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Numerical solution to Volterra integral equations of the first kind by implicit Runge–Kutta method of high accuracy

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The analog of the implicit Runge–Kutta method applied to Volterra integral equations of the first kind is considered. It allows to obtain the results of high accuracy under a sufficient simplicity and stability of used algorithm. The estimation of numerical results for a fixed time step is performed. A special choice of integration's nodes and quadrature coefficients makes it possible to receive the error estimation, decreasing exponentially under increasing of a number of method's stages. That creates a good premise for using the method of high accuracy and permits to integrate with a big time step. The stability of method to variations of kernel and the right-hand side of the equation is proved. The above theoretical conclusions are confirmed by numerical experiments for Volterra equations of the first kind with different types of kernels and equivalent Volterra equation of the second kind.

Introduction

It is known that the integral Volterra equations of the first kind refer to the conditionally-correct tasks type. From one side, the Volterra equations of the first kind are a special case of the Fredholm equations, and hence, regularization methods may be applied for their stable numerical solving. From the other side, under some limitations on smoothness of kernel and the right-hand side of equation, they may be referred to correct type tasks and may admit a possibility of applying the methods, basing on discretization of initial equation. One may use also the discretization procedure as a regularization factor for solving the equations of this type. All these approaches are not without shortcomings, since the first one may lead to loss of volterness for regularizated equation, and others can lead to unstability of approximate solution from errors of initial data, especially under high accuracy methods applying.

One may be acquainted with sufficiently complete review for Volterra equations solving in work [7].

In this work the analog of implicit Runge–Kutta method for numerical solving of Volterra equations is chosen. This choice is not accidental because it allows to get results of high accuracy under sufficient simplicity of used algorithm, that distinguish such method among many ones, when the least solution error is required.

The implicit Runge–Kutta method was considered and applied for solving of Volterra integral equations of the first kind, apparently for the first time, in work [3], where the convergence and asymptotic stability to the rounding errors were proved. However, in real calculations one has to integrate often over a few fixed time step, for which the converge conditions may be unrealized. In this case the problem of estimation of the obtained solution for that time step value appears. Such estimation will be obtained in this work. Moreover, there will be shown that under appropriate choice of quadrature coefficients and nodes one can obtain the estimation of approximate error, decreasing exponentially under a growth of a number of stages in the method, and that will allow to integrate with a big time step. Thus, a high accuracy of obtained results will be reached not at a sacrifice of a small time step, as a solution error decreasing, being caused by the specific character of chosed method.

1. Statement of the problem

Consider the numerical solution to Volterra integral equation of the first kind:

$$\int_{t_0}^{t} K(t,s,y(s))\,ds = f(t), \quad t_0 \le t \le T. \tag{1}$$

Note that one of the main special cases for (1) is the linear Volterra equation, that will be also used in this work:

$$\int_{t_0}^{t} k(t,s)y(s)\,ds = f(t), \quad t_0 \le t \le T. \tag{2}$$

Introduce the grid on the segment $[t_0, T]$ with the nodes t_i:

$$t_i = ih, \quad i = 0,\dots,I, \quad h = \frac{T-t_0}{I},$$

and the subgrid partition

$$t_{ij} = t_i + \lambda_j h, \quad 0 \le \lambda_1 \le \lambda_2 \le \dots \le \lambda_m = 1.$$

Let us assume that for

$$\forall\,\tau \in [t_0,T]: \ |y(\tau)| < L, \quad K(\tau,\cdot,\cdot) \in C^m([t_0,T]\times[-L,L]).$$

In the points t_{ij} equation (1) will be as follows:

$$\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} K(t_{ij},s,y(s))\,ds + \int_{t_i}^{t_{ij}} K(t_{ij},s,y(s))\,ds = f(t_{ij}). \tag{3}$$

We approximate every integral under the sum sign in (3) by the quadrature formula

$$\int_{t_k}^{t_{k+1}} \varphi(s)\,ds \approx h \sum_{l=1}^{m} b_l \varphi(t_{kl})$$

and the rest integral in (3) by the formula

$$\int_{t_i}^{t_{ij}} \varphi(s)\,ds \approx h \sum_{l=1}^{m} b_{jl} \varphi(t_{il}).$$

Then, under approximation of (3), we obtain the following system of equations:

$$\sum_{k=0}^{i-1} \sum_{l=1}^{m} h b_l K(t_{ij}, t_{kl}, Y_{kl}) + \sum_{l=1}^{m} h b_{jl} K(t_{ij}, t_{il}, Y_{il}) = f_{ij}, \tag{4}$$
$$i = 0, \dots, I-1; \quad j = 1, \dots, m,$$

where $f_{ij} = f(t_{ij})$, and Y_{pq} denotes the finite-dimensional approximation for $y(t_{pq})$.

