D from the functions $\tau_k(x^0, x)$, $(x^0, x) \in S \times S$, $k = 1, 2, 3$, and subsequent determination of the still unknown combinations of the sought functions from the functions $\boldsymbol{\alpha}_n^{i1}(x^0,x)$, $\boldsymbol{\beta}_n^{i1}(x^0,x)$, and $\boldsymbol{\gamma}_n^{i1}(x^0,x)$ given on $S\times S$. The theory of solution of the inverse kinematic problem in the case of isotropic media is developed sufficiently well (see, e.g., [39]). In particular, it was established that in the case of simple metrics, the values of the functions $\tau_k(x^0,x)$, $k=1,2,3$, on the set $S\times S$ uniquely define the functions $c_k(x)$, $k=1,2,3$, inside the domain D. Thus, we can think that the velocities $c_k(x)$, $k=1,2,3$, were determined inside the domain D as a result of solution to the corresponding inverse kinematic problems. If we know the functions $c_k(x)$, $k=1,2,3$, then we know three nonlinear combinations of parameters of the medium (see formulas (2.16)). Since the coefficient μ is assumed to be known and constant everywhere, then the coefficient ε is uniquely defined from these relations, and the elastic moduli can be expressed in terms of the density of the medium ρ and the known velocities c_2 and c_3 only. Thus, among the sought for parameters only the electrical conductivity σ and the density of the medium ρ remain unknown. To find them, we use the functions $\boldsymbol{\alpha}_n^{i1}(x^0,x)$, $\boldsymbol{\beta}_n^{i1}(x^0,x)$, and $\boldsymbol{\gamma}_n^{i1}(x^0,x)$.

It should be noted that this method of reducing an inverse problem to solution of the inverse kinematic problem and the subsequent use of amplitudes for determining the rest of the relations between the desired coefficients was earlier applied by Romanov [38, 39, 40] and Yakhno [49] for investigation of simpler inverse problems for hyperbolic equations of second order and for equations arising in the theory of elasticity.

Consider one possible statement of the inverse problem of reconstructing the parameters σ and ρ; we will use the function $\boldsymbol{\alpha}_{-2}^{11}(x^0,x)$ for its solution. Assume that $\boldsymbol{f}^0=0$ and $\sigma(x)\equiv\sigma_0$ for $x\in\mathbb{R}^3\setminus D$ and $\alpha\neq 0$ for $x\in D$. We will show that the problem under consideration can be reduced to the following integral geometry problem. Consider the integrals

$$\int_{\Gamma_1(x^0,x)}(a\eta_1+b)\,ds=p_1(x^0,x),\quad (x^0,x)\in S\times S, \tag{3.29}$$

$$\int_{\Gamma_1(x^0,x)}a\eta_2\,ds=p_2(x^0,x),\quad (x^0,x)\in S\times S. \tag{3.30}$$

Assume that we know the functions $p_k(x^0,x)$, $k=1,2$, with the weights η_k, $k=1,2$, also known. It is required to determine the coefficients a and b inside the domain D if they are known outside of D. The coefficients a and b are related to the parameters of the medium σ and ρ by the formulas

$$a=\frac{\sigma\alpha}{2\rho c_1\epsilon^2\left(c_1^2-c_2^2\right)},\qquad b=-\frac{\sigma}{2c_1\epsilon}.$$

The weight functions η_k, $k=1,2$, are defined by the relations

$$\eta_1 = (\boldsymbol{h}^0 \cdot \boldsymbol{w}^0)^2 \, |\boldsymbol{w}^0|^{-2} + \xi \, (\boldsymbol{h}^0 \cdot \boldsymbol{\nu})^2,$$
$$\eta_2 = (\boldsymbol{h}^0 \cdot \boldsymbol{w}^0) \left(\boldsymbol{h}^0 \cdot (\boldsymbol{\nu} \times \boldsymbol{w}^0)\right) |\boldsymbol{w}^0|^{-2},$$

where $\xi = (c_1^2 - c_2^2)/(c_1^2 - c_3^2)$, $\xi \in (0,1)$; $\boldsymbol{\nu}$ is a unit vector tangential to $\Gamma_1(x^0, x)$ at the point x; and $\boldsymbol{w}^0$ is the solution to problem (3.46)–(3.48); this solution is completely defined by specifying the quantities $\boldsymbol{j}^0$ and $c_1(x)$.

Remark 3.4. The functions a and b uniquely define only σ and σ/ρ. Thus, the density of a medium can be uniquely defined only at those points of the domain D, where $\sigma \neq 0$. This characteristic feature of the problem concerned is a consequence of the fact that we use the information about the coefficient $\boldsymbol{\alpha}_{-2}^{11}$. One can show that in the case, when $\boldsymbol{\alpha}_0^{31}(x^0, x)$, $(x^0, x) \in S \times S$, is used as additional information (provided $\boldsymbol{f}^0 \neq 0$), the density of a medium is uniquely defined in the domain D.

The fact that equalities (3.29) and (3.30) contain the weight functions η_k, $k = 1, 2$, complicates the solution of the corresponding integral geometry problem in the general case. So, we give only three special formulations of the original problem.

1. Let $c_1 = \text{const} = c_{10}$, $\boldsymbol{j}^0 = (1,0,0) \cdot 4\pi c_{10}^2$, $\boldsymbol{h}^0 = (1,0,1)$. Let $S(z)$ and $D(z)$ denote cross-sections of the surface S and the domain D by the plane $x_3 = z$. In this case, the integral geometry problem splits to a one-parameter family of planar problems for each cross-section of the domain D by the plane $x_3 = z$. The vector $\boldsymbol{\nu}$ of the corresponding line $\Gamma_1(x^0, x)$, $(x^0, x) \in S(z) \times S(z)$, can be characterized in this case by the angular coordinate φ: $\boldsymbol{\nu} = (\cos\varphi, \sin\varphi, 0)$. This coordinate is constant along $\Gamma_1(x^0, x)$ and, consequently, depends only on the boundary points $(x^0, x) \in S(z) \times S(z)$. Therefore, the weight functions η_k, $k = 1, 2$, can be written down as $\eta_1 = 1 + \xi \cos^2\varphi$, $\eta_2 = \sin\varphi$. Since the weight function η_2 is constant along $\Gamma_1(x^0, x)$, equation (3.30) can be written down as

$$\int_{\Gamma_1(x^0,x)} a \, ds = p_2'(x^0, x), \quad (x^0, x) \in S \times S, \tag{3.31}$$

where $p_2' = p_2/\sin\varphi$. The problem of determining the function a from equation (3.31) is a standard integral geometry problem. Having found the function a, we can calculate the integral of the product $a\eta_1$ along $\Gamma_1(x^0, x)$ in formula (3.29). This yields a similar integral geometry problem for determining the coefficient b. The uniqueness of its solution is evident. A conditional stability estimate is less evident because in the general case the weight function η_1 is not constant on $\Gamma_1(x^0, x)$.

The following theorem holds.

Theorem 3.1. *Let* (a, b) *and* $(\bar{a}, \bar{b})$ *be two solutions to equations* (3.29), (3.30) *corresponding to the data* (p_1, p_2) *and* $(\bar{p}_1, \bar{p}_2)$*, respectively, and coinciding outside the domain* D*. Then the following estimates hold for the differences* $\tilde{a} = a - \bar{a}$*,* $\tilde{b} = b - \bar{b}$*,* $\tilde{p}_1 = p_1 - \bar{p}_1$*, and* $\tilde{p}_2 = p_2 - \bar{p}_2$*:*

$$\int_{D(z)} \tilde{a}^2\, d\bar{x} \le \frac{1}{2\pi} \int_{S(z)} dl' \int_{S(z)} \left| \frac{\partial \tilde{p}_2}{\partial l} \cdot \frac{\partial \tilde{p}_2}{\partial l'} \right| dl, \tag{3.32}$$

$$\int_{D(z)} \tilde{b}^2\, d\bar{x} \le \frac{1}{\pi} \int_{S(z)} dl' \int_{S(z)} \left\{ \frac{23}{8} \left| \frac{\partial \tilde{p}_2}{\partial l} \cdot \frac{\partial \tilde{p}_2}{\partial l'} \right| + \left| \frac{\partial \tilde{p}_1}{\partial l} \cdot \frac{\partial \tilde{p}_1}{\partial l'} \right| \right\} dl, \tag{3.33}$$

where $d\bar{x} = dx_1 dx_2$*;* $\partial/\partial l$ *and* $\partial/\partial l'$ *denote derivatives in the directions tangential to* $S(z)$ *at the points* x^0 *and* x*, respectively; and* dl *and* dl' *are the length elements calculated at these points.*

It is more convenient to give the proof of Theorem 3.1 later.

2. Consider a slightly modified 2D version of the problem. Namely, let D be an infinite cylindrical domain with the ruling parallel to the axis x_3: $D = D_0 \times \mathbb{R}$, where D_0 is a cross-section of the domain D by the plane $x_3 = 0$. Assume that coefficients of the problem are independent of the variable x_3. Let Γ be a simple smooth closed curve enclosing the domain D_0; and let the vectors $\boldsymbol{j}^0$ and $\boldsymbol{h}^0$ be the same as in the previous case. In this case, the geodesics $\Gamma_1(x^0, x)$, $(x^0, x) \in S \times S$, are planar curves. Denoting the unit vector tangential to $\Gamma_1(x^0, x)$ at the point x by $\boldsymbol{\nu} = (\cos\varphi, \sin\varphi, 0)$, we conclude that in this case we also have $\eta_2 = \sin\varphi$ and $\eta_1 = 1 + \xi\cos^2\varphi$. There is an essential distinction between this and the previous cases; namely, the weight function is not constant along a geodesic line. In this case, equation (3.30) can be written down as

$$\int_{\Gamma_1(x^0, x)} a\, dx_2 = p_2(x^0, x), \quad (x^0, x) \in S \times S. \tag{3.34}$$

Thus, p_2 is an integral along $\Gamma_1(x^0, x)$ of the simplest form of the first degree. The problems of finding a form of the first degree from its integrals along a family of geodesics of a Riemannian metric were considered in [4, 39, 45].

Assume that there are two solutions a and $\bar{a}$ to equation (3.34) corresponding to the function p_2 and coinciding outside the domain D_0. Then their difference $\tilde{a} = a - \bar{a}$ solves a homogeneous equation. It follows from [4] that $\partial\tilde{a}/\partial x_1 = 0$, $x \in D_0$. Since $\tilde{a} = 0$ outside the domain D_0, we conclude that $\tilde{a} = 0$, $x \in D_0$, i.e., $a = \bar{a}$. Thus, the following uniqueness theorem holds:

Theorem 3.2. *Specifying the functions* p_1 *and* p_2 *for* $(x^0, x) \in S \times S$ *uniquely defines the functions* a *and* b *in the domain* D_0*.*

It should be noted that certain estimates of the conditional stability of the solution to equations (3.29), (3.34) can be obtained from the estimates given in [45].

3. Let us turn to the original formulation of the problem. We consider the case, when the velocity c_1 essentially depends on all three variables x_1, x_2, and x_3. Let $\boldsymbol{j}^0$ be a constant nonzero vector. We also assume that the vector $\boldsymbol{h}^0$ may have two different values $\boldsymbol{h}^{0k}$, $k = 1, 2$, where $\boldsymbol{h}^{02} = r\boldsymbol{h}^{01}$, $r^2 \neq 1$, and that for each vector $\boldsymbol{h}^{0k}$ the function $\boldsymbol{\alpha}^1_{-2}(x^0, x)$ and, respectively, the functions $p_k(x^0, x)$, $(x^0, x) \in S \times S$, are known. We denote by p_{1k}, $k = 1, 2$, the function p_1 corresponding to the vectors $\boldsymbol{h}^{0k}$, $k = 1, 2$. From equality (3.29) for $\boldsymbol{h}^0 = \boldsymbol{h}^{0k}$, $k = 1, 2$, we readily obtain

$$\int_{\Gamma_1(x^0,x)} a\eta_0\, ds = \bar{p}_1(x^0, x), \quad (x^0, x) \in S \times S, \tag{3.35}$$

$$\int_{\Gamma_1(x^0,x)} b\, ds = \bar{p}_2(x^0, x), \quad (x^0, x) \in S \times S, \tag{3.36}$$

where

$$\eta_0 = (\boldsymbol{h}^{01} \cdot \boldsymbol{w}^0)^2\, |\boldsymbol{w}^0|^{-2} + \xi\, (\boldsymbol{h}^{01} \cdot \boldsymbol{\nu})^2,$$

$$\bar{p}_1 = \frac{p_{11} - p_{12}}{1 - r^2}, \qquad \bar{p}_2 = \frac{p_{12} - p_{11} r^2}{1 - r^2}.$$

Thus, using the data corresponding to two different values of the vector $\boldsymbol{h}^0$, we can split the original problem in two problems independent of each other. The problem of determining the function b from equation (3.36) is an integral geometry problem which has been studied in full (see Romanov, 1987).

The following conditional stability theorem (an analogue of estimate (3.54) from [39]) holds for this problem.

Theorem 3.3. *Let b and b' be two solutions to equation* (3.36) *corresponding to the right-hand sides $\bar{p}_2$ and $\bar{p}_2'$. Then the following inequality holds for the differences $\tilde{b} = b - b'$ and $\tilde{p}_2 = \bar{p}_2 - \bar{p}_2'$:*

$$\int_D c_1^{-1} \tilde{b}^2 dx \le \frac{1}{8\pi} \int_S \int_S |\mathrm{grad}_x\, \tilde{p}_2 \times \boldsymbol{n}(x)| \cdot |\mathrm{grad}_{x^0}\, \tilde{p}_2 \times \boldsymbol{n}(x^0)| \times \left(\sum_{k,j=1}^{3} \left(\frac{\partial^2 \tau_1}{\partial x_k\, \partial x_j^0} \right)^2 \right)^{1/2} dS_x dS_{x^0},$$

where $\boldsymbol{n}(x)$ is a normal vector to the surface S at the point x.

As concerns the properties of equation (3.35), their investigation requires additional efforts.

