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One-dimensional dynamic inverse problem for a system of Hopf-type equations

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Abstract. A dynamic inverse problem for a one-dimensional system of the Hopf-type equations is considered. A theorem on local solvability in the class of functions analytic in the variable x is proved.

Keywords: Two-velocity hydrodynamics, viscous fluid, relative velocity, direct problem, inverse problem, Darcy coefficient.

Introduction

The theory of two-phase filtration finds important application in solving problems of petroleum engineering, soil science, biomechanics and others practical areas. Increasing attention is being paid to modeling of multiphase flows in connection with burial radioactive waste. Simulation and numerical analysis of two-phase filtration in elastically deformable porous media are important element in the development of cost-effective and safe cleaning devices, reducing the number of laboratory and field experiments, identify the main mechanisms, optimize existing strategies and evaluate possible risks. In recent years, interest in processes has significantly increased of multiphase filtration in low-permeability fractured porous collectors. One of the important reasons for this is the fact that fractured hydrocarbon deposits contain more than 20 % of world oil reserves [1].

1. Hopf-type system of equations

A subclass of the system of two-speed hydrodynamics in the case of constant phase saturation in the dissipative case are systems of the Hopf-type equations. In the one-dimensional case, in the absence of mass forces, this system has the form [2–5]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -b(u-v), \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \varepsilon b(u-v), \tag{2}$$

where u and v are the corresponding phase velocity components with partial densities ρ_1, ρ_2, $\varepsilon = \rho_1/\rho_2$ — dimensionless positive constant.

System (1), (2) differs from the system of two-speed hydrodynamics in the dissipative case, due to the friction coefficient, the absence of pressure and the incompressibility condition. For this reason, the problems associated with a Hopf-type system, the system will sometimes be called two-speed hydrodynamics without pressure. When the friction coefficient disappears ($b = 0$), system (1), (2) goes over to the well-known Hopf equation [6].

It is known that, in a certain sense, the Cauchy problem for the Hopf equation with initial data $w|_{t=0} = f(x)$ reduces to the theorem on implicit functions, since locally

$$w(t,x) = f(x - wt) \quad \leftrightarrow \quad \frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = 0. \tag{3}$$

Global solvability of the Cauchy problem with bounded initial data $f(x)$ was proven in the 50s of last century (see bibliography in [7]). The Hopf equation was replaced in the half-plane $t > 0$ by a discrete equation for the functions w_n^k on the lattice:

$$\begin{aligned} w_n^{k+1} &= w_{n+1}^k\Big(\frac{1}{2} + \frac{h}{4l}w_{n+1}^k\Big) - w_{n-1}^k\Big(\frac{1}{2} + \frac{h}{4l}w_{n-1}^k\Big), \\ w_n^0 &= \frac{1}{2l}\int_{(n-1)l}^{(n+1)l} w(0,x)\,dx, \end{aligned} \tag{4}$$

where the replacement $w(t,x) \to w(kh,nl) = w_n^k$ is made. Another version of the proof of the theorem for the existence of a generalized solution to the Cauchy problem was based on the replacement hyperbolic Hopf equation by a parabolic equation with a small parameter at the highest derivative:

$$\frac{\partial w_\varepsilon}{\partial t} + w_\varepsilon\frac{\partial w_\varepsilon}{\partial x} = \frac{\partial^2 w_\varepsilon}{\partial x^2}, \quad w_\varepsilon(0,x) = w(0,x) \in L_\infty. \tag{5}$$

For piecewise linear initial data, formula (3) and the method of characteristics allow us to construct a family of exact solutions of the Hopf equation with discontinuities. This makes it possible, in particular, to verify the non-uniqueness of the solution to the Cauchy problem in the class of generalized functions. The question of the coincidence of solutions to the Cauchy problem constructed in different ways depends one way or another on specifying suitable conditions on the fault lines. Let us recall in this connection that for the Hopf equation there is arbitrariness in the choice of the "fundamental" conservation law $\frac{\partial \rho(w)}{\partial t} + \frac{\partial \sigma(w)}{\partial x} = 0$, so that for a generalized solution, the conditions on the break line depend on the specific choice this conservation law.