In the work [3], it was proved that under the following conditions for problem (2):

$$f(t) \in C^{m+2}[t_0, T]; \quad k(t,s) \in C^{m+2}([t_0,T] \times [t_0,T]);$$

$k(t,t) \neq 0$ for $t \in [t_0, T]$, and $\det(B) \neq 0$, where $B = (b_{ij})$, scheme (4) converges with the order m, i.e.,

$$\|\varepsilon_i\| \leq Ch^m, \quad C = \text{const}, \quad i = 0, \dots, I-1.$$

Here $\varepsilon_i = (\varepsilon_{i1}, \dots, \varepsilon_{im})$, $\varepsilon_{ij} = Y_{ij} - y(t_{ij})$.

The numerical stability of the method was also proved, when the values Y_{0j}, $j = 1, \dots, m$, were calculated with errors.

These theorems may be unconstructive in practice, because often we have to calculate with a fixed time step h, or to decrease it until some limited value h_0, for which the theorem conditions may be invalid. By virtue of that, the problem of estimation of the numerical solution error for the fixed time step h appears.

In this work, such estimation, together with estimation for quadrature formula (4), under a special choice of subgrid nodes and quadrature coefficients, will be obtained. The analysis of error variation under increasing of method stages m (this is the number of subgrid nodes), also will be carried out.

2. The approximation error on segment $[t_i, t_{ij}]$

Consider the approximation error for the integral

$$J_{ij} = \int_{t_i}^{t_{ij}} K(t_{ij}, s, y(s))\,ds,$$

by the quadrature formula

$$I_{ij} = \sum_{l=1}^{m} hb_{jl}K(t_{ij}, t_{il}, Y_{il}), \quad j = 1, \ldots, m.$$

We have

$$J_{ij} - I_{ij} = \sum_{l=1}^{m} hb_{jl}(K(t_{ij}, t_{il}, y_{il}) - K(t_{ij}, t_{il}, Y_{il})) + h\rho_{ij},$$

where

$$y_{il} = y(t_{il}), \quad \rho_{ij} = \frac{1}{h}\Big(\int_{t_i}^{t_{ij}} K(t_{ij}, s, y(s))\, ds - \sum_{l=1}^{m} hb_{jl}K(t_{ij}, t_{il}, y_{il})\Big).$$

Since the function $K(t, s, z)$ has the continuous derivative by the last value in the region of changing the former ones, we obtain:

$$\sum_{l=1}^{m} hb_{jl}(K(t_{ij}, t_{il}, y_{il}) - K(t_{ij}, t_{il}, Y_{il})) = \sum_{l=1}^{m} hb_{jl}\bar{k}_{ijl}(y_{il} - Y_{il}),$$

where

$$\bar{k}_{ijl} = \frac{\partial K(t, s, z)}{\partial z}\Big|_{(t_{ij}, t_{il}, \xi_{il})}, \quad \xi_{il} \in [Y_{il}, y_{il}].$$

To find the values ρ_{ij}, and accounting that $K(t, \cdot, \cdot) \in C^{(m)}([t_0, T] \times [-L, L])$, we expand the kernel of the integral J_{ij} in the Taylor series:

$$\begin{aligned} K(t_{ij}, s, y(s)) = & \sum_{k=0}^{m-1} \frac{1}{k!}K^{(k)}(t_{ij}, t_i, y(t_i))(s - t_i)^k + \\ & \frac{1}{m!}K^{(m)}(t_{ij}, \tau_i, y(\tau_i))(s - t_i)^m, \end{aligned}$$

where $\tau_i \in [t_i, t_{i+1}]$. Then

$$\begin{aligned} J_{ij} = & \sum_{k=0}^{m-1} \frac{1}{(k+1)!}K^{(k)}(t_{ij}, t_i, y(t_i))(h\lambda_j)^{k+1} + \\ & \frac{1}{(m+1)!}K^{(m)}(t_{ij}, \tau_i, y(\tau_i))(h\lambda_j)^{m+1}. \end{aligned}$$