Reduction to an integral geometry problem

The differential and the integral equations considered here correspond to Inverse Problem 3.4, which was formulated earlier. As the first step, let us derive a differential equation for finding the function $\boldsymbol{\alpha}^{11}_{-2}(x)$. Let $\boldsymbol{f}^0 = 0$. Then $\boldsymbol{u}^0 \equiv 0$ and, as consequence, $\boldsymbol{\alpha}^{31}_k = \boldsymbol{\beta}^{31}_k = \boldsymbol{\gamma}^{31}_k \equiv 0$ for $k < 0$. This means that the function $\boldsymbol{u}$ has no singular part. Let us substitute representation (2.21) into equations (2.8)–(2.15). Equating the coefficients at $\theta_n(S_1)$, $n = -3, -2$, we obtain the following relationships between $\boldsymbol{\alpha}^{10}_{-2}$, $\boldsymbol{\alpha}^{20}_{-2}$, $\boldsymbol{\alpha}^{10}_{-1}$, $\boldsymbol{\alpha}^{20}_{-1}$ and $\boldsymbol{\alpha}^{11}_{-2}$, $\boldsymbol{\alpha}^{21}_{-2}$, $\boldsymbol{\alpha}^{11}_{-1}$, $\boldsymbol{\alpha}^{21}_{-1}$, $\boldsymbol{\alpha}^{31}_{0}$:

$$\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{10}_{-2} + \varepsilon\boldsymbol{\alpha}^{20}_{-2} = 0, \qquad \operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{20}_{-2} - \mu\boldsymbol{\alpha}^{10}_{-2} = 0, \tag{3.37}$$

$$\begin{aligned} &\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{10}_{-1} + \varepsilon\boldsymbol{\alpha}^{20}_{-1} - \operatorname{rot}\boldsymbol{\alpha}^{10}_{-2} = 0, \\ &\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{20}_{-1} - \mu\boldsymbol{\alpha}^{10}_{-1} - \operatorname{rot}\boldsymbol{\alpha}^{20}_{-2} = 0, \end{aligned} \tag{3.38}$$

$$\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{11}_{-2} + \varepsilon\boldsymbol{\alpha}^{21}_{-2} = 0, \qquad \operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{21}_{-2} - \mu\boldsymbol{\alpha}^{11}_{-2} = 0, \tag{3.39}$$

$$\begin{aligned} &\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{11}_{-1} + \varepsilon\boldsymbol{\alpha}^{21}_{-1} - \operatorname{rot}\boldsymbol{\alpha}^{11}_{-2} + \alpha\boldsymbol{\alpha}^{31}_{0} \times \boldsymbol{h}^0 + \sigma\boldsymbol{\alpha}^{20}_{-2} = 0, \\ &\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{21}_{-1} - \mu\boldsymbol{\alpha}^{11}_{-1} - \operatorname{rot}\boldsymbol{\alpha}^{21}_{-2} = 0, \end{aligned} \tag{3.40}$$

$$\begin{aligned} &\rho\boldsymbol{\alpha}^{31}_{0} - (\lambda + 2\varkappa)\,(\boldsymbol{\alpha}^{31}_{0} \cdot \operatorname{grad}\tau_1)\operatorname{grad}\tau_1 + \\ &\qquad \varkappa\big(\operatorname{grad}\tau_1 \times (\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{31}_{0})\big) - \sigma\mu\boldsymbol{\alpha}^{20}_{-2} \times \boldsymbol{h}^0 = 0. \end{aligned} \tag{3.41}$$

Using formulas (2.18), we can transform relations (3.37) and (3.39) to the equalities

$$\begin{aligned} &\operatorname{grad}\tau_1 \cdot \boldsymbol{\alpha}^{10}_{-2} = 0, \qquad \operatorname{grad}\tau_1 \cdot \boldsymbol{\alpha}^{20}_{-2} = 0, \\ &\operatorname{grad}\tau_1 \cdot \boldsymbol{\alpha}^{11}_{-2} = 0, \qquad \operatorname{grad}\tau_1 \cdot \boldsymbol{\alpha}^{21}_{-2} = 0, \end{aligned}$$

which express the principle of orthogonality of the electromagnetic field to the direction of propagation of its main singularity. To derive equations for the coefficient $\boldsymbol{\alpha}^{10}_{-2}$ we proceed as follows. We take the vector product of the first relation from (3.38) by $\operatorname{grad}\tau_1$ and then eliminate the term $\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{20}_{-1}$ with the help of the second relation from (3.38). Using the first equality from (3.37), we eliminate $\boldsymbol{\alpha}^{20}_{-2}$ and, finally, obtain

$$\begin{aligned} &\operatorname{grad}\tau_1 \times (\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{10}_{-1}) + \varepsilon\mu\boldsymbol{\alpha}^{10}_{-1} - \\ &\qquad \varepsilon\operatorname{rot}\big(\varepsilon^{-1}(\operatorname{grad}\tau_1 \times \boldsymbol{\alpha}^{10}_{-2})\big) - \operatorname{grad}\tau_1 \times \operatorname{rot}\boldsymbol{\alpha}^{10}_{-2} = 0. \end{aligned}$$

For further simplification, we use the following equalities, which are well known from the vector analysis:

$$\begin{aligned} \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) &= \boldsymbol{a}\,(\boldsymbol{c} \cdot \boldsymbol{b}) - \boldsymbol{b}\,(\boldsymbol{c} \cdot \boldsymbol{a}), \\ \operatorname{rot}(\boldsymbol{a} \times \boldsymbol{b}) &= \boldsymbol{a} \cdot \operatorname{div}\boldsymbol{b} - \boldsymbol{b} \cdot \operatorname{div}\boldsymbol{a} + (\boldsymbol{b} \cdot \operatorname{grad})\boldsymbol{a} - (\boldsymbol{a} \cdot \operatorname{grad})\boldsymbol{b}, \\ \boldsymbol{a} \times \operatorname{rot}\boldsymbol{b} &= \operatorname{grad}(\boldsymbol{a} \cdot \boldsymbol{b}) - (\boldsymbol{b} \cdot \operatorname{grad})\boldsymbol{a} - (\boldsymbol{a} \cdot \operatorname{grad})\boldsymbol{b} - \boldsymbol{b} \times \operatorname{rot}\boldsymbol{a}. \end{aligned}$$

Taking into account formulas (2.18) for τ_1 and the equality $\operatorname{grad}\tau_1 \cdot \boldsymbol{\alpha}^{10}_{-1} = \operatorname{div}\boldsymbol{\alpha}^{10}_{-2}$, which follows from the second relation from (3.38), we obtain

$$L\boldsymbol{\alpha}^{10}_{-2} \equiv 2\,\mathrm{grad}\,\boldsymbol{\alpha}^{10}_{-2}\cdot \mathrm{grad}\,\tau_1 + \boldsymbol{\alpha}^{10}_{-2}\big(\Delta\tau_1 - \mathrm{grad}\,\ln\varepsilon\cdot \mathrm{grad}\,\tau_1\big) + (\mathrm{grad}\,\ln\varepsilon\cdot\boldsymbol{\alpha}^{10}_{-2})\,\mathrm{grad}\,\tau_1 = 0. \tag{3.42}$$

The relations for $\boldsymbol{\alpha}^{11}_{-2}$ are derived in a similar way:

$$L\boldsymbol{\alpha}^{11}_{-2} = 2c_1^{-1}\boldsymbol{F}, \tag{3.43}$$

where

$$\boldsymbol{F} \equiv -\frac{c_1}{2}\,\mathrm{grad}\,\tau_1 \times (\alpha\boldsymbol{\alpha}^{31}_0 \times \boldsymbol{h}^0 + \sigma\boldsymbol{\alpha}^{20}_{-2}).$$

Using formula (3.41), we can find $\boldsymbol{\alpha}^{31}_0$:

$$\boldsymbol{\alpha}^{31}_0 = \frac{\sigma}{\rho}\Big[\frac{\boldsymbol{h}^0\cdot\boldsymbol{\alpha}^{10}_{-2}}{c_1^2-c_2^2}\,\mathrm{grad}\,\tau_1 - \frac{\boldsymbol{h}^0\cdot\mathrm{grad}\,\tau_1}{c_1^2-c_3^2}\boldsymbol{\alpha}^{10}_{-2}\Big].$$

Thus, the final formula for the function $\boldsymbol{F}$ takes the form

$$\boldsymbol{F} = a\Big\{(\boldsymbol{h}^0\cdot\boldsymbol{\alpha}^{10}_{-2})\,\big[\boldsymbol{h}^0 - c_1^2\,(\boldsymbol{h}^0\cdot\mathrm{grad}\,\tau_1)\,\mathrm{grad}\,\tau_1\big] + \xi c_1^2\,(\boldsymbol{h}^0\cdot\mathrm{grad}\,\tau_1)^2\boldsymbol{\alpha}^{10}_{-2}\Big\} + b\boldsymbol{\alpha}^{10}_{-2}, \tag{3.44}$$

where

$$a = \frac{\sigma\alpha}{2\rho\,(c_1^2-c_2^2)\,\varepsilon^2 c_1}, \qquad b = -\frac{\sigma}{2c_1\varepsilon}, \qquad \xi = \frac{c_1^2-c_2^2}{c_1^2-c_3^2}, \quad \xi\in(0,1). \tag{3.45}$$

Equations (3.44) and (3.45) imply that the coefficient $\boldsymbol{\alpha}^{11}_{-2}$ linearly depends on σ and σ/ρ. Consequently, in this case the inverse problem can be treated as problem of determining σ and σ/ρ or the coefficients a and b, which is the same.

The equations for the geodesics $\Gamma_1(x^0,x)$ can be written down as

$$\frac{dx}{ds} = \boldsymbol{\nu}, \qquad \frac{d\boldsymbol{\nu}}{ds} = -\,\mathrm{grad}\,\ln c_1 + (\mathrm{grad}\,\ln c_1\cdot\boldsymbol{\nu})\,\boldsymbol{\nu},$$

where s is the Euclidean length of the curve passing through the point x_0 and $\boldsymbol{\nu} = c_1\,\mathrm{grad}\,\tau_1$ is the unit vector tangential to $\Gamma_1(x^0,x)$ at the point x and oriented in the direction of increasing s.

Let us show that along the geodesic $\Gamma_1(x^0,x)$ equations (3.42) and (3.43) are ordinary differential equations. Indeed,

$$2\sum_{k=1}^{3}\frac{\partial\boldsymbol{\alpha}^{10}_{-2}}{\partial x_k}\frac{\partial\tau_1}{\partial x_k} = 2c_1^{-1}\frac{d\boldsymbol{\alpha}^{11}_{-2}}{ds}.$$

Since the magnetic permeability μ is constant, $\mathrm{grad}\,\ln\varepsilon = -\,\mathrm{grad}\,\ln c_1^2$. Further in the treatment we use the equality

$$\Delta\tau_1 + \operatorname{grad} \ln c_1^2 \cdot \operatorname{grad} \tau_1 = 2c_1^{-1} \frac{d}{ds} \ln \left[\tau_1 \Big(\det \frac{\partial \boldsymbol{g}(x^0, x)}{\partial x} \Big)^{-1/2} \right],$$

which follows immediately from [39, p. 116, Eq. (4.18)]. The function $\boldsymbol{g}(x^0, x)$ defines the Riemannian coordinates of the point x in the metric $d\tau_1 = c_1^{-1} ds$, which corresponds to the point x^0 and in our case can be found by the formula

$$\boldsymbol{g}(x^0, x) = -(2c_1^2(x^0))^{-1} \operatorname{grad}_{x^0} \tau_1^2(x^0, x).$$

Under the above assumptions, this function is smooth in both variables, namely $\boldsymbol{g}(x^0, x) \in C^{m-2}(\mathbb{R}^6)$. In a homogeneous space we have $\boldsymbol{g}(x^0, x) \equiv x - x^0$.

Let us introduce the new functions

$$\boldsymbol{w}^0(x^0, x) = \boldsymbol{\alpha}^{10}_{-2}(x^0, x)\, \tau_1(x^0, x) \Big(\det \frac{\partial \boldsymbol{g}(x^0, x)}{\partial x} \Big)^{-1/2},$$
$$\boldsymbol{w}(x^0, x) = \boldsymbol{\alpha}^{11}_{-2}(x^0, x)\, \tau_1(x^0, x) \Big(\det \frac{\partial \boldsymbol{g}(x^0, x)}{\partial x} \Big)^{-1/2}.$$

Then the differential equations for the functions $\boldsymbol{w}^0(x^0, x)$ and $\boldsymbol{w}(x^0, x)$ along the geodesic $\Gamma_1(x^0, x)$ take the form:

$$\frac{d\boldsymbol{w}^0}{ds} - (\boldsymbol{w}^0 \cdot \operatorname{grad} \ln c_1)\, \boldsymbol{\nu} = 0, \qquad \frac{d\boldsymbol{w}}{ds} - (\boldsymbol{w} \cdot \operatorname{grad} \ln c_1)\, \boldsymbol{\nu} = \boldsymbol{F}, \tag{3.46}$$

where

$$\boldsymbol{F} = a \Big\{ (\boldsymbol{h}^0 \cdot \boldsymbol{w}^0) \left[\boldsymbol{h}^0 - \boldsymbol{\nu}\, (\boldsymbol{h}^0 \cdot \boldsymbol{\nu}) \right] + \xi\, (\boldsymbol{h}^0 \cdot \boldsymbol{\nu})^2\, \boldsymbol{w}^0 \Big\} + b\boldsymbol{w}^0. \tag{3.47}$$

The initial conditions at the point $s = 0$ can be found by comparing the singularities of the solutions $(\boldsymbol{H}^0, \boldsymbol{E}^0, \boldsymbol{u}^0)$ and $(\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{u})$ in a homogeneous medium. Since the function $\boldsymbol{H}^0$ in a homogeneous medium is calculated by the formula

$$\boldsymbol{H}^0 = \operatorname{rot} \left[\frac{\boldsymbol{j}^0}{4\pi\, |x - x^0|}\, \theta_{-1}(S_1) \right],$$

then it follows that

$$\boldsymbol{w}^0 = (4\pi c_{10})^{-1} \boldsymbol{j}^0 \times \operatorname{grad} \tau_1 \big|_{x = x^0} = (4\pi c_{10}^2)^{-1} (\boldsymbol{j}^0 \times \boldsymbol{\nu}^0), \tag{3.48}$$

where $\boldsymbol{\nu}^0$ is the unit vector tangential to $\Gamma_1(x^0, x)$ at the point x^0 and c_{10} is the value of the velocity c_1 outside the domain D. Analyzing the solution to problem (2.12)–(2.15), corresponding to the homogeneous medium (see [41, Sec. 4]), we derive the following initial data for $\boldsymbol{w}$:

$$\boldsymbol{w}\big|_{s=0} = (4\pi c_{10}^2)^{-1} (\boldsymbol{j}^0 \times \boldsymbol{\nu}^0).$$

It should be noted that the vector-functions $\boldsymbol{w}^0$ and $\boldsymbol{w}$ are orthogonal to the vector $\boldsymbol{\nu}(x^0,x) = c_1 \operatorname{grad} \tau_1(x^0,x)$. The first equation from (3.46) implies that $d|\boldsymbol{w}^0|^2/ds = 0$, i.e., the function $|\boldsymbol{w}^0|$ is constant along the geodesic $\Gamma_1(x^0,x)$:

$$|\boldsymbol{w}^0| = (4\pi c_{10}^2)^{-1}\,|\boldsymbol{j}^0 \times \boldsymbol{\nu}^0|.$$

Let us represent the function $\boldsymbol{w}$ in the form

$$\boldsymbol{w} = (1+p_1)\boldsymbol{w}^0 + p_2(\boldsymbol{\nu}\times\boldsymbol{w}^0).$$

Trivial calculations bring about the following equations for defining the functions p_1 and p_2:

$$\frac{dp_1}{ds} = a\eta_1 + b, \qquad p_1\big|_{s=0} = 0, \tag{3.49}$$

$$\frac{dp_2}{ds} = a\eta_2, \qquad p_2\big|_{s=0} = 0, \tag{3.50}$$

where

$$\eta_1 = (\boldsymbol{h}^0\cdot\boldsymbol{w}^0)^2\,|\boldsymbol{w}^0|^{-2} + \xi\,(\boldsymbol{h}^0\cdot\boldsymbol{w}^0)^2,$$
$$\eta_2 = (\boldsymbol{h}^0\cdot\boldsymbol{w}^0)\,\big(\boldsymbol{h}^0\cdot(\boldsymbol{\nu}\times\boldsymbol{w}^0)\big)\,|\boldsymbol{w}^0|^{-2}.$$

Integrating equations (3.49) and (3.50) along $\Gamma_1(x^0,x)$, $(x^0,x)\in S\times S$, we arrive at:

$$\int_{\Gamma_1(x^0,x)} (a\eta_1 + b)\,ds = p_1(x^0,x), \quad (x^0,x)\in S\times S, \tag{3.51}$$

$$\int_{\Gamma_1(x^0,x)} a\eta_2\,ds = p_2(x^0,x), \quad (x^0,x)\in S\times S. \tag{3.52}$$

Since the function $\boldsymbol{\alpha}_{-2}^{11}$ is known for $(x^0,x)\in S\times S$, therefore, the functions $p_k(x^0,x)$, $k=1,2$, are also known for $(x^0,x)\in S\times S$. Consequently, the problem of determining the functions a and b from equations (3.51) and (3.52) is an integral geometry problem. In a special case, when $c_1 = \text{const} = c_{10}$ and the geodesics $\Gamma_1(x^0,x)$ are straight lines, this problem is a problem of tomography.