Although the above methods for constructing approximate solutions using equations (4), (5) are constructive, it is unclear, however, what information can be extracted from them about the qualitative behavior of solutions after the occurrence of discontinuities caused by the intersection of characteristics (see explicit formula (3)) [8]. Moreover, the implicit function theorem is not applicable for the Hopf equation with a source and for system (1), (2) with the Cauchy data

$$u|_{t=0} = u_1^0(x), \quad v|_{t=0} = u_2^0(x). \tag{6}$$

Following [9], we denote by X_s, $0 \le s < 1$, the space of functions $u(z)$ holomorphic in the domain $\Omega_s = \{z \in \mathbb{C},\ |z| < sR\}$, R is some constant. Let $\|u\|_s = \sup_{\Omega_s} |u(z)|$. Let us denote $D_1 = \{x,\ |x| < R_0\}$, $D_2 = \{z \in \mathbb{C},\ |z| < R\}$, $C^k([0, R_0]; D_1)$ — space of k times continuously differentiable functions with respect to t for $0 \le t \le R_0$ with values in the space of functions that are holomorphic in a neighborhood of D_1, $0 < R_0 \le R$.

2. The Cauchy problem for a Hopf-type system

Consider in the region $D = \{(x,t),\ 0 \le t \le R,\ |x| < R\}$ systems of the Hopf type (1), (2) with the Cauchy data (6).

Problem (1), (2), (6) is equivalent to the solvability of the following system of nonlinear integrodifferential equations:

$$u_1(t,x) = u_1^0(x) - \int_0^t u_1(\tau,x)\frac{\partial u_1(\tau,x)}{\partial x}\,d\tau - b\int_0^t (u_1(\tau,x) - u_2(\tau,x))\,d\tau,$$
$$u_2(t,z) = u_2^0(x) - \int_0^t u_2(\tau,x)\frac{\partial u_2(\tau,x)}{\partial x}\,d\tau + \varepsilon b\int_0^t (u_1(\tau,x) - u_2(\tau,x))\,d\tau,$$

where $u_1(t,x) = u(t,x)$, $u_2(t,x) = v(t,x)$.

Let us assume it is done

Condition A. $u_k^0(x)$, $k = 1,2$, are holomorphic functions in a neighborhood of D_1 admitting analytic continuations to the neighborhood of D_2 and

$$\|u_1^0\|_s + \|u_2^0\|_s \le N.$$

We denote the continuations of the functions $u_k^0(x)$, $u_k(t,x)$ from D_1 to D_2 by $u_k^0(z)$, $u_k(t,z)$. Next, we will use the ideas of works [9–12]. In terms of the function $\psi_k = u_k - u_k^0$, $k = 1,2$, the latter system will take the form

$$\psi_1(t,z) = -\int_0^t \psi_1(\tau,z)\frac{\partial\psi_1(\tau,z)}{\partial z}\,d\tau - \int_0^t u_1^0(z)\frac{\partial\psi_1(\tau,z)}{\partial z}\,d\tau -$$
$$\int_0^t \psi_1(\tau,z)\frac{\partial u_1^0(\tau,z)}{\partial z}\,d\tau - b\int_0^t (\psi_1(\tau,z) - \psi_2(\tau,z))\,d\tau - \Phi_1(z)t, \tag{7}$$

$$\psi_2(t,z) = -\int_0^t \psi_2(\tau,z)\frac{\partial\psi_2(\tau,z)}{\partial z}\,d\tau - \int_0^t u_2^0(z)\frac{\partial\psi_2(\tau,z)}{\partial z}\,d\tau -$$
$$\int_0^t \psi_2(\tau,z)\frac{\partial u_2^0(\tau,z)}{\partial z}\,d\tau + \varepsilon b\int_0^t (\psi_1(\tau,z)-\psi_2(\tau,z))\,d\tau - \Phi_2(z)t, \quad (8)$$

where

$$\Phi_1(z) = u_1^0(z)\frac{\partial u_1^0(z)}{\partial z} + b(u_1^0(z)-u_2^0(z)),$$
$$\Phi_2(z) = u_2^0(z)\frac{\partial u_2^0(z)}{\partial z} - \varepsilon b(u_1^0(z)-u_2^0(z)),$$
$$\frac{\partial u(t,z)}{\partial z} = \frac{1}{2}(u_x - iu_y),$$