Proof of Theorem 3.1. Note that estimate (3.32) was obtained in [32], so we only need to prove estimate (3.33). Following the technique developed in [32], we can write down the inequality as

$$\int_{D(z)}\int_0^{2\pi}\Big[(\tilde{a}\eta_1+\tilde{b})^2 - \Big(\tilde{a}\frac{\partial\eta_1}{\partial\varphi}\Big)^2\Big]\,d\varphi\,d\bar{x} \le \int_{S(z)}\int_{S(z)}\Big|\frac{\partial\tilde{p}_1}{\partial l}\cdot\frac{\partial\tilde{p}_1}{\partial l'}\Big|\,dl\,dl',$$

which is an analogue of inequality (3.32). Since

$$(\tilde{a}\eta_1 + \tilde{b})^2 - \Big(\tilde{a}\frac{\partial\eta_1}{\partial\varphi}\Big)^2 \geq \frac{1}{2}\tilde{b}^2 - \tilde{a}^2\,(\eta_1^2 + \eta_{1\varphi}^2)$$
$$\geq \frac{1}{2}\tilde{b}^2 - \tilde{a}^2\left[(1+\cos^2\varphi)^2 + \sin^2 2\varphi\right],$$

we have

$$\int_0^{2\pi}\Big[(\tilde{a}\eta_1 + \tilde{b})^2 - \Big(\tilde{a}\frac{\partial\eta_1}{\partial\varphi}\Big)^2\Big]\,d\varphi \geq \pi\Big(\tilde{b}^2 - \frac{23}{4}\tilde{a}^2\Big).$$

Consequently,

$$\int_{D(z)}\Big(\tilde{b}^2 - \frac{23}{4}\tilde{a}^2\Big)\,d\bar{x} \leq \frac{1}{\pi}\int_{S(z)}\int_{S(z)}\Big|\frac{\partial\tilde{p}_1}{\partial l}\cdot\frac{\partial\tilde{p}_1}{\partial l'}\Big|\,dl\,dl'. \tag{3.53}$$

Estimate (3.33) follows immediately from inequalities (3.32) and (3.53). This proves the theorem. □

3.3. An inverse problem for electromagnetoelasticity equations with partially nonlinear interaction

In this section, following the original work [27], we present some results of solution to inverse problems for a system of equations of electromagnetoelasticity with partially nonlinear interaction between electromagnetic and elastic fields. The results of the first initial boundary-value problem solution presented in Section 2.2 will be sufficiently used.

Formulation of inverse problem

Our main inverse problem consists in determining the function f from equations (2.26)–(2.33) by appropriate additional information about the solution to direct problem 2.1.

Now we can formulate the inverse problem.

Inverse Problem 3.5. *Determine a set of the functions*

$$\boldsymbol{u} : [0,T]\times\Omega_2 \to \mathbb{R}^3, \quad \boldsymbol{E},\boldsymbol{H} : [0,T]\times\Omega \to \mathbb{R}^3, \quad f : [0,T] \to \mathbb{R}$$

such that

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} = \operatorname{Div} T + f(t)\,\boldsymbol{g}(t,x), \quad (t,x) \in (0,T)\times\Omega_2, \tag{3.54}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), \quad \frac{\partial\boldsymbol{u}}{\partial t}(0,x) = \boldsymbol{u}_1(x), \quad x \in \Omega_2, \tag{3.55}$$

$$\boldsymbol{u}(t,x) = 0, \quad (t,x) \in (0,T)\times\partial\Omega_2, \tag{3.56}$$

$$\varepsilon\frac{\partial \boldsymbol{E}}{\partial t} + \sigma\boldsymbol{E} + \sigma\mu\frac{\partial \widetilde{\boldsymbol{u}}}{\partial t} \times \boldsymbol{H} = \operatorname{rot}\boldsymbol{H}, \quad (t,x) \in (0,T) \times [\Omega_1 \cup \Omega_2], \tag{3.57}$$

$$\mu\frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{rot}\boldsymbol{E} = 0, \qquad \operatorname{div}\mu\boldsymbol{H} = 0, \quad (t,x) \in (0,T) \times [\Omega_1 \cup \Omega_2], \tag{3.58}$$

$$\boldsymbol{E}(0,x) = \boldsymbol{E}_0(x), \quad \boldsymbol{H}(0,x) = \boldsymbol{H}_0(x), \quad x \in \Omega, \tag{3.59}$$

$$\boldsymbol{n} \times \boldsymbol{E} = 0, \quad (t,x) \in (0,T) \times \partial\Omega, \tag{3.60}$$

$$[\boldsymbol{E} \times \boldsymbol{n}]_{\partial\Omega_2} = [\boldsymbol{H} \times \boldsymbol{n}]_{\partial\Omega_2} = 0, \quad (t,x) \in (0,T) \times \partial\Omega_2, \tag{3.61}$$

$$\Phi[\boldsymbol{E}] = \phi(t), \quad t \in [0,T]. \tag{3.62}$$

It is assumed that the functions $\varepsilon, \mu : \overline{\Omega} \to \mathbb{R}_+$, $\sigma : \overline{\Omega} \to \overline{\mathbb{R}}_+$, and $\boldsymbol{E}_0, \boldsymbol{H}_0 : \Omega \to \mathbb{R}^3$ are continuous in the domain $\overline{\Omega} \setminus \partial\Omega_2$ with possible jumps on the surface $\partial\Omega_2$. We also assume that the functions $\boldsymbol{g} : [0,T] \times \Omega_2 \to \mathbb{R}^3$, $\boldsymbol{u}_0, \boldsymbol{u}_1 : \Omega_2 \to \mathbb{R}^3$, and $\phi : [0,T] \to \mathbb{R}$ are given and have sufficient smoothness. As concerns the functional Φ, we assume that it is linear and depends only on the spatial variables. For example, we may assume that the functional Φ has the form

$$\Phi[\boldsymbol{E}] = \int_\Omega \boldsymbol{K}(x) \cdot \boldsymbol{E}(x)\, dx, \tag{3.63}$$

where $\boldsymbol{K} : \Omega \to \mathbb{R}^3$ is a given sufficiently smooth vector-function.

Remark 3.5. It is the additional information (3.62) only that connects the solutions to the Lamé equations and the Maxwell equations. If instead of the additional information (3.62) we specify a certain information about the vector-function $\boldsymbol{u}$, for example, of the form

$$\Phi[\boldsymbol{u}] = \phi(t), \quad t \in [0,T], \tag{3.64}$$

then Inverse Problem 3.5 splits to the inverse problem of determining the function $f(t)$ from equations (3.54)–(3.56) and (3.64) and then the direct problem (3.57)–(3.62).

Solution to inverse problem (3.54)–(3.62)

We now can prove solvability of the inverse problem. We assume that the function f in the free term of equation (3.54) is unknown and the function $\boldsymbol{g}$ is known. By Theorems 2.2 and 2.3, our inverse problem (3.54)–(3.62) is equivalent to the following problem: find a function $f \in L^p((0,T);\mathbb{R})$ such that

$$\Phi[\boldsymbol{E}(f)(t,\cdot)] = \phi(t), \quad \forall t \in [0,T]. \tag{3.65}$$

To prove solvability of problem 3.65 we require the fulfillment of the following regularity conditions for the data ϕ, the function $\boldsymbol{g}$, and the kernel $\boldsymbol{K}$ of the functional Φ:

$$\begin{gathered} \phi \in W^{2,p}\big((0,T_0);\mathbb{R}\big), \\ \boldsymbol{g} \in L^{p'}\big((0,T_0); H^2(\Omega_2;\mathbb{R}^3) \cap H_0^1(\Omega_2;\mathbb{R}^3)\big), \quad T_0 > 0; \end{gathered} \tag{3.66}$$

$$\|\boldsymbol{g}(t,\cdot)\|_{2,2,\Omega} \le C_6(T_1)\, t^{-\alpha} \tag{3.67}$$

for almost all $t \in (0, T_1)$ and certain $\alpha \in (0, 1/p')$ and $T_1 \in (0, T_0)$; and

$$\varepsilon^{-1}\boldsymbol{K} \in H(\mathrm{rot};\Omega), \qquad \boldsymbol{n} \times \boldsymbol{K} = 0, \quad \forall x \in \partial\Omega. \tag{3.68}$$

Remark 3.6. The first assumption in (3.68) is equivalent to the requirement that $\boldsymbol{K} \in H(\mathrm{rot};\Omega_1) \cap H(\mathrm{rot};\Omega_2)$ and $\boldsymbol{n} \times [\varepsilon^{-1}\boldsymbol{K}]_{\partial\Omega_2} = 0$.

Using the results of [46] and formulas (3.63) and (3.68) we easily obtain the relation

$$\Phi[\varepsilon^{-1} \operatorname{rot} \boldsymbol{H}(t,\cdot)] = \int_\Omega \boldsymbol{H}(t,x) \cdot \mathrm{rot}(\varepsilon^{-1}\boldsymbol{K})(x) dx := \Phi_1[\boldsymbol{H}(t,\cdot)]. \tag{3.69}$$

Replacing $\boldsymbol{u}$ in equations (3.54) and (3.57) by $\boldsymbol{u}(f)$, applying the linear functional Φ to both sides of equation (3.57), and using the information (3.65), we obtain

$$\begin{aligned} \phi'(t) = \Phi_1[\boldsymbol{H}(f)(t,\cdot)] - \Phi\Big[\frac{\sigma}{\varepsilon}\boldsymbol{E}(f)(t,\cdot)\Big] - \\ \Phi_2\Big[\frac{\sigma\mu}{\varepsilon}\frac{\partial}{\partial t}\boldsymbol{U}(f)(t,\cdot) \times \boldsymbol{H}(f)(t,\cdot)\Big], \quad \forall t \in [0,T], \end{aligned} \tag{3.70}$$

where

$$\Phi_2[\boldsymbol{E}] = \int_{\Omega_2} \boldsymbol{K}(x) \cdot \boldsymbol{E}(x)\, dx. \tag{3.71}$$

Equations (3.65) and (3.70) imply that the data $(\boldsymbol{u}_1, \boldsymbol{E}_0, \boldsymbol{H}_0, \phi)$ should satisfy the following consistency conditions:

$$\phi(0) = \Phi[\boldsymbol{E}_0], \qquad \phi'(0) = \Phi_1[\boldsymbol{H}_0] - \Phi\Big[\frac{\sigma}{\varepsilon}\boldsymbol{E}_0\Big] - \Phi_2\Big[\frac{\sigma\mu}{\varepsilon}\boldsymbol{U}_1 \times \boldsymbol{H}_0\Big]. \tag{3.72}$$

On the contrary, if the function $f \in L^p((0,T);\mathbb{R})$ solves equation (3.70), then, using formulas (3.69) and (3.71), we can conclude that the function f solves the equation

$$\frac{\partial}{\partial t}\big\{\Phi[\boldsymbol{E}(f)(t,\cdot)] - \phi(t)\big\} = 0, \quad \forall t \in [0,T]. \tag{3.73}$$

Using equation (3.73) and the first condition from (3.72), we can easily show that the function f is a solution to equation (3.65). Differentiating equation (3.70) once again with respect to the variable t and taking into account relation (3.54), we finally obtain

$$\begin{aligned}\phi''(t) = \Phi_1\Big[\frac{\partial}{\partial t}\boldsymbol{H}(f)(t,\cdot)\Big] - \Phi\Big[\frac{\sigma}{\varepsilon}\frac{\partial}{\partial t}\boldsymbol{E}(f)(t,\cdot)\Big] - \\ \Phi_2\Big[\frac{\sigma\mu}{\varepsilon}\frac{\partial}{\partial t}\boldsymbol{u}(f)(t,\cdot) \times \frac{\partial}{\partial t}\boldsymbol{H}(f)(t,\cdot)\Big] - \\ \Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\operatorname{Div} T(\boldsymbol{u}(f)(t,\cdot)) \times \boldsymbol{H}(f)(t,\cdot)\Big] - \\ f(t)\,\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot) \times \boldsymbol{H}(f)(t,\cdot)\Big], \quad \text{for a.a. } t \in (0,T). \quad (3.74)\end{aligned}$$

Note that according to the assumptions (3.66)–(3.68) and Theorems 2.2 and 2.3, each term in (3.74) has a sense.

On the contrary, if $f \in L^p((0,T);\mathbb{R})$ is a solution to equation (3.74), then f should be a solution to the equation

$$\begin{aligned}\frac{\partial}{\partial t}\Big\{\phi'(t) - \Phi_1[\boldsymbol{H}(f)(t,\cdot)] + \Phi\Big[\frac{\sigma}{\varepsilon}\boldsymbol{E}(f)(t,\cdot)\Big] + \\ \Phi_2\Big[\frac{\sigma\mu}{\varepsilon}\frac{\partial}{\partial t}\boldsymbol{U}(f)(t,\cdot) \times \boldsymbol{H}(f)(t,\cdot)\Big]\Big\} = 0, \quad \text{for a.a. } t \in (0,T). \quad (3.75)\end{aligned}$$

However, as follows from (3.75) and from the last condition in (3.72), the function f is a solution to equation (3.70). Thus, we have shown that the solutions to equations (3.65) and (3.74) are equivalent.

In order that it were possible to rewrite equation (3.74) in the form of an equation with a stationary point for f, we assume that $\boldsymbol{g}$ and $\boldsymbol{H}_0$ satisfy the condition

$$\begin{aligned}\Big|\int_{\Omega_2} \frac{\sigma(x)\,\mu(x)}{\varepsilon(x)\,\rho(x)}\boldsymbol{K}(x)\cdot\big[\boldsymbol{g}(t,x) \times \boldsymbol{H}_0(x)\big]\,dx\Big| \\ = \Big|\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot) \times \boldsymbol{H}_0\Big]\Big| \geq 2m, \quad \text{for a.a. } t \in (0,T), \quad (3.76)\end{aligned}$$

where m is a certain positive constant.

Now we can formulate our main result.

Theorem 3.4. *Let $\rho, \lambda, \varkappa$ and ε, μ, σ satisfy conditions* (2.34)*,* (2.35)*, and* (2.42)*; and let the functions $\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{E}_0, \boldsymbol{H}_0$ and $\boldsymbol{g}, \phi, \boldsymbol{K}$ satisfy conditions* (2.36)*–*(2.38)*,* (2.43)*,* (2.44)*,* (3.66)*–*(3.68)*, and* (3.72)*. Then there exists a number $T^* \in (0, \min\{T_0, T_1\})$ for which Inverse Problem* (3.54)*–*(3.62) *has the unique solution $(\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{H}, f)$ satisfying conditions* (2.36)*,* (2.39)*,* (2.45)*, and* (2.46) *for every $T \in (0, T^*)$.*

Remark 3.7. We can also show that the solution $(\boldsymbol{u}, \boldsymbol{E}, \boldsymbol{H}, f)$ of Inverse Problem (3.54)–(3.62) continuously depends on the data $(\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{g}, \phi)$ in the norms of respective spaces. For this, we need to prove a continuous dependence of the solution on the data, just as was done in the case of solutions to direct problems (3.54)–(3.56) and (3.57)–(3.61). Since the proof is very bulky, we omit this part.

Proof of Theorem 3.4. Introduce the family $X(M,T)$ of complete metric subspaces in $L^p((0,T);\mathbb{R})$, this family depending on the two positive constants M and T:

$$X(M,T) = \big\{ f \in L^p((0,T);\mathbb{R}) \ \big|\ \|f\|_{T,0,p} \le M \big\}.$$

If we assume that $f \in X(M,T)$ is a solution to the operator equation (3.74), then we can write it in the following form which is convenient for application of the theorem about a stationary point:

$$\begin{aligned} f(t) = {} & \Big(\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot)\times \boldsymbol{H}(f)(t,\cdot)\Big]\Big)^{-1} \times \\ & \Big\{-\phi''(t) + \Phi_1\Big[\frac{\partial}{\partial t}\boldsymbol{H}(f)(t,\cdot)\Big] - \Phi\Big[\frac{\sigma}{\varepsilon}\,\frac{\partial}{\partial t}\boldsymbol{E}(f)(t,\cdot)\Big] - \\ & \Phi_2\Big[\frac{\sigma\mu}{\varepsilon}\,\frac{\partial}{\partial t}\boldsymbol{u}(f)(t,\cdot)\times \frac{\partial}{\partial t}\boldsymbol{H}(f)(t,\cdot)\Big] - \\ & \Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\operatorname{Div} T(\boldsymbol{u}(f)(t,\cdot))\times \boldsymbol{H}(f)(t,\cdot)\Big]\Big\} := N(f)(t) \end{aligned} \tag{3.77}$$

for a.a. $t \in (0,T)$.

Our task is to show the local solvability of equation (3.77). For this, we set $T_3(M) = \min\{T_0, T_1, T_2(M)\}$, where $T_2(M)$ is the unique positive root of the equation

$$\|\boldsymbol{K}\|_{0,2,\Omega_2}\big[C_2(T_0) + TC_3(T_0,M)\big]\, C_6\,\gamma\rho_0^{-1}\,\|\sigma\|_{0,\infty,\Omega_2}\,\|\mu\|_{0,\infty,\Omega_2}\,T^{1-\alpha} = m.$$

Using estimate (2.47) and condition (3.76), we can show that each function $f \in X(M,T)$ satisfies the following basic inequality for a.a. $t \in (0,T) \subset (0, T_3(M))$:

$$\begin{aligned} & \Big|\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot)\times \boldsymbol{H}(f)(t,\cdot)\Big]\Big| \\ & \quad \ge \Big|\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot)\times \boldsymbol{H}_0\Big]\Big| - \Big|\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot)\times \int_0^t \frac{\partial}{\partial t}\boldsymbol{H}(f)(s,\cdot)ds\Big]\Big| \\ & \quad \ge 2m - \|\boldsymbol{K}\|_{0,2,\Omega_2}\,\gamma\rho_0^{-1}\,\|\sigma\|_{0,\infty,\Omega_2}\,\|\mu\|_{0,\infty,\Omega_2}\,\|\boldsymbol{g}(t,\cdot)\|_{0,\infty,\Omega_2} \times \\ & \qquad \Big\|\frac{\partial}{\partial t}\boldsymbol{H}(f)(t,\cdot)\Big\|_{T,\infty,0,2}\cdot t \\ & \quad \ge 2m - \|\boldsymbol{K}\|_{0,2,\Omega_2}\big[C_2(T_0) + TC_3(T_0,M)\big]C_6\gamma\rho_0^{-1}\|\sigma\|_{0,\infty,\Omega_2} \times \\ & \qquad \|\mu\|_{0,\infty,\Omega_2}\,T^{1-\alpha} \ge m. \end{aligned} \tag{3.78}$$

Taking into account formulas (2.39), (2.47), and (3.78) and the embedding $H^1(\Omega_2;\mathbb{R})\cdot H^1(\Omega_2;\mathbb{R}) \hookrightarrow L^2(\Omega_2;\mathbb{R})$ (where the dot denotes the functional product), we can estimate the nonlinear operator N:

$$\|N(f)\|_{T,0,p} \le \frac{1}{m}\Big\{\|\phi''\|_{T_0,0,p} + \|\mathrm{rot}(\varepsilon^{-1}\boldsymbol{K})\|_{0,2,\Omega}\Big\|\frac{\partial}{\partial t}\boldsymbol{H}(f)(t,\cdot)\Big\|_{0,2,\Omega} + \gamma\,\|\sigma\|_{0,\infty,\Omega}\,\|\boldsymbol{K}\|_{0,2,\Omega}\Big\|\frac{\partial}{\partial t}\boldsymbol{E}(f)(t,\cdot)\Big\|_{0,2,\Omega} + \gamma\,\|\sigma\|_{0,\infty,\Omega_2}\,\|\mu\|_{0,\infty,\Omega_2}\,\|\boldsymbol{K}\|_{0,2,\Omega}\bigg(\Big\|\frac{\partial}{\partial t}\boldsymbol{u}(f)(t,\cdot)\Big\|_{0,\infty,\Omega_2}\Big\|\frac{\partial}{\partial t}\boldsymbol{H}(f)(t,\cdot)\Big\|_{0,2,\Omega_2} + \rho_0^{-1}C_7\big(\|\lambda\|_{0,\infty,\Omega_2},\|\varkappa\|_{0,\infty,\Omega_2},\Omega_2\big)\,\|\boldsymbol{u}(f)(t,\cdot)\|_{3,2,\Omega_2}\,\|\boldsymbol{H}(f)(t,\cdot)\|_{1,2,\Omega_2}\bigg)\Big\}$$
$$\le \frac{1}{m}\Big\{\|\phi''\|_{T_0,0,p} + \big(1 + \|\boldsymbol{g}\|_{T_0,p',2,2} + \|\boldsymbol{K}\|_{0,2,\Omega_2} + \|\mathrm{rot}(\varepsilon^{-1}\boldsymbol{K})\|_{0,2,\Omega}\big)\times \big[C_8(T_0) + TC_9(T_0,M)\big]\Big\}, \quad \forall T \in (0, T_3(M)]. \quad (3.79)$$

Let us choose a pair of numbers (M, T^*) from the conditions

$$M = \frac{2}{m}\Big\{\|\phi''\|_{T_0,0,p} + C_8(T_0)\,\big(1 + \|\boldsymbol{g}\|_{T,0,p',2,2} + \|\boldsymbol{K}\|_{0,2,\Omega} + \|\mathrm{rot}(\varepsilon^{-1}\boldsymbol{K})\|_{0,2,\Omega}\big)\Big\}, \quad (3.80)$$

$$\frac{1}{m}\,\big(1 + \|\boldsymbol{g}\|_{T_0,p',2,2} + \|\boldsymbol{K}\|_{0,2,\Omega} + \|\mathrm{rot}(\varepsilon^{-1}\boldsymbol{K})\|_{0,2,\Omega}\big)\,C_9(T_0,M)\,T^* \le M, \quad T^* \in (0, T_3(M)]. \quad (3.81)$$

Formulas (3.79)–(3.81) imply that the operator N maps $X(M,T)$ into itself for all $T \in (0, T^*]$.

Now we estimate the difference $N(f_2) - N(f_1)$ for arbitrary functions $f_1, f_2 \in X(M,T)$, $T \in (0, T^*]$. To this end, we consider the equation

$$N(f_2)(t) - N(f_1)(t) = \Big(\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot)\times\boldsymbol{H}(f_2)(t,\cdot)\Big]\Big)^{-1}\Big\{\Phi_1\Big[\frac{\partial}{\partial t}\boldsymbol{H}(f_2)(t,\cdot) - \frac{\partial}{\partial t}\boldsymbol{H}(f_1)(t,\cdot)\Big] - \Phi\Big[\frac{\sigma}{\varepsilon}\Big(\frac{\partial}{\partial t}\boldsymbol{E}(f_2)(t,\cdot) - \frac{\partial}{\partial t}\boldsymbol{E}(f_1)(t,\cdot)\Big)\Big] - \Phi_2\Big[\frac{\sigma\mu}{\varepsilon}\Big(\frac{\partial}{\partial t}\boldsymbol{U}(f_2)(t,\cdot) - \frac{\partial}{\partial t}\boldsymbol{u}(f_1)(t,\cdot)\Big)\times\frac{\partial}{\partial t}\boldsymbol{H}(f_2)(t,\cdot)\Big] - \Phi_2\Big[\frac{\sigma\mu}{\varepsilon}\,\frac{\partial}{\partial t}\boldsymbol{u}(f_1)(t,\cdot)\times\Big(\frac{\partial}{\partial t}\boldsymbol{H}(f_2)(t,\cdot) - \frac{\partial}{\partial t}\boldsymbol{H}(f_1)(t,\cdot)\Big)\Big] - \Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\,\mathrm{Div}\,T\big(\boldsymbol{u}(f_2)(t,\cdot) - \boldsymbol{u}(f_1)(t,\cdot)\big)\times\boldsymbol{H}(f_2)(t,\cdot)\Big] - \Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\,\mathrm{Div}\,T(\boldsymbol{u}(f_1)(t,\cdot))\times\big[\boldsymbol{H}(f_2)(t,\cdot) - \boldsymbol{H}(f_1)(t,\cdot)\big]\Big]\Big\} -$$

$$\Big(\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot)\times\boldsymbol{H}(f_2)(t,\cdot)\Big]\Big)^{-1}\times$$
$$\Phi_2\Big[\frac{\sigma\mu}{\varepsilon\rho}\boldsymbol{g}(t,\cdot)\times\big[\boldsymbol{H}(f_2)(t,\cdot)-\boldsymbol{H}(f_1)(t,\cdot)\big]\Big]N(f_1)(t), \tag{3.82}$$

for a.a. $t\in(0,T)$.

Using formulas (2.40), (2.41), (2.47), (2.48), (3.79), and (3.82), we can easily derive the estimate

$$\begin{aligned}&\big|N(f_2)(t)-N(f_1)(t)\big|\\&\quad\le C_{10}(T_0,M,m^{-1})\left(1+\|\boldsymbol{g}\|_{T_0,p',2,2}+\|\boldsymbol{K}\|_{0,2,\Omega_2}+\|\mathrm{rot}(\varepsilon^{-1}\boldsymbol{K})\|_{0,2,\Omega}\right)\times\\&\qquad\Big(\int_0^t h_1(t-s)\,\|f_2-f_1\|_{s,0,p}\,ds+\|f_2-f_1\|_{t,0,p}\Big),\quad\forall t\in[0,T],\end{aligned} \tag{3.83}$$

where

$$h_1(t)=\exp\big\{t\,\|\sigma\|_{0,\infty,\Omega}+C_6(T_0)M\|\boldsymbol{g}\|_{T,0,p',2,2,\Omega_2}\big\}.$$

Estimate (3.83) implies the inequality

$$\begin{aligned}&\big\|N(f_2)-N(f_1)\big\|_{t,0,p}\\&\quad\le C_{10}(T_0,M,m^{-1})\left(1+\|\boldsymbol{g}\|_{T_0,p',2,2}+\|\boldsymbol{K}\|_{0,2,\Omega_2}+\|\mathrm{rot}(\varepsilon^{-1}\boldsymbol{K})\|_{0,2,\Omega}\right)\times\\&\qquad[1+T_0h_1(T_0)]\int_0^t\|f_2-f_1\|_{s,0,p}\,ds\\&\quad:=C_{11}(T_0,M,m^{-1})\int_0^t\|f_2-f_1\|_{s,0,p}\,ds,\quad\forall t\in[0,T].\end{aligned} \tag{3.84}$$

Inequality (3.84) enables us to establish the following estimate for the iterations N^n of the operator N:

$$\big\|N^n(f_2)-N^n(f_1)\big\|_{t,0,p}\le\frac{C_{11}^n(T_0,M,m^{-1})}{(n-1)!}\int_0^t(t-s)^{n-1}\|f_2-f_1\|_{s,0,p}\,ds$$
$$\text{for a.a. } t\in(0,T),\quad\forall n\in\mathbb{N}.$$

We now apply the generalized contraction mapping principle (see [43, p. 103]). This ensures equation (3.74) to have a unique solution $f\in X(M,T)$ for every $T\in(0,T^*)$, which proves the theorem. □

3.4. An inverse problem for electromagnetoelasticity equations with complete nonlinear interaction

In this section, we present the new results of the solution to an inverse problem for a electromagnetoelasticity system in the case of complete nonlinear interaction of electromagnetic and elastic waves. The results of the first initial boundary value problem solution, presented in Section 2.3, will be sufficiently used. Some of these results were announced in short communication [36].

Formulation of an inverse problem

Let us consider one of possible formulations of inverse problems for the direct problem earlier considered in Section 2.3. Let us now formulate the inverse problem to be be studied.

Inverse Problem 3.6. *Determine a set of the functions*

$$h : Q_T \to \mathbb{R}, \quad u : Q_T \to \mathbb{R}, \quad \phi : [0,T] \to \mathbb{R}$$

such that

$$h_t = (rh_z)_z - (hu_t)_z - (rj)_z, \quad (z,t) \in Q_T, \tag{3.85}$$

$$u_{tt} = (\nu^2 u_z)_z - phh_z + \phi(t)g(z,t), \quad (z,t) \in Q_T, \tag{3.86}$$

$$h(\pm l, t) = 0, \qquad u(\pm l, t) = 0, \quad t \in [0,T], \tag{3.87}$$

$$h(z,0) = h_0(z), \quad u(z,0) = u_0(z), \quad u_t(z,0) = u_1(z), \quad z \in \Omega, \tag{3.88}$$

$$\int_\Omega \rho(z) h h_z dz = -\frac{1}{2}\int_\Omega \rho_z h^2 dz = \psi(t), \quad t \in [0,T], \tag{3.89}$$

where $\rho \in \overset{\circ}{W}{}^1_2(\Omega)$, $Q_T = \Omega \times [0,T]$ *and* $\Omega = (-l, l)$.

The functions r, ν, ϕg, j are supposed to be smooth functions with possible jumps at the points $z_m : -l < z_1 < z_2 < \ldots < z_m < l$, $r(z) \geq r_0 > 0$; $\nu(z) \geq \nu_0 > 0$, p is a positive number; and

$$\int_\Omega \rho(z) g(z,t)\, dz \geq \rho_0 > 0, \quad t \in [0,T]. \tag{3.90}$$

At the points of discontinuity we assume the fulfillment of the following transmission conditions

$$[h(z,t)]_{z=z_i} = 0, \quad [u(z,t)]_{z=z_i} = 0, \tag{3.91}$$

$$\begin{gathered} [r(z)(h_z(z,t) - j(z,t))]_{z=z_i} = 0, \\ [\nu^2(z) u_z(z,t)]_{z=z_i} = 0, \quad i = 1, 2, \ldots, m. \end{gathered} \tag{3.92}$$

The solution of the direct problem (3.85)–(3.88) and the transmission conditions (3.92) are understood in the generalized sense (see Section 2.3). Conditions (3.91) are valid in the classical sense because h, u are continuous functions. Let us introduce the function $f(z,t) = \phi(t)g(z,t)$ and repeat the formulation of the direct problem.

Definition 3.2. The functions $h(z,t) \in \overset{\circ}{V}_2(Q_T), u(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ are called the generalized solution of the direct problem (3.85)–(3.88), (3.91), (3.92) if for almost all $t_1 \in [0,T]$ they satisfy the equalities

$$\begin{aligned} -\int_{Q_{t_1}} h\eta_t\,dz\,dt + \int_{Q_{t_1}} rh_z\eta_z\,dz\,dt - \int_{Q_{t_1}} hu_t\eta_z\,dz\,dt \\ = \int_{Q_{t_1}} rj\eta_z\,dz\,dt + \int_\Omega h_0(z)\eta(z,0)\,dz - \int_\Omega h(z,t_1)\eta(z,t_1)\,dz, \end{aligned} \tag{3.93}$$

$$\begin{aligned} -\int_{Q_{t_1}} u_t\zeta_t\,dz\,dt + \int_{Q_{t_1}} \nu^2 u_z\zeta_z\,dz\,dt + \int_{Q_{t_1}} phh_z\zeta\,dz\,dt \\ = \int_{Q_{t_1}} f\zeta\,dz\,dt + \int_\Omega u_1\zeta(z,0)\,dz\,dt - \int_\Omega u_t(z,t_1)\zeta(z,t_1)\,dz, \end{aligned} \tag{3.94}$$

$u(z,0) = u_0(z)$, $z \in \Omega$, where $Q_{t_1} = \Omega \times (0,t_1)$, $\eta(z,t), \zeta(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$.