$z = x+iy$, $\operatorname{Im}\psi_k|_{D_1} = 0$. We obtain a solution to system (7), (8) as the limit of the sequence ψ_k^n, defined inductively by the equalities

$$\psi_k^0 = 0, \quad \psi_k^{n+1} = \psi_k^n + w_k^n,$$

where w_k^n are defined as follows:

$$w_1^n(t,z) = -\int_0^t \psi_1^n(\tau,z)\frac{\partial\psi_1^n(\tau,z)}{\partial z}\,d\tau - \int_0^t u_1^0(z)\frac{\partial\psi_1^n(\tau,z)}{\partial z}\,d\tau -$$
$$\int_0^t \psi_1^n(\tau,z)\frac{\partial u_1^0(\tau,z)}{\partial z}\,d\tau - b\int_0^t (\psi_1^n(\tau,z)-\psi_2^n(\tau,z))\,d\tau -$$
$$\psi_1^n - \Phi_1(z)t \equiv S_1(\psi_1^n,\psi_2^n) - \psi_1^n - \Phi_1(z)t, \quad (9)$$
$$w_2^n(t,z) = -\int_0^t \psi_2^n(\tau,z)\frac{\partial\psi_2^n(\tau,z)}{\partial z}],d\tau - \int_0^t u_2^0(z)\frac{\partial\psi_2^n(\tau,z)}{\partial z}\,d\tau -$$
$$\int_0^t \psi_2^n(\tau,z)\frac{\partial u_2^0(\tau,z)}{\partial z}]\,d\tau + \varepsilon b\int_0^t (\psi_1^n(\tau,z)-\psi_2^n(\tau,z))\,d\tau -$$
$$\psi_2^n - \Phi_2(z)t \equiv S_2(\psi_1^n,\psi_2^n) - \psi_2^n - \Phi_2(z)t, \quad (10)$$

for $t < b_n(1-s)$, $b_{n+1} = b_n[1-(n+2)^{-2}]$, $n = 0,1,2,\ldots$.

Let B be the space of functions $u_k(t,z)$, $k = 1,2$, which for any $s \in [0,1]$ are continuous functions of t, $t < b_\infty(l-s)$, with values in X_s that

$$k[u_1,u_2] = \sup_{\substack{t<b_\infty(1-s),\\ 0\le s<1}} (\|u_1\|_s^2 + \|u_2\|_s^2)\Big(\frac{b_\infty(1-s)}{t} - 1\Big)^2 < \infty,$$

$$\operatorname{Im} u_k|_{\omega_s} = 0, \quad b_\infty = b_0\prod_{n=0}^{\infty}[1-(n+2)^{-2}], \quad \omega_s = \{x \in R^1,\ |x| < sR\}.$$

Let for $j \le n$ be given w_m^j, $m = 1, 2$, such that

$$\eta_j = \sup_{\substack{t < b_\infty(1-s),\\ 0 \le s < 1}} (\|w_1^j\|_s^2 + \|w_2^j\|_s^2)\Big(\frac{b_j(1-s)}{t} - 1\Big)^2 < \infty,$$
$$\|w_1^j\|_s^2 + \|w_2^j\|_s^2 < N.$$

Let us prove that w_m^{n+1}, $m = 1, 2$, are defined for $t < b_{n+1}(l - s)$. From the construction it follows that $\operatorname{Im} w_k^0|_{\omega_s} = 0$. Thus, if w_m^n are defined, then

$$\operatorname{Im} w_m^n|_{\omega_s} = 0.$$

From the definition of ψ_m^{n+1}, ψ_m^n it follows that

$$\|w_m^{n+1}\|_s \le \sum_{j=0}^{n} \|w_m^j\|_s, \quad m = 1, 2.$$

It is necessary that

$$\|w_1^{n+1}\|_s^2 + \|w_2^{n+1}\|_s^2 < N.$$

Then the function w_m^{n+1} will be defined. So, from (9), (10) it follows

$$w_1^{n+1}(t, z) = S_1(\psi_1^{n+1}, \psi_2^{n+1}) - S_1(\psi_1^n, \psi_2^n), \tag{11}$$
$$w_2^{n+1}(t, z) = S_2(\psi_1^{n+1}, \psi_2^{n+1}) - S_2(\psi_1^n, \psi_2^n), \tag{12}$$