As was demonstrated in Section 2.3, direct problem (3.85)–(3.88), (3.91), (3.92) for the known function $\phi(t)$ has the solution $h(z,t) \in \overset{\circ}{V}_2\,(Q_T)$, $u(z,t) \in \overset{\circ}{W}{}_2^{1,1}(Q_T)$ satisfying the inequalities

$$\begin{aligned} \frac{1}{2}\int_\Omega \{ph^2(z,t_1) + \nu^2(z)u_z^2(z,t_1) + u_t^2(z,t_1)\}\,dz + \frac{1}{2}\int_{Q_{t_1}} r(z)h_z^2(z,t)\,dz\,dt \\ \le C_1\left(1 + t_1\int_{Q_{t_1}} f^2(z,t)\,dz\,dt\right) \end{aligned}$$

for almost all $t_1 \in [0,T]$ and

$$\|u(z,t)\|_{\overset{\circ}{W}{}_2^{1,1}(Q_T)} \le C_1(1 + \|f\|_{2,Q_T}), \qquad \max_{Q_T} |h(z,t)| \le C_1.$$

In addition, the integral equality (3.86)–(3.88) can be rewritten in a different form if $u_{tt} \in L_2(0,T;H^{-1}(\Omega))$. Here $H^{-1}(\Omega)$ is the space dual to $H_0^1(\Omega) = \overset{\circ}{W}{}_2^1\,(\Omega)$. Setting with $\langle\cdot,\cdot\rangle$ the scalar product between elements of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ it is easy to establish from (3.94) the following equalities

$$\begin{aligned} \langle u_{tt}, \xi(z)\rangle + \int_\Omega \nu^2(z)u_z(z,t)\xi(z)\,dz + \int_\Omega ph(z,t)h_z(z,t)\xi(z)\,dz \\ = \int_\Omega f(z,t)\xi(z)dz, \qquad u_t(z,0) = u_1(z), \quad u(z,0) = u_0(z). \end{aligned} \tag{3.95}$$

Note that from conditions (3.95) follows the estimate [19, Chapter VII, p. 318–385]

$$\max_{t\in[0,T]}\{\|u\|_{H_0^1(\Omega)} + \|u_t\|_{2,\Omega} + \|u_{tt}\|_{L^2(0,T;H^{-1}(\Omega))}\}$$
$$\leq C_2\{\|hh_z\|_{2,Q_T} + \|f\|_{2,Q_T} + \|u_0\|_{H_0^1(\Omega)} + \|u_1\|_{2,\Omega}\}. \quad (3.96)$$

Consider the direct problem (3.85)–(3.88), (3.91), (3.92) for two different functions $f_k, k = 1, 2$. The solutions corresponding to these functions will be marked as $h_k(z,t)$, $u_k(z,t)$, $k = 1, 2$. To solve the Inverse Problem 3.6 we need to estimate the differences $h_2 - h_1$ and $u_2 - u_1$. As was done in Theorem 2.5, we introduce two functions v, w by the formulas

$$v = (h_2 - h_1)e^{-\lambda t}, \qquad w = (u_2 - u_1)e^{-\lambda t},$$

where λ is a positive number. The functions v, w are the generalized solution of the problem

$$v_t + \lambda v = (rv_z)_z - (h_2 w_t + \lambda h_2 w + u_{1,t} v)_z, \quad (3.97)$$
$$w_{tt} + 2\lambda w_t + \lambda^2 w = (\nu^2 w_z)_z - ph_2 v_z - ph_{1,z} v + (f_2 - f_1)e^{-\lambda t}, \quad (3.98)$$

with boundary conditions (3.87), zero initial data and transmission conditions (3.91), (3.92) formulated for the functions v, w. This difference was estimated when demonstrating of the uniqueness theorem 2.5 for the case $f_2 - f_1 \equiv 0$. Setting $t_1 = \tau$ we obtain the formula similar to (2.82) with the right-hand side $\int_{Q_\tau} w_t(f_2 - f_1)\, dz\, dt$:

$$\frac{1}{2}\|v(z,\tau)\|^2_{2,\Omega} + \frac{1}{2}\|w_t(z,\tau)\|^2_{2,\Omega} + \frac{\lambda^2}{2}\|w(z,\tau)\|^2_{2,\Omega} + 2\lambda\|w_t(z,\tau)\|^2_{2,\Omega} +$$
$$\lambda p\|v\|^2_{2,Q_\tau} + pr_0\|v_z\|^2_{2,Q_\tau} - pC_1 \max_{Q_\tau}|v|\|v_z\|^2_{2,Q_\tau} -$$
$$\lambda p\|v_z\|^2_{2,Q_\tau}\|w\|_{2,Q_\tau} - pC_1 \max_{Q_\tau}|v|\|w_t\|^2_{2,Q_\tau}$$
$$\leq \left|\int_{Q_\tau} w_t(f_2 - f_1)\, dz\, dt\right| \leq \frac{1}{2\epsilon}\|w_t\|^2_{2,Q_\tau} + \epsilon\|f_2 - f_1\|^2_{2,Q_\tau}.$$

We have the following inequalities $\|w_t\|_{2,Q_\tau} \leq \|w_t\|_{2,Q_{t_1}}$ and $\|f_2 - f_1\|_{2,Q_\tau} \leq \|f_2 - f_1\|_{2,Q_{t_1}}$ for $\tau \in [0, t_1]$. Applying Theorem 7.1 from [25, Chapter III, p. 181] for parabolic equation (3.97) we obtain

$$\max_{Q_\tau}|v(z,t)| \leq C_3(\|w_t^2\|_{1,r,Q_\tau} + \lambda\|w^2\|_{1,r,Q_\tau}),$$

where $r = 2/(1 - 2\kappa)$, $\kappa \in (0, 1/2)$. However the first term in the right-hand side of the latter inequality can be estimated as follows:

$$\|w_t^2\|_{1,r,Q_\tau} \leq \|w_t^2\|_{1,r,Q_{t_1}} = \left(\int_0^{t_1}\left(\int_\Omega w_t^2 dz\right)^r dt\right)^{1/r} = \left(\int_0^{t_1}\|w_t(z,\tau)\|^{2r}_{2,\Omega} d\tau\right)^{1/r}$$
$$\leq t_1^{1/r} \max_{\tau\in[0,t_1]}\|w_t(z,\tau)\|^2_{2,\Omega}.$$

In the same manner we obtain

$$\|w^2\|_{1,r,Q_\tau} \le t_1^{1/r} \max_{\tau\in[0,t_1]} \|w(z,\tau)\|^2_{2,\Omega},$$

$$\|w_t\|_{2,Q_{t_1}} \le t_1^{1/2} \max_{\tau\in[0,t_1]} \|w_t(z,\tau)\|_{2,\Omega}, \qquad \|v_z\|_{2,Q_{t_1}} \le 2C_1.$$

By the above reason the following inequality takes place

$$\max_{\tau\in[0,t_1]}\Big\{\frac{p}{2}\|v(z,\tau)\|^2_{2,\Omega}+\frac{1}{2}\|w_t(z,\tau)\|^2_{2,\Omega}+\frac{\nu_0^2}{2}\|w_z(z,\tau)\|^2_{2,\Omega}+\lambda^2\|w(z,\tau)\|^2_{2,\Omega}\Big\}+$$
$$2\lambda\|w_t\|^2_{2,Q_{t_1}}+pr_0\|v_z\|^2_{2,Q_{t_1}}+\lambda p\|v\|^2_{2,Q_{t_1}}-C_4t^{1/r}\lambda \max_{\tau\in[0,t_1]}\|w_t(z,\tau)\|^2_{2,\Omega}-$$
$$C_4t^{1/r}\lambda \max_{\tau\in[0,t_1]}\|w(z,\tau)\|^2_{2,\Omega} - 2C_1^2p\lambda t_1^{1/2} \max_{\tau\in[0,t_1]}\|w(z,\tau)\|^2_{2,\Omega}$$
$$\le \frac{t_1}{2\epsilon}\max_{\tau\in[0,t_1]}\|w_t(z,\tau)\|^2_{2,\Omega}+\epsilon\|f_2-f_1\|^2_{2,Q_{t_1}}.$$

Let us choose t_1, λ satisfying the inequalities

$$\frac{\lambda^2}{2} > C_4pt^{1/r}\lambda + 2C_1^2pt^{1/2}\lambda, \qquad \frac{1}{8} > C_4pt_1, \qquad 2\epsilon = \sqrt{t_1} < \frac{1}{8}.$$

For these values of t_1, λ the following inequality is valid

$$\max_{\tau\in[0,t_1]}\Big\{\frac{p}{2}\|v(z,\tau)\|^2_{2,\Omega}+\frac{1}{2}\|w_t(z,\tau)\|^2_{2,\Omega}+\frac{\nu_0^2}{2}\|w_z(z,\tau)\|^2_{2,\Omega}+\lambda^2\|w(z,\tau)\|^2_{2,\Omega}\Big\}+$$
$$2\lambda\|w_t\|^2_{2,Q_{t_1}}+pr_0\|v_z\|^2_{2,Q_{t_1}} \le \sqrt{t_1}\|f_2-f_1\|^2_{2,Q_{t_1}}.$$

Therefore the following estimate holds:

$$\max_{t_1\in[0,T]}\{\|u_2-u_1\|_{H_0^1(\Omega)}+\|u_{2,t}-u_{1,t}\|_{2,\Omega}+p\|h_2-h_1\|_{2,\Omega}+\|h_{2,z}-h_{1,z}\|_{2,\Omega}\}$$
$$\le C_5\sqrt{T}\|f_2-f_1\|_{2,Q_T}. \tag{3.99}$$

The following inequality takes place for any function $\xi \in H_0^1(\Omega)$, $\|\xi\|_{H_0^1(\Omega)} \le 1$:

$$\begin{aligned}\langle u_{2,tt}-u_{1,tt},\xi\rangle &\le C_3T\|\phi_2-\phi_1\|_2 + |\langle f_2-f_1,\xi\rangle| \\ &\le C_3T\|\phi_2-\phi_1\|_2 + C_4\sqrt{T}\|\phi_2-\phi_1\|_2 \\ &\le C_6\sqrt{T}\|\phi_2-\phi_1\|_2, \end{aligned}\tag{3.100}$$

where $\|\cdot\|_2$ means the usual $L_2(0,T)$-norm. From formulas (3.89), (3.90), (3.95) we obtain the following equation for the definition of the function $\phi(t)$:

$$\phi(t) = \frac{\langle u_{tt}, \rho(z)\rangle + \int_\Omega \nu^2(z) u_z(z,t)\rho'(z)\,dz + \int_\Omega p h(z,t) h_z(z,t)\rho(z)\,dz}{\int_\Omega \rho(z) g(z,t)\,dz}. \tag{3.101}$$

Let us mark as $W(\phi) : L_2(0,T) \mapsto L_2(0,T)$ the operator defined by the formula

$$W(\phi) = \frac{\langle u_{tt}, \rho\rangle + \int_\Omega \nu^2 u_z \rho'\,dz + p\psi}{\int_\Omega \rho g\,dz}, \tag{3.102}$$

where $u = u(z,t;\phi)$, $h = h(z,t;\phi)$ are the solution of Inverse Problem 3.6 and ψ is the additional information (3.89).

The main results

Now we are ready to formulate and prove our main results.

Theorem 3.5. *Let the function $\phi(t)$ be a fixed point of the operator $W(\phi)$, i.e., $\phi = W(\phi)$. Then the functions $u(z,t;\phi)$, $h(z,t;\phi)$, $\phi(t)$ are the solution of Inverse Problem 3.6. The reciprocal statement is valid: let $u(z,t;\phi)$, $h(z,t;\phi)$, $\phi(t)$ be the solution of Inverse Problem 3.6, then $\phi = W(\phi)$.*

Proof. Let the function ϕ be a fixed point of the operator W. Then for the function $\phi(t)$ equality (3.101) is valid. Taking into account (3.101) we obtain equality (3.89), i.e., the functions $u(z,t;\phi)$, $h(z,t;\phi)$, $\phi(t)$ are the solution of Inverse Problem 3.6. The opposite statement is obtained in a similar way. □

The following theorem of existence and uniqueness holds.

Theorem 3.6. *For sufficiently small values $T > 0$, Inverse Problem 3.6 has the unique solution, which can be obtained by the method of successive approximations.*

Proof. Using formula (3.102) we arrive at:

$$W(\phi_2) - W(\phi_1) = \frac{\langle \tilde{u}_{tt}, \rho\rangle + \int_\Omega \nu^2 \tilde{u}_z \rho'\,dz}{\int_\Omega \rho g\,dz},$$

where

$$\tilde{u} = u(z,t;\phi_2) - u(z,t;\phi_1).$$

Application of estimates (3.96), (3.99), (3.100) implies that for sufficiently small values $T > 0$ this mapping is a contracted one, which together with Theorem 3.5 proves our statement. □

Chapter 4

Numerical solution of inverse problems

In this chapter, we present some results of the numerical solution to inverse problems for a system of equations describing the linear processes of interaction of electromagnetic and elastic waves based on motion of particles.

Basic equations

The first attempts to apply theory of electromagnetoelasticity to the study of the process of wave propagation in elastic conductive media were made in [16, 23]. Knopoff studied the influence of electromagnetic fields on the propagation of elastic waves and arrived at the conclusion that in the class of geophysical problems the effect of electromagnetic phenomena on the process of propagation of elastic waves is negligible, at least, in case of comparatively small electromagnetic disturbances.

We assume that the model under consideration satisfies the basic hypotheses of continuum mechanics: continuity, Euclidity, and absoluteness of time. The first hypothesis means that an uninterrupted continuum is considered, the second one implies the possibility to introduce a Cartesian frame of reference for all points, and according to the third hypothesis relativistic effects are not taken into account. Moreover, the model is inapplicable in the case of strong magnetic fields. We also assume that electromagnetoelastic waves arise due to the action of mechanical perturbations, and that one can neglect the effect of electromagnetic waves on the process of propagation of elastic oscillations and also neglect the displacement currents as compared with conduction currents. Finally, we will consider the case of small perturbations.

Now we can write the governing equations. The assumption that we consider the fields of small perturbations allows us to consider the linearized statement of the problem when the vector of the magnetic field intensity, the vector of the electric field intensity, and the displacement vector can be represented in the form

$$(\boldsymbol{h}^0, \mathbf{0}, \mathbf{0}) + (\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{u}),$$

where $(\boldsymbol{h}^0, \mathbf{0}, \mathbf{0})$ is the value related to the unperturbed state of the medium ($\boldsymbol{h}^0$ is a constant vector); and the vectors $\boldsymbol{H} = (H_1, H_2, H_3)$,

$\boldsymbol{E} = (E_1, E_2, E_3)$, and $\boldsymbol{u} = (u_1, u_2, u_3)$ correspond to small perturbations of the electromagnetic and elastic fields. Besides, in view of our assumptions, one can reckon that the process of elastic wave propagation is described by the usual system of differential equations of the theory of elasticity:

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \operatorname{Div} T, \tag{4.1}$$

where the stress tensor T is defined in terms of the components u_i of the displacement vector and in the case of an isotropic electromagnetoelastic medium such tensor has the form

$$\begin{gathered} T = \lambda \operatorname{tr} S \cdot I + 2\varkappa S, \\ S_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3. \end{gathered} \tag{4.2}$$

Propagation of electromagnetic waves through an elastic conductive medium is described in our case by the following system:

$$\operatorname{rot} \boldsymbol{H} = \boldsymbol{J}, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = -\operatorname{rot} \boldsymbol{E}, \qquad \operatorname{div} \boldsymbol{B} = 0, \tag{4.3}$$

where, in virtue of our assumptions, the constitutive relations are written as

$$\boldsymbol{B} = \mu(\boldsymbol{h}^0 + \boldsymbol{H}), \qquad \boldsymbol{J} = \sigma\Big(\boldsymbol{E} + \mu \frac{\partial \boldsymbol{u}}{\partial t} \times \boldsymbol{h}^0\Big). \tag{4.4}$$

Now we proceed to the statement of the direct problem for differential equations (4.1)–(4.4). Consider the rectangular Cartesian frame of reference $(x_1, x_2, x_3) = x$. Let the plane $x_3 = 0$ be the interface of two media of the types “air” ($x_3 < 0$) and “conductive ground” ($x_3 > 0$). Electromagnetic and elastic characteristics of the ground are described by piecewise constant functions with break planes parallel to the plane $x_3 = 0$. We assume that elastic oscillations arise under the action of a force source concentrated at the origin

$$T_{k3}\big|_{x_3=0} = \delta_{k3} f(t)\, \delta(x_1, x_2), \qquad k = 1, 2, 3, \tag{4.5}$$

where $\delta(\cdot)$ stands for the Dirac’s mass.

As concerns the force source and initial data, we assume that the function $f(t)$ and the electromagnetoelastic field are absent before the moment $t = 0$, i.e.,

$$(\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{u}, f)\big|_{t<0} \equiv 0. \tag{4.6}$$

To single out the unique solution to the direct problem, it is necessary to add the radiation condition at infinity:

$$|\boldsymbol{H}| \to 0, \quad |\boldsymbol{E}| \to 0 \quad \text{for} \quad |x| \to \infty. \tag{4.7}$$

Moreover, on the planes where the coefficients of the problem have breaks we assume the standard interface conditions

$$[H_k] = [E_k] = [u_m] = [T_{m3}] = 0, \quad k = 1, 2, \quad m = 1, 2, 3. \tag{4.8}$$

Thus, the direct problem consists in finding the vector functions $\boldsymbol{H}$, $\boldsymbol{E}$, and $\boldsymbol{u}$ which satisfy equations (4.1)–(4.8), provided the elastic and electromagnetic characteristics of the medium, and the constant vector $\boldsymbol{h}^0$ characterizing the external magnetic is known.

Our main task consists in showing the possibility of applying the optimization approach to the simultaneous determination of electromagnetic and elastic characteristics of the medium and the function $f(t)$ from system (4.1)–(4.8), basing on some additional information about the components of the vector functions $\boldsymbol{E}$ and $\boldsymbol{u}$. We shall study a special case of this problem which, however, will reflect many principal points of the more general case.

As concerns the form of a sensing signal (i.e., the function $f(t)$), in most cases of real geophysical investigations, is either unknown or is given only approximately, while its accurate estimate is necessary for practical solution of many inverse problems.

4.1. The first inverse problem

Now let us state the first inverse problem (see Avdeev, Goryunov, and Primenko, 1996, 1997). Let z denote the variable x_3. Consider the functions

$$v(z) = \Big(\frac{\lambda + 2\varkappa}{\rho}\Big)^{1/2}, \qquad c(z) = \Big(\frac{1}{\sigma\mu}\Big)^{1/2},$$

where $v(z)$ is the velocity of longitudinal waves and $c(z)$ is the rate of the diffusion process of electromagnetic waves.

We shall say that the functions $v(z)$, $c(z)$, and $f(t)$ belong to the class $\mathfrak{M}$ if there exist positive constants v_m, c_m, f_m, z_m, z'_m, and t_m such that

$$V(z) = \begin{cases} v_m, & z \in (z'_{m-1}, z'_m), \quad m = \overline{1, k+1}, \\ v_{k+1}, & z > z'_{k+1}, \end{cases}$$

$$c(z) = \begin{cases} c_m, & z \in (z_{m-1}, z_m), \quad m = \overline{1, n+1}, \\ c_{n+1}, & z > z_{n+1}, \end{cases}$$

$$f(t) = \begin{cases} f_m, & t \in (t_{m-1}, t_m), \quad m = \overline{1, l+1}, \\ 0, & t > t_{m+1}, \end{cases}$$

where $z_0 = z'_0 = t_0 = 0$ and $n, k, l \in \mathbb{N}$.

Henceforth we will always assume that the functions $v(z)$, $c(z)$, and $f(t)$ belong to the class $\mathfrak{M}$.

Consider the functions

$$u(z,t) = \Re F_{x_1x_2}(u_3)\big|_{\nu_1=\nu_2=0}, \qquad e(z,t) = \Re F_{x_1x_2}(E_1)\big|_{\nu_1=\nu_2=0},$$

where $F_{x_1x_2}(\cdot)$ stands for the generalized Fourier transform with respect to the variables x_1, x_2; and (ν_1, ν_2) are the dual variables. Starting with equations (4.1)–(4.8), we can write down the system of relations for the functions u and e in the domain $z \geq 0$ as follows:

$$\frac{\partial^2 u}{\partial t^2} = v^2(z)\frac{\partial^2 u}{\partial z^2}, \qquad (t,z) \in \mathbb{R} \times \Omega', \tag{4.9}$$

$$u\big|_{t<0} \equiv 0, \qquad \frac{\partial u}{\partial z}\bigg|_{z=0} = F(t), \tag{4.10}$$

$$[u]_{z=z'_m} = \left[\frac{\partial u}{\partial z}\right]_{z=z'_m} = 0, \qquad m = \overline{1, k+1}, \tag{4.11}$$

$$\frac{\partial e}{\partial t} = c^2(z)\frac{\partial^2 e}{\partial z^2} + \mu h^0 \frac{\partial^2 u}{\partial t^2}, \qquad (t,z) \in \mathbb{R} \times \Omega, \tag{4.12}$$

$$e\big|_{t<0} \equiv 0, \qquad \lim_{z\to\infty} e = 0, \qquad \frac{\partial e}{\partial z}\bigg|_{z=0} = 0, \tag{4.13}$$

$$[e]_{z=z_m} = \left[\frac{\partial e}{\partial z}\right]_{z=z_m} = 0, \qquad m = \overline{1, n+1}, \tag{4.14}$$

where

$$\Omega' = \mathbb{R}_+ \setminus \{z = z'_m,\ m = \overline{1, k+1}\},$$
$$\Omega = \mathbb{R}_+ \setminus \{z = z_m,\ m = \overline{1, n+1}\},$$

$F(t) = (\lambda(0) + 2\varkappa(0))^{-1} f(t)$, and h is a constant characterizing the external magnetic field.

Now let us formulate the first inverse problem that will be studied below.

Inverse Problem 4.1. *Find the functions $v(z)$, $c(z)$, $f(t) \in \mathfrak{M}$ (i.e., a set of the numbers v_m, c_m, and f_m) if the following additional information on the solutions to problems (4.9)–(4.11) and (4.12)–(4.14) is known:*

$$u\big|_{z=0} = u_0(t), \tag{4.15}$$

$$e\big|_{z=0} = e_0(t), \qquad t \in \overline{\mathbb{R}}_+, \tag{4.16}$$

and the numbers μ and h are known, too.

Remark 4.1. Without loss of generality, we assume that $\mu = \mu_0$, where μ_0 is the magnetic permeability of vacuum.

To solve Inverse Problem 4.1 numerically, an optimization approach was used based on minimizing data misfit functionals.

At the first stage, the initial boundary value problem (4.9)–(4.11) describing elastic waves propagation in a vertically inhomogeneous medium was considered.

In this model, the medium is assumed to be a stack of homogeneous layers over a homogeneous half-space.

Concerning system (4.9)–(4.11), we considered the inverse problem of reconstructing the functions $v(z)$, $f(t) \in \mathfrak{M}$ from the additional information (4.15).

Applying the Fourier transform with respect to the variable t, we rewrite the original statement (4.9)–(4.11), (4.15) in the following form:

$$\frac{d^2}{dz^2}u(z,\omega) + \nu^2 u(z,\omega) = 0, \qquad z \in \Omega', \tag{4.17}$$

$$\frac{du(z,\omega)}{dz}\Big|_{z=0} = F(\omega), \tag{4.18}$$

$$[u(z,\omega)]_{z=z'_m} = \Big[\frac{du(z,\omega)}{dz}\Big]_{z=z'_m} = 0, \qquad m = \overline{1,k+1}, \tag{4.19}$$

where $\nu^2 = \omega^2 v^{-2}(z)$ and

$$F(\omega) = \int_0^{+\infty} F(t)\exp(-\mathrm{i}\omega t)\,dt.$$

To single out the unique solution, we assume the principle of the limit absorption to be satisfied, i.e.,

$$u(z,\omega) = \lim_{\varepsilon\to+0} u(z,\omega - \mathrm{i}\varepsilon), \tag{4.20}$$

where

$$\lim_{z\to+\infty} u(z,\omega - \mathrm{i}\varepsilon) = 0. \tag{4.21}$$

The additional information (4.15) can be represented as $u(z,\omega)\big|_{z=0} = u_0(\omega)$.

We seek for the solution to the inverse problem as a minimum point of the functional

$$\Phi_1[n(z),F(\omega)] = \int_{\omega_1}^{\omega_2} \big|u_0(\omega) - B_1[n(z),F(\omega)](\omega)\big|^2 d\omega, \tag{4.22}$$

where (ω_1,ω_2) is the range of temporal frequencies defined by the spectral contents $F(\omega)$ of a sensing signal, and $B_1[n(z),F(\omega)]$ is a nonlinear operator mapping the functions $n(z) = v^{-2}(z)$ and $F(\omega)$ into the trace of the solution to the direct problem (4.17)–(4.21) at the point $z = 0$.

One can prove the Fréchet differentiability of functional (4.22) with respect to its arguments $n(z)$ and $F(\omega)$ and then obtain the following expressions for its gradients:

$$\nabla_{n(z)}\Phi_1[n(z),F(\omega)](\xi) = -2\Re\int_{\omega_1}^{\omega_2}(\omega+\mathrm{i}\varepsilon)^2 F(\omega)\times \big[u_0(\omega)-B_1[n(z),F(\omega)](\omega)\big]\,\bar{\mathcal{G}}_1(\xi,\omega)\,d\omega, \tag{4.23}$$

$$\nabla_{F(\omega)}\Phi_1[n(z),F(\omega)](\omega) = -2\Re\big[u_0(\omega)-B_1[n(z),F(\omega)](\omega)\big]\,\bar{\mathcal{G}}_1(\xi,\omega) - 2\mathrm{i}\Im\big[u_0(\omega)-B_1[n(z),F(\omega)](\omega)\big]\,\bar{\mathcal{G}}_1(\xi,\omega), \tag{4.24}$$

where $\mathcal{G}_1(\xi,\omega)$ is the solution to problem (4.17)–(4.21) with $F(\omega)\equiv 1$ and the bar over the symbol of the function denotes complex conjugation.

Assume that there exists a point (n_s, F_s), at which the gradients of the functional vanish. Then from (4.23) and (4.24) one can easily obtain the following expression:

$$F_s(\omega) = \frac{\bar{\mathcal{G}}_1(0,\omega)\,u_0(\omega)}{|\mathcal{G}_1^2(0,\omega)|^2}, \tag{4.25}$$

where $\mathcal{G}_1(0,\omega)$ is the trace of the solution to problem (4.17)–(4.21) with $F(\omega)\equiv 1$ and $n(z) = n_s(z)$ calculated at the point $z = 0$.

In [15], it was proposed to use a formula similar to (4.21) for calculation of the impulse $F_k(\omega)$ on the k-th iteration when solving the inverse problem of the VSP (vertical seismic profiling). In [6], application of this algorithm to solution of the inverse dynamic seismic problem with an unknown source in the case, where the whole wave field is measured on the free surface $z = 0$, was described.

Using expressions (4.23) and (4.25), we can apply the optimization methods of steepest descent of first order to search for a minimum point of functional (4.22), i.e., to reconstruct the unknown functions $v(z)$ and $F(t)$. If we succeed in reconstructing these functions, then, having solved direct problem (4.17)–(4.21), we can determine the spectrum of the wave field $u(z,\omega)$ in the whole of the half-space under study, i.e., find the right-hand side in the differential equation for the electric field in problem (4.12)–(4.14).

At the second stage, the initial boundary value problem (4.12)–(4.14) is considered, which, in terms of Fourier images with respect to the variable t, can be written down as

$$\frac{d^2}{dz^2}e(z,\omega) + \eta^2(z)e(z,\omega) = \mathrm{i}\omega\mu_0 h^0\eta^2(z)\,u(z,\omega), \quad z\in\Omega, \tag{4.26}$$

$$\frac{de(z,\omega)}{dz}\Big|_{z=0} = 0, \tag{4.27}$$

$$[e(z,\omega)]_{z=z_m} = \left[\frac{de(z,\omega)}{dz}\right]_{z=z_m} = 0, \quad m = \overline{1, n+1}, \tag{4.28}$$

where $\eta^2(z) = -\mathrm{i}\omega c^{-2}(z)$.

The additional information (4.16) is rewritten as

$$e(z,\omega)\big|_{z=0} = E_0(\omega). \tag{4.29}$$

We seek the solution to inverse problem (4.26)–(4.29) as a minimum point of the object functional

$$\Phi_2[\sigma(z)] = \int_{\omega_1}^{\omega_2} \big|e_0(\omega) - B_2[\sigma(z)](\omega)\big|^2 d\omega, \tag{4.30}$$

where $B_2[\sigma(z)]$ is a nonlinear operator mapping the function $\sigma(z)$ (the "test" value of conductivity) into the trace of the solution to the direct problem (4.26)–(4.28) at $z = 0$.

The gradient of the object functional (4.30) with respect to conductivity is written down as follows:

$$\nabla_\sigma \Phi_2[\sigma(z)](\xi) = A_1(\xi) + A_2(\xi), \tag{4.31}$$

where

$$A_1(\xi) = 2\mu_0^2 H \Re \int_{\omega_1}^{\omega_2} \omega^2 \left[e_0(\omega) - B_2[\sigma(z)](\omega)\right] \bar{\mathcal{G}}_2(\xi,\omega)\, \bar{u}(\xi,\omega) d\omega, \tag{4.32}$$

$$A_2(\xi) = 2\mu_0^2 H \Im \int_{\omega_1}^{\omega_2} \omega^3 \left[e_0(\omega) - B_2[\sigma(z)](\omega)\right] \times$$
$$\bar{\mathcal{G}}_2(\xi,\omega) \int_0^{+\infty} \sigma(\tau)\, \bar{\mathcal{G}}_2(\tau,\omega)\, \bar{u}(\tau,\omega) d\tau d\omega, \tag{4.33}$$

and $\mathcal{G}_2(\xi,\omega)$ is the Green function for problem (4.26)–(4.28).

Using formulas (4.31)–(4.33), we can apply the optimization methods of the first order for the search for the minimum point of functional (4.30), i.e., for reconstruction of unknown conductivity $\sigma(z)$.

To carry out numerical experiments rather a complex model of a vertically inhomogeneous medium was chosen. This model incorporated sharp changes in the values of parameters. The reconstruction of the medium was carried out up to a depth of 1.75 km. The medium below this depth was assumed to be homogeneous. All the medium from the surface to the depth 1.75 km was partitioned into 9 layers of equal width.

As a sensing signal, an impulse with a "bell-shaped envelope" was chosen (the dominating frequency $f = 20$ Hz):

$$F(\omega) = \left[\exp\Big(-\Big(\frac{\omega - 2\pi f}{\pi f}\Big)^2\Big) + \exp\Big(-\Big(\frac{\omega + 2\pi f}{\pi f}\Big)^2\Big)\right] \exp(-\mathrm{i}\cdot 1.75\,\omega/f).$$

Computations were made for temporal frequencies from 5 to 40 Hz.

To compute the whole wave field $u(z,\omega)$ and the electrical field intensity $E(z,\omega)$, the numerical-analytical method was used.

To calculate the impulse $F_j(\omega)$ on the j-th iteration, we use the condition of the vanishing gradient of functional (4.22) with respect to the function F_j on the current velocity $v_j(z)$, i.e., expression (4.25).

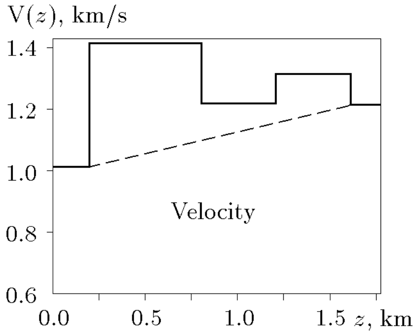

Figure 4.1

In Figure 4.1, the velocity model of a medium (solid line) and the initial approximation (dashed line) are shown. Figure 4.2 represents the spectrum $F(\omega)$ of the input signal impulse $F(t)$ (thick line) and its first approximation (thin line) obtained by formula (4.25). In Figure 4.3, the function $F(t)$ (thick line) and its first approximation (thin line) are shown. As a result of 35 iterations by the conjugate gradients method we succeeded in reconstructing with a good accuracy both the velocity distribution for this medium and the functions $F(\omega)$ and $F(t)$. The results of calculations are plotted in Figures 4.4–4.6.

Here and in the sequel, by pointing out the number of iterations made, we mean a practically complete stop of the iteration process at the stage concerned. The quality of the approximations obtained was estimated by closeness of the values of the corresponding functional to zero.