Repeating the reasoning from [11] taking into account $\|\frac{\partial u}{\partial z}\|_{s'} \le \frac{1}{R(s-s')}\|u\|_s$ [10] with simple transformations from (11), (12), we obtain

$$\|w_1^{n+1}\|_s^2 + \|w_2^{n+1}\|_s^2 \le \frac{2^4(N+C)}{R^2} \frac{b_0^3 k[w_1^n, w_2^n]}{(b_{n+1}(1-s)/t - 1)^2} +$$
$$(N + C + 2(1+\varepsilon b)) \int_0^t (\|w_1^n(\tau, z)\|_s^2 + \|w_2^n(\tau, z)\|_s^2)]\, d\tau$$

or

$$\eta_{n+1} \le q\eta_n,$$

where

$$q = \Big(\frac{2^4(N+C)}{R^2} b_0^2 + N + C + 2(1+\varepsilon b)\Big) b_0 < 1$$

due to the smallness of b_0,

$$C = \max\Big\{\|u_1^0\|_s, \|u_2^0\|_s, \Big\|\frac{\partial u_1^0}{\partial z}\Big\|_s, \Big\|\frac{\partial u_2^0}{\partial z}\Big\|_s\Big\}.$$

Whence, due to the smallness of b_0, we have

$$\|\psi_1^{n+1} - \psi_1^n\|_s^2 + \|\psi_2^{n+1} - \psi_2^n\|_s^2 \leq \text{const} \cdot q^n (b_\infty(1-s)/t - 1)^{-2}$$

for $t < b_\infty(1-s)$, since $\sum_{n=0}^{\infty} q^n < \infty$. From this we conclude that the functions ψ_m^k, $m = 1, 2$, converge to some function ψ_k. The uniqueness of the solution is established in the usual way. The following theorem has been proven

Theorem 1. *Let conditions A be satisfied. Then the Cauchy problem* (1), (2), (6) *has a unique solution* $u_1(z,t)$, $u_2(z,t)$ *which for any positive* $s \in (0,1)$ *is once continuously differentiable function on* t *for* $t < b(l-s)$ *with values in* X_s, *and*

$$\|u_1\|_s + \|u_2\|_s \leq N.$$

3. Inverse source problem for a Hopf-type system

In the one-dimensional case in the presence of mass forces, this system has the form [3, 4, 12]:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = -b(u_1 - u_2) + f(x) g_1(t), \tag{13}$$

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} = \varepsilon b(u_1 - u_2) + f(x) g_2(t), \tag{14}$$

where $f(x)g_1(t)$ and $f(x)g_2(t)$ are mass forces.

Let the Cauchy data (6) and additional information be given

$$u_k|_{x=0} = \varphi_k(t), \quad k = 1, 2. \tag{15}$$

The inverse problem consists of determining the function

$$(u_1(x,t), u_2(x,t), g_1(t), g_2(t))$$

from (13)–(15), (6). In this case, the function $f(x)$ is known and separated from zero, the matching condition is satisfied

$$u_k^0(0) = \varphi_k(0), \quad k = 1, 2. \tag{16}$$

Let us apply the operator $\frac{\partial}{\partial x} f^{-1}(x)$ to both sides of equalities (13) and (14). After simple transformations we obtain

$$\frac{\partial u_1}{\partial t} - \varphi_1'(t) = -b(u_1 - u_2) + b(\varphi_1 - \varphi_2) + b\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1 - u_2)d\xi + \int_0^x \frac{f'(\xi)}{f(\xi)}\Big(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial \xi}\Big)d\xi - \int_0^x \Big(\Big(\frac{\partial u_1}{\partial \xi}\Big)^2 + u_1\frac{\partial^2 u_1}{\partial \xi^2}\Big)d\xi, \tag{17}$$

$$\frac{\partial u_2}{\partial t} - \varphi_2'(t) = \varepsilon b(u_1 - u_2) - \varepsilon b(\varphi_1 - \varphi_2) - \varepsilon b\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1 - u_2)d\xi + \int_0^x \frac{f'(\xi)}{f(\xi)}\Big(\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial \xi}\Big)d\xi - \int_0^x \Big(\Big(\frac{\partial u_2}{\partial \xi}\Big)^2 + u_2\frac{\partial^2 u_2}{\partial \xi^2}\Big)d\xi. \tag{18}$$