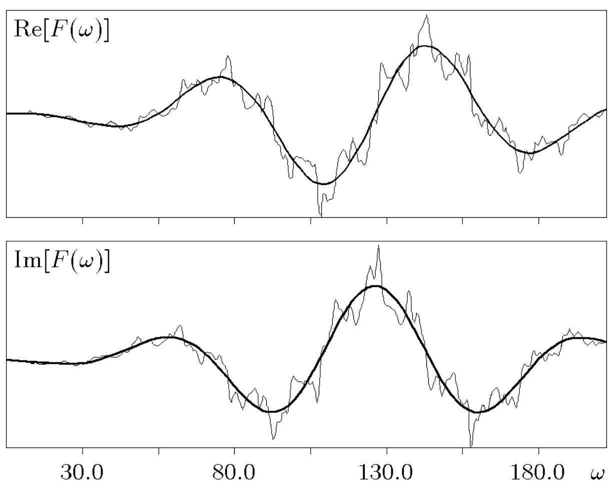

Figure 4.2

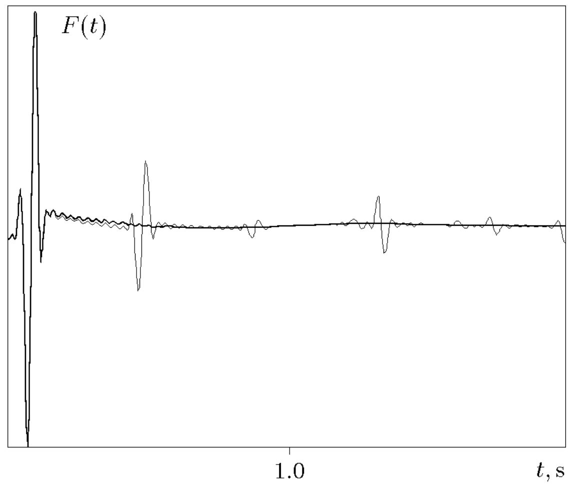

Figure 4.3

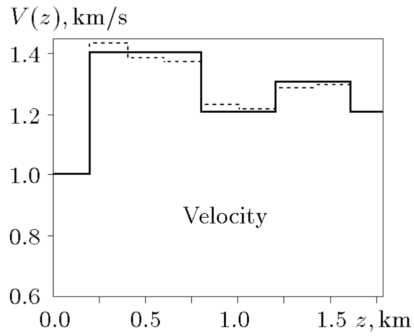

Figure 4.4

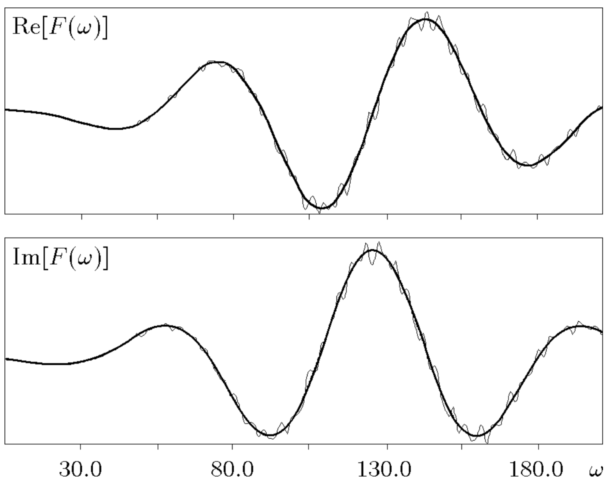

Figure 4.5

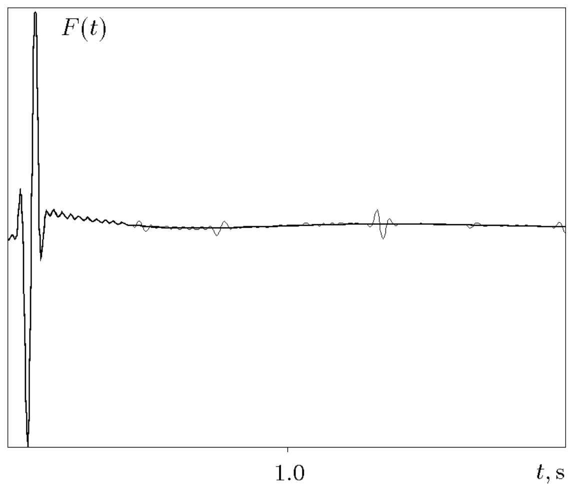

Figure 4.6

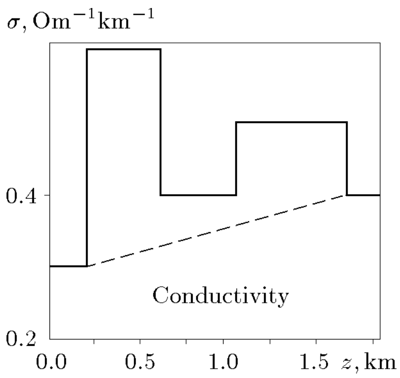

Figure 4.7

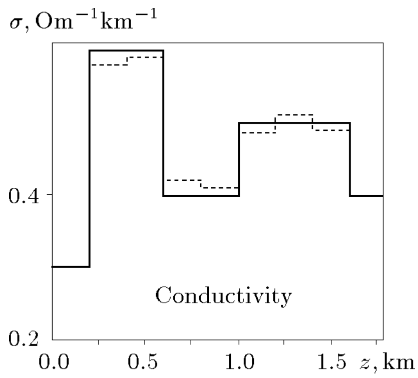

Figure 4.8

At the next stage, using the reconstructed functions $v(z)$ and $F(t)$, we calculated the spectrum of the wave field $u(z, \omega)$ in the whole of the half-space under study, i.e., the right-hand side in problem (4.26)–(4.28) was determined.

In Figure 4.7, the "true" function $\sigma(z)$ (solid line, see also the same solid line in Figure 4.8) and its initial approximation (dashed line) are shown. The final approximation computed by 68 iterations of the conjugate gradient method is plotted in Figure 4.8 (dashed line).

4.2. The second inverse problem

Let z denote the variable x_3. Consider the functions

$$v_p(z) = \sqrt{\frac{\lambda(z) + 2\varkappa(z)}{\rho(z)}}, \qquad v_s(z) = \sqrt{\frac{\varkappa(z)}{\rho(z)}}, \qquad c(z) = \sqrt{\frac{1}{\sigma(z)\,\mu(z)}},$$

where $v_p(z)$ is the velocity of longitudinal waves, $v_s(z)$ is the velocity of transverse waves, $c(z)$ is the rate of the diffusion process of electromagnetic waves.

Now let us formulate the inverse problem to be studied below (for details, see [9, 11, 12]).

Inverse Problem 4.2. *Find the functions $v_p(z)$, $v_s(z)$, and $c(z)$ if the following additional information on the solution to the direct problem is known:*

$$\boldsymbol{u}|_{z=0} = \boldsymbol{u}_0(x_1, x_2, t), \qquad \boldsymbol{H}|_{z=0} = \boldsymbol{H}_0(x_1, x_2, t).$$

We assume that μ and $\boldsymbol{h}^0$ are also known.

Remark 4.2. Without loss of generality, we will assume that $\mu = \mu_0$, where μ_0 is magnetic permeability of the air, and ρ is a known constant.

To solve the direct problem, the following "numerical-analytical" algorithms are used [31].

Applying the Fourier transforms with respect to the variables x_1, x_2, and t (marked by $\sim$ over corresponding functions), we rewrite the original system of equations (4.1)–(4.4) in the following form:

$$\tilde{u}_{1,zz} - \Big(\frac{v_p^2}{v_s^2}\nu_1^2 + \nu_2^2 - \frac{\omega^2}{v_s^2}\Big)\tilde{u}_1 - \nu_1\nu_2\frac{v_p^2 - v_s^2}{v_s^2}\,\tilde{u}_2 + \mathrm{i}\nu_1\frac{v_p^2 - v_s^2}{v_s^2}\,\tilde{u}_{3,z} = 0, \quad (4.34)$$

$$\tilde{u}_{2,zz} - \nu_1\nu_2\frac{v_p^2 - v_s^2}{v_s^2}\,\tilde{u}_1 - \Big(\nu_1^2 + \frac{v_p^2}{v_s^2}\nu_2^2 - \frac{\omega^2}{v_s^2}\Big)\tilde{u}_2 + \mathrm{i}\nu_2\frac{v_p^2 - v_s^2}{v_s^2}\,\tilde{u}_{3,z} = 0, \quad (4.35)$$

$$\tilde{u}_{3,zz} + \frac{v_p^2 - v_s^2}{v_s^2}\,\mathrm{i}(\nu_1\tilde{u}_{1,z} + \nu_2\tilde{u}_{2,z}) - \Big(\frac{v_s^2}{v_p^2}(\nu_1^2 + \nu_2^2) - \frac{\omega^2}{v_p^2}\Big)\tilde{u}_3 = 0, \quad (4.36)$$

$$\widetilde{H}_{1,zz} + r^2\widetilde{H}_1 = \omega c^{-2}\big(\nu_2(h_1^0\tilde{u}_2 - h_2^0\tilde{u}_1) + \mathrm{i}(h_3^0\tilde{u}_{1,z} - h_1^0\tilde{u}_{3,z})\big), \quad (4.37)$$

$$\widetilde{H}_{2,zz} + r^2\widetilde{H}_2 = \omega c^{-2}\big(\nu_1(h_2^0\tilde{u}_1 - h_1^0\tilde{u}_2) + \mathrm{i}(h_3^0\tilde{u}_{2,z} - h_2^0\tilde{u}_{3,z})\big), \quad (4.38)$$

$$\widetilde{H}_{3,zz} + r^2\widetilde{H}_3 = \omega c^{-2}\big(\nu_1(h_3^0\tilde{u}_1 - h_1^0\tilde{u}_3) + \nu_2(h_3^0\tilde{u}_2 - h_2^0\tilde{u}_3)\big). \quad (4.39)$$

The solution to system (4.37)–(4.39) is sought for in the form

$$\widetilde{H}_l = C_{1l}^j \mathrm{e}^{\tau_j z} + C_{2l}^j \mathrm{e}^{-\tau_j z} + \varphi_l^j, \qquad l = 1, 2, 3,$$

where j is the number of a layer.

If the solutions φ_l^j, $l = 1, 2, 3$, to the system are known, the constants C_{kl}^j, $k = 1, 2$, can be determined with the help of wide spread (for such problems) recurrent formulas through the boundary conditions and the interface conditions for layers.

It is non-trivial to find a particular solution of system (4.34)–(4.36); the difficulties of construction of an analytical, in every layer, solution to equations of elasticity by the matrix methods are well known [31]. To solve equations (4.34)–(4.36), a modification of the factorization method is used.

Numerical solution of the direct problem enables us to consider some dynamic features of seismomagnetic waves. Each kind of seismic waves generates an electromagnetic wave associated with it and propagating with the same velocity. The electromagnetic wave, generated by a seismic wave of a given kind, is called the seismomagnetic wave of the same kind (e.g., Rayleigh seismomagnetic wave, longitudinal seismomagnetic wave, transverse seismomagnetic wave, etc.). As compared to the longitudinal wave, the seismomagnetic wave is transverse, just as any other electromagnetic wave is. However, the longitudinal seismomagnetic wave propagates with a velocity close to that of the longitudinal seismic wave. The basic dynamic features of seismomagnetic waves for homogeneous elastic media were considered in [31]. Further, we convert components of both seismic and seismomagnetic fields from x_1, x_2, z to the spherical coordinate system for stratified elastic media. We transform all the components into dimensionless form. We normalize all the components of the seismomagnetic field and the seismic field to have a unit maximum amplitude in both cases; for this, we divide them by proper numbers.

Figure 4.9 shows the radial and the tangential components of the elastic wave displacement at the point $r_0 = 3\lambda$, where λ is the dominant P-wavelength in the elastic medium, and the radial and tangential components of the seismomagnetic field at the same point for different angles $\hat{\theta}$. Here $\hat{\theta}$ is the angle between the strength vector of the external magnetic field $\boldsymbol{h}^0$ and the vertical axis z. The elastic model is used with an explosive point source located near the point $z = 0$; in this case, the transverse components of all waves are equal to zero. The parameters of the model are the following: $v_{p1} = 1000$ m/s, $v_{p2} = 2000$ m/s, $v_{si} = V_{pi}/1.73$, $i = 1, 2$, the depth of the layer $h = \lambda$, the strength of the geomagnetic field $h^0 = 40$ A/m, and the conductivity $\sigma = 0.01$ S/m. Figure 4.9 shows that the phase and the first arrivals of geomagnetic variations coincide with the analogous characteristics of the seismic waves. The first wave is the longitudinal wave P, the second wave is the Rayleigh wave, and the third wave is the wave reflected from the boundary of the layer. The radial and the tangential components of the P- and the Rayleigh seismomagnetic waves have the same circular polarization. The amplitude of the P-seismomagnetic wave decreases with an increase in the angle $\hat{\theta}$, while the amplitude of the Rayleigh wave increases.

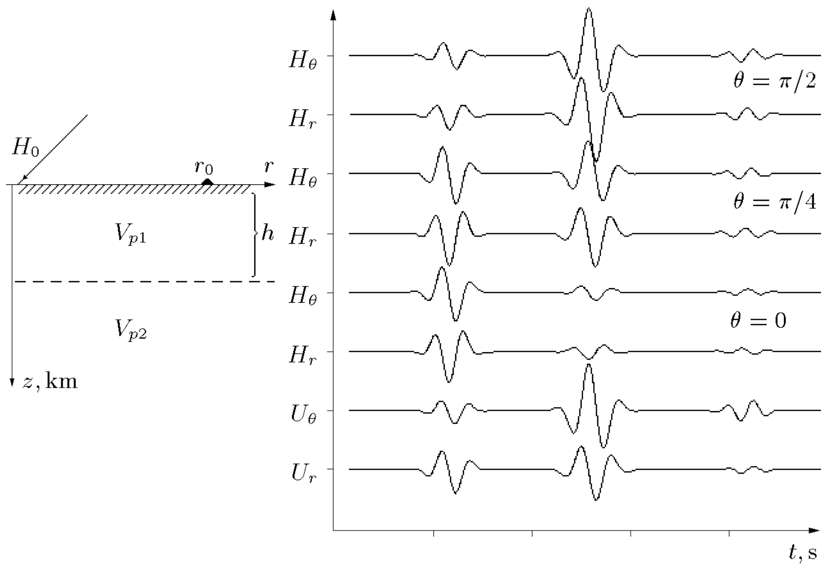

Figure 4.9

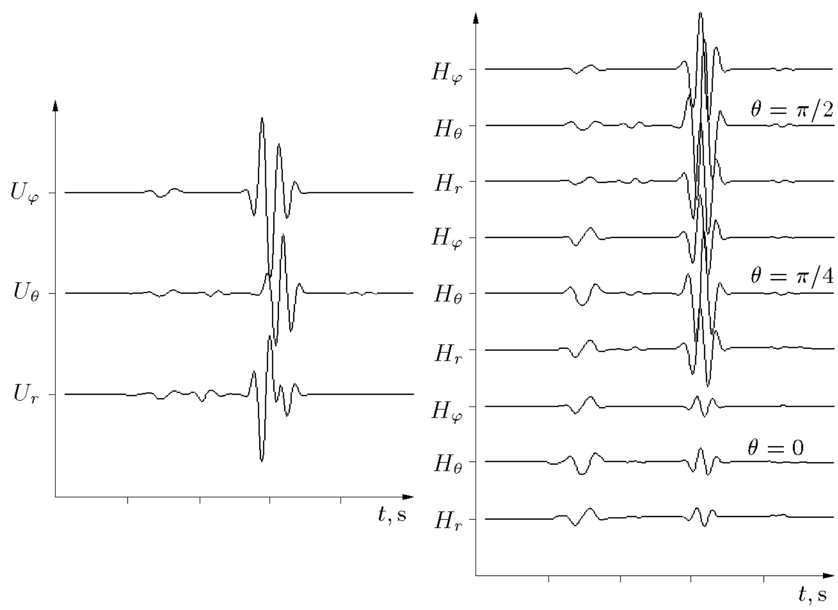

Figure 4.10

Figure 4.10 shows radial, tangential, and transverse components of the elastic displacements and seismomagnetic waves. The elastic model and its parameters are the same as in Figure 4.9. A point source located near $z = 0$ is a source of the horizontal type force. In this case we have nonzero radial, tangential, and transverse components of all waves. The amplitude of Pseismomagnetic wave decreases with an increase in the angle $\hat{\theta}$, while the amplitude of the Rayleigh wave increases.

To solve the inverse problem numerically, an optimization approach was used, which is based on minimizing the data misfit functionals of the observed data and the data computed when solving "test" direct problems.

We will seek for the functions $v_p(z)$ and $v_s(z)$ as a minimum point of the functional

$$\Phi_1[v_p, v_s] = \int_{\omega_1}^{\omega_2} \int_{\nu_{1,1}}^{\nu_{1,2}} \int_{\nu_{2,1}}^{\nu_{2,2}} \left| \widetilde{\boldsymbol{u}}_0(\nu_1, \nu_2, \omega) - B_1[v_p, v_s](\nu_1, \nu_2, \omega) \right|^2 d\nu_1 d\nu_2 d\omega,$$

where (ω_1, ω_2) is the range of temporal frequencies defined by the spectral contents $f(\omega)$ of a sensing signal, $(\nu_{1,1}, \nu_{1,2})$ and $(\nu_{2,1}, \nu_{2,2})$ are the ranges of spatial frequencies, and $B_1[v_p, v_s]$ is a nonlinear operator mapping the functions $v_p(z)$ and $v_s(z)$ into the solution of the appropriate direct problem at the point $z = 0$.

If we succeed in reconstructing the functions $v_p(z)$ and $v_s(z)$, then, having solved the direct problem, we can determine the spectrum of the wave vector $\widetilde{\boldsymbol{u}}(\nu_1, \nu_2, z, \omega)$ in the whole of the half-space under study, i.e., we can find the right-hand side in the system of differential equations for the magnetic fields.

Then we seek for the conductivity function $\sigma(z)$ as a minimum point of the object functional

$$\Phi_2[c(z)] = \int_{\omega_1}^{\omega_2} \int_{\nu_{1,1}}^{\nu_{1,2}} \int_{\nu_{2,1}}^{\nu_{2,2}} \left| \widetilde{\boldsymbol{H}}_0(\nu_1, \nu_2, \omega) - B_2[c(z)](\nu_1, \nu_2, \omega) \right|^2 d\nu_1 d\nu_2 d\omega,$$

where $B_2[c(z)]$ is a nonlinear operator mapping the function $c(z)$ (the "test" value of conductivity) into the solution of the appropriate direct problem at the point $z = 0$.

To arrange the interactive process of the search for the minimum points of the object functionals we used the quasi-Newton method.

The reconstruction of the medium was carried out up to a depth of 0.5 km. The medium below this depth was assumed to be homogeneous. The whole medium from the surface to the depth of 0.5 km was partitioned into layers of equal width.

As a sensing signal, an impulse with a "bell-shaped envelope" was chosen with the dominating frequency $f = 25$ Hz. All computations were made for temporal frequencies from 5 to 50 Hz.

The real model distributions for the functions $v_p(z)$, $v_s(z)$, and $\sigma(z)$ are shown in Figure 4.11 by solid lines. The initial approximations for the functions $v_p(z)$, $v_s(z)$, and $\sigma(z)$ are shown by dashed lines.

The results of reconstruction are presented in Figure 4.12. These results were obtained for the functions $v_p(z)$ and $v_s(z)$ (59 iterations) and the function $\sigma(z)$ (38 iterations).

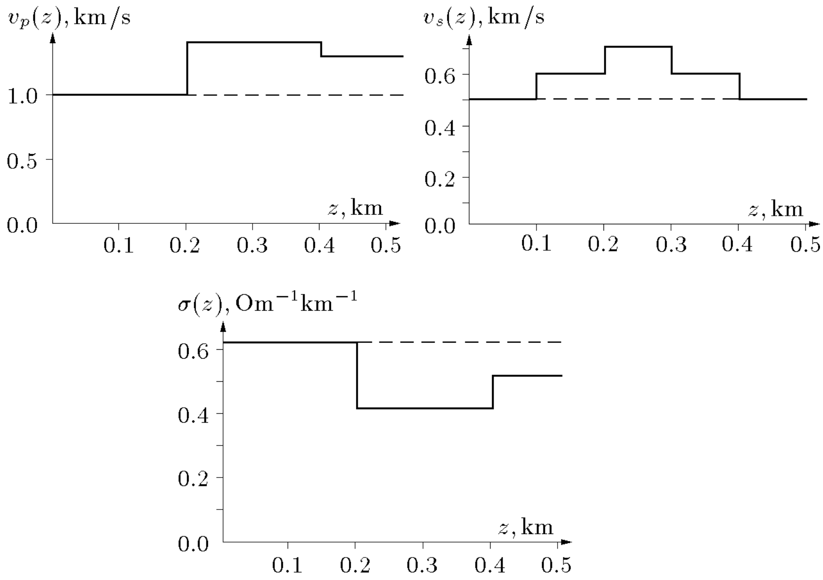

Figure 4.11

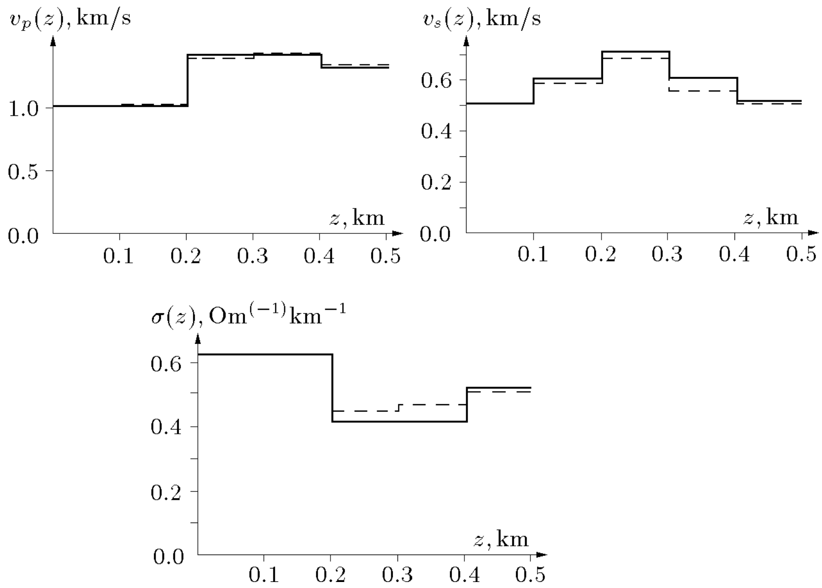

Figure 4.12

Bibliography

[1] Alekseev A.S. On statement of combined inverse geophysics problems // Sib. J. Comp. Math. – 1992. – Vol. 1, No. 1. – P. 47–58.

[2] Alvén H. Cosmical Electrodymanics. – Oxford: Clarendon Press, 1950.

[3] Andrè A.B.Q., Priimenko V.I. Modelagem matemàtica e numérica da propagaço de ondas electromagnticas acopladas a ondas elásticas, considerando-se efeito sismomagnètico // Proceedings of 7th International Congress of Geophysics. – Salvador, Brazil, 2001.

[4] Anikonov Yu.E. Some Methods for Studying Multidimensional Inverse Problems for Differential Equations. – Novosibirsk: Nauka, 1978 (in Russian).

[5] Anisimov S.V., Gokhberg M.B., Ivanov E.A., Pedanov M.V., Rusakov N.N., Troizhkya V.A., Goncharov V.E. Short period oscillations of electromagnetic field of the Earth after explosion // Dokl. Acad. Nauk SSSR. – 1985. – Vol. 281, No. 3. – P. 556–559.

[6] Avdeev A.V. Simultaneous determination of sounding signal and wave propagation velocity in a vertically-nonhomogeneous medium // Computerized Tomography: Proc. of the Fourth Intern. Symp., Novosibirsk, August 10–14, 1993. – The Netherlands, Utrecht: VSP, 1995. – P. 76–84.

[7] Avdeev A.V., Goryunov E.V., Priimenko V.I. An inverse problem of electromagnetoelasticity with unknown source of elastic oscillations. – Novosibirsk, 1996. – (Preprint / RAS. Sib. Branch. Comp. Cent; 1074).

[8] Avdeev A.V., Goryunov E.V., Priimenko V.I. An inverse problem of electromagnetoelasticity with unknown source of elastic oscillations // Matem. Modelir. – 1997. – Vol. 9, No. 10. – P. 50–62 (in Russian).

[9] Avdeev A.V., Goryunov E.V., Soboleva O.N., Priimenko V.I. Numerical solution of some direct and inverse problems of electromagnetoelasticity // J. Inv. Ill-Posed Problems. – 1999. – Vol. 7, No. 5. – P. 453–462.

[10] Avdeev A.V., Lavrentiev M.M.-jr., Priimenko V.I. Inverse Problems and Some Applications. – Novosibirsk: ICM&MG Publ., 1999.

[11] Avdeev A.V., Priimenko V.I, Goryunov E.V., Zvjagin D.V. Direct and inverse problems of electromagnetoelasticity // Proceedings of 5th International Congress of Geophysics. – São-Paulo, Brazil, 1997. – P. 657–661.

[12] Avdeev A.V., Priimenko V.I, Goryunov E.V., Zvjagin D.V. Direct and inverse problems for the system of differential equations of ectromagnetoelasticity // Proceedings of the International Conference PRISM'97. – The Netherlands, Nijmegen, 1997. – P. 1–8.

[13] Balakirev M.K., Gilinskii I.A. Waves in Piezocrystals. – Novosibirsk: Nauka, 1982 (in Russian).

[14] Burdakova O.A., Yakhno V.G. A one-dimensional inverse problem of electroelasticity // Questions of Correctness of the Problems of Analysis. – Novosibirsk: Institute of Mathematics SB USSR Acad, Sci., 1989. – P. 33–43 (in Russian).

[15] Cheverda V.A., Voronina T.A. An optimizational approach to process VSP data // J. Inv. Ill-Posed Problems. – 1994. – Vol. 2, No. 3. – P. 42–54.

[16] Dunkin J.W., Eringen A.C. On the propagation of waves in an electromagnetic elastic solid // J. Engng. Sci. – 1963. – Vol. 1. – P. 461–495.

[17] Eringen A.C. Nonlinear Theory of Continuous Media. – New York: McGraw-Hill, 1962.

[18] Eringen A.C., Maugin G.A. Electrodynamics of Continua. Vol. I, II. Foundations and Solid Media. – Springer-Verlag, 1990.

[19] Evans L.C. Partial Differential Equations / Graduate Studies in Mathematics. Vol. 19. – Providence: Amer. Math. Soc., 1998.

[20] Imomnazarov Kh.Kh. On one class of coupled one-dimensional problems for Maxwell's equations and the equations of continual filtration theory // Proc. of the Institute of Computational Mathematics and Mathematical Geophysics (Computing Center). Ser. "Mathematical Modelling in Geophysics". – 1998. – Issue 5. – P. 61–73 (in Russian).

[21] Jackson J.D. Classical Electrodynamics. – New York: John Wiley&Sons, Inc., 1999.

[22] Klimenko O.A. An algorithm for solving a problem in elastoelectrodynamics // Computerized Tomography. Proc. of the Fourth Intern. Symp., August 10–14, Novosibirsk, 1993 / Ed. M.M. Lavrent'ev. – The Netherlands, Utrecht: VSP, 1995. – P. 283–287.

[23] Knopoff L. The interaction between elastic wave motion and a magnetic field in electrical conductors // J. Geophys. Res. – 1995. – Vol. 60. – P. 441–456.

[24] Lavrent'ev M.M.-jr., Priimenko V.I. Simultaneous determination of elastic and electromagnetic medium parameters // Computerized Tomography. Proc. of the Fourth Intern. Symp., August 10–14, Novosibirsk, 1993 / Ed. M.M. Lavrent'ev. – The Netherlands, Utrecht: VSP, 1995. – P. 302–308.

[25] Ladyzhenskaya O.A., Solonnikov V.A., Uralceva M.N. Linear and Quasilinear Equations / Translations of Mathematical Monographs. Vol. 23. – Providence: Amer. Math. Soc., 1968.

[26] Landau L.D., Lifshitz E.M. Course of Theoretical Physics. Vol. 8. Electrodynamics of Continuous Media. – Oxford-Elmsford, New York: Pergamon Press, 1984.

[27] Lorenzi A., Priimenko V.I. Identification problems related to electro-magneto-elastic interactions // J. Inv. Ill-Posed Problems. – 1996. – Vol. 4, No. 2. – P. 115–143.

[28] Lorenzi A., Romanov V.G. Identification of an electromagnetic coefficient connected with deformation currents // Inverse Problems. – 1993. – Vol. 9, No. 2. – P. 301–319.

[29] Maugin A.G. Continuum Mechanics of Electromagnetoelastic Solids. – Amsterdam, New York: North-Holland Publ. Co., 1988.

[30] Merazhov I.Z., Yakhno V.G. Direct and inverse problems for system of electroelasticity equations // Computerized Tomography. Proceedings of the Fourth Int. Symposium, Novosibirsk, 1993. – The Netherlands, Utrecht: VSP, 1995. – P. 332–335.

[31] Mikhailenko B.G., Soboleva O.N. Mathematical modeling of seismomagnetic effects arising in the seismic wave motion in the Earth's constant magnetic field // Appl. Math. Lett. – 1997. – Vol. 10, No. 3. – P. 47–51.

[32] Mukhometov R.G. The problem of recovering a two-dimensional Riemannian metric and integral geometry // Dokl. Akad. Nauk SSSR. – 1977. – Vol. 232, No. 1. – P. 32–35 (in Russian).

[33] Parton V.Z., Kudrjavtsev B.A. Electromagnetoelasticity of Piezoelectric and Electrical-Conducting Bodies. – Moscow: Nauka, 1988 (in Russian).

[34] Pride S.R. Governing equations for the coupled electromagnetoelastic and acoustic of porous media // Phys. Rev. B. Third Series. – 1994. – Vol. 50, No. 21. – P. 15678–15696.

[35] Priimenko V.I., Vishnevskii M.P. The Cauchy problem for a nonlinear model system of electromagnetoelasticity // 56th Seminàrio Brasileiro de Análise. – Niteroi, Brasil, 2002. – P. 395–413.

[36] Priimenko V.I., Vishnevskii M.P. Direct and inverse problems for a nonlinear model electromagnetoelasticity system // Abstracts of International Conference "Ill-Posed and Inverse Problems" in honour of the 70th anniversary of the birth of Prof. M.M. Lavrentiev, August 5–9, 2002. – Novosibirsk, 2002.

[37] Priimenko V.I., Vishnevskii M.P. Direct and inverse problems of electromagnetoelasticity: theoretical and numerical aspects // 57th Seminàrio Brasileiro de Análise. – Viçosa, Brasil, 2003. – P. 117–213.

[38] Romanov V.G. Integral Geometry and Inverse Problems for Hyperbolic Equations. – Springer-Verlag, 1974.

[39] Romanov V.G. Inverse Problems of Mathematical Physics. – The Netherlands, Utrecht: VSP, 1987.

[40] Romanov V.G. Uniqueness theorems in inverse problems for some second-order equations // Soviet Math. Dokl. – 1992. – Vol. 44, No. 3. – P. 678–682.

[41] Romanov V.G. Structure of a solution to the Cauchy problem for the system of the equations of electrodynamics and elasticity in the case of point sources // Siberian Math. J. – 1995. – Vol. 36, No. 3. – P. 541–561.

[42] Romanov V.G. On an inverse problem for a coupled system of equations of electrodynamics and elasticity // J. Inv. Ill-Posed Problems. – 1995. – Vol. 3, No. 4. – P. 321–332.

[43] Schwartz L. Analyse Mathématique. Vol. I, II. – Paris: Hermann, 1967.

[44] Sedov L.I. Mechanics of Continua. Vol. I, II. – Moscow: Nauka, 1973 (in Russian).

[45] Sharafutdinov V.A. Integral Geometry of Tensor Fields. – The Netherlands, Utrecht: VSP, 1994.

[46] Sheen D. A generalized Green's theorem // Appl. Math. Lett. – 1992. – Vol. 5, No. 4. – P. 95–98.

[47] Sobolev S.L. Some Applications of Functional Analysis in Mathematical Physics / Translations of Mathematical Monographs. Vol. 90. – Providence: Amer. Math. Soc., 1991.

[48] Tamm I.E. Foundations of the Theory of Electricity. – Moscow: Nauka, 1976 (in Russian).

[49] Yakhno V.G. Inverse Problems for Differential Equations of Elasticity. – Moscow: Nauka, 1990 (in Russian).

[50] Yakhno V.G. Inverse problem for differential equations system of electromagnetoelasticity in linear approximation // Inverse Problems, Tomography, and Image Processing. – New York: Plenum Press, 1998. – P. 211–239.

[51] Yakhno V.G., Merazhov I.Z. Direct problems and an one-dimensional inverse problem of electroelasticity for "slow" waves // Siberian Advances in Math. – 2000. – Vol. 10, No. 1. – P. 87–150; Translated from: Matem. Trudy, 1999. – Vol. 2, No. 2. – P. 148–213 (in Russian).