Thus, the study of the solvability of the inverse problem (13)–(15), (6) was reduced to the study of the solvability of the direct problem (17), (18), (6). This problem is equivalent to a system of integrodifferential equations Volterra of the second kind

$$\begin{aligned} u_1(x,t) = {}& u_1^0(x) + \varphi_1(t) - \varphi_1(0) - \\ & b\int_0^t (u_1(x,\tau) - u_2(x,\tau) - \varphi_1(\tau) + \varphi_2(\tau))\,d\tau + \\ & \int_0^x \frac{f'(\xi)}{f(\xi)}(u_1(\xi,t) - u_1^0(\xi))\,d\xi + \int_0^t\int_0^x u_1(\xi,\tau)\frac{\partial u_1(\xi,\tau)}{\partial \xi}\,d\xi\,d\tau - \\ & \int_0^t\int_0^x \Big(\Big(\frac{\partial u_1(\xi,\tau)}{\partial \xi}\Big)^2 + u_1(\xi,\tau)\frac{\partial^2 u_1(\xi,\tau)}{\partial \xi^2}\Big)d\xi\,d\tau + \\ & b\int_0^t\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1(\xi,\tau) - u_2(\xi,\tau))d\xi\,d\tau, \\ u_2(x,t) = {}& u_2^0(x) + \varphi_2(t) - \varphi_2(0) + \\ & \varepsilon b\int_0^t (u_1(x,\tau) - u_2(x,\tau) - \varphi_1(\tau) + \varphi_2(\tau))\,d\tau + \\ & \int_0^x \frac{f'(\xi)}{f(\xi)}(u_2(\xi,t) - u_2^0(\xi))\,d\xi + \int_0^t\int_0^x u_2(\xi,\tau)\frac{\partial u_2(\xi,\tau)}{\partial \xi}\,d\xi\,d\tau - \\ & \int_0^t\int_0^x \Big(\Big(\frac{\partial u_2(\xi,\tau)}{\partial \xi}\Big)^2 + u_2(\xi,\tau)\frac{\partial^2 u_2(\xi,\tau)}{\partial \xi^2}\Big)d\xi\,d\tau - \\ & \varepsilon b\int_0^t\int_0^x \frac{f'(\xi)}{f(\xi)}(u_1(\xi,\tau) - u_2(\xi,\tau))d\xi\,d\tau. \end{aligned}$$

The solvability of this system in the class of continuous in t and analytic in x is carried out in the same way as in Theorem 1. In this way it is proved

Theorem 2. *Let conditions A and agreement* (16) *be satisfied, the function* $f(x)$ *be analytic in some neighborhood of zero,* $f(x) \neq 0$. *Then the inverse problem* (13)–(15), (6) *has a unique solution* $u_1(z,t)$, $u_2(z,t)$, $g_1(t)$, $g_2(t)$

which for any positive $s \in (0,1)$ *is once continuously differentiable function on* t *for* $t < b(l-s)$ *with values in* X_s*, and*

$$\|u_1\|_s + \|u_2\|_s \le N.$$

The functions $g_1(t)$ *and* $g_2(t)$ *are determined by the formulas*

$$g_1(t) = \frac{1}{f(0)}\left(\frac{\partial \varphi_1(t)}{\partial t} + \varphi_1(t)\frac{\partial u_1}{\partial x}\Big|_{x=0} + b(\varphi_1(t) - \varphi_2(t))\right),$$
$$g_2(t) = \frac{1}{f(0)}\left(\frac{\partial \varphi_2(t)}{\partial t} + \varphi_2(t)\frac{\partial u_2}{\partial x}\Big|_{x=0} - \varepsilon b(\varphi_1(t) - \varphi_2(t))\right).$$

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