From the other hand,

$$\begin{aligned} \sum_{l=1}^{m} hb_{jl}K(t_{ij}, t_{il}, y_{il}) = & \sum_{l=1}^{m} hb_{jl}\Big[\sum_{k=0}^{m-1} \frac{1}{k!}K^{(k)}(t_{ij}, t_i, y(t_i))(h\lambda_l)^k + \\ & \frac{1}{m!}K^{(m)}(t_{ij}, \tau_i, y(\tau_i))(h\lambda_l)^m\Big]. \end{aligned}$$

$$= \sum_{k=0}^{m-1} \frac{h}{k!} K^{(k)}(t_{ij}, t_i, y(t_i)) \sum_{l-1}^{m} b_{jl}(h\lambda_l)^k + \frac{h}{m!} K^{(m)}(t_{ij}, \tau_i, y(\tau_i)) \sum_{l=1}^{m} b_{jl}(h\lambda_l)^m.$$

Form here we obtain

$$\rho_{ij} = \sum_{k=0}^{m-1} \frac{h^k}{k!} K^{(k)}(t_{ij}, t_i, y(t_i))\delta_{jk} + \frac{h^m}{m!} K^{(m)}(t_{ij}, \tau_i, y(\tau_i))\delta_{jm}, \quad (5)$$

where

$$\delta_{jk} = \frac{1}{(k+1)} \lambda_j^{k+1} - \sum_{l=1}^{m} b_{jl}\lambda_l^k, \quad k = 0, \ldots, m. \quad (6)$$

Choose the quadrature coefficients b_{ij} such that $\delta_{jk} = 0$, for $k = 0, \ldots, m-1$. Then we obtain the following system of equations:

$$B\Lambda^{k-1}e = \frac{1}{k}\Lambda^k e, \qquad k = 1, \ldots, m, \quad (7)$$

where $B = (b_{jl})$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$, $e = (1, \ldots, 1)^T$, and this formula may be rewritten in the form:

$$BW = \Lambda WL, \quad (8)$$

where $L = \mathrm{diag}(1, 1/2, 1/3, \ldots, 1/m)$, W is the Vandermonde matrix:

$$W = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{m-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \lambda_m & \lambda_m^2 & \ldots & \lambda_m^{m-1} \end{bmatrix}.$$

Because all the nodes λ_j are different, the matrix W is nonsingular and the system of equations (8) has the unique solution.

From (5) and the special choice of the coefficients b_{ij}, we have

$$\rho_{ij} = \frac{h^m}{m!} K^{(m)}(t_{ij}, \tau_i, y(\tau_i))\delta_{jm}. \quad (9)$$

Now we find the error $\delta_m = (\delta_{1m}, \ldots, \delta_{mm})^T$, denoting it in explicit form as a function of subgrid nodes λ_j.

We choose the nodes λ_j to be equal to the nodes of the highest accuracy Radau quadrature formula with the fixed last node $\lambda_m = 1$.

For the matrix Λ, from the Cayley–Hamilton theorem, we have

$$\Lambda^m = \sum_{j=1}^{m} q_j \Lambda^{m-j}, \tag{10}$$

where q_j are the coefficients of the characteristic polynomial for the matrix Λ. In the work [2] the values q_j for the above choice of nodes were found in the explicit form:

$$q_j = (-1)^{j-1} \binom{m}{j}^2 \Big/ \binom{2m-1}{j}. \tag{11}$$

Multiplying both parts of equality (10) on matrix B from the left-hand side, and on the vector e from the right-hand side, we obtain

$$B\Lambda^m e = \sum_{j=1}^{m} q_j B\Lambda^{m-j} e$$

From the other hand, multiplying both parts of the same equality on the matrix Λ and vector e, we have

$$\Lambda^{m+1} e = \sum_{j=1}^{m} q_j \Lambda^{m-j+1} e.$$

Hence

$$\delta_m = B\Lambda^m e - \frac{1}{m+1}\Lambda^{m+1} e = \sum_{j=1}^{m} q_j (B\Lambda^{m-j} - \frac{1}{m+1}\Lambda^{m-j+1}) e$$

and, taking into account the system of equations (7) and equalities (11), we have

$$\delta_m = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j}^2 \binom{2m-1}{j}^{-1} \Big(\frac{1}{m-j+1} - \frac{1}{m+1}\Big) \Lambda^{m-j+1} e. \tag{12}$$

Carrying out the substitution of indexes $k = m - j + 1$ in (12), and using the identity $\binom{m}{m-k+1} = \binom{m}{k-1}$, finally, we obtain

$$\delta_{jm} = \frac{1}{m+1} \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k-1} \binom{2m-1}{m-k+1}^{-1} \frac{m-k+1}{k} \lambda_j^k. \tag{13}$$

The explicit dependence for the norm of the vector δ_m from the stage number m will be considered below.

3. The approximation error on segment $[t_k, t_{k+1}]$

Consider the approximation error for the integral

$$J_{ij}^{(k)} = \int_{t_k}^{t_{k+1}} K(t_{ij}, s, y(s))\, ds$$

by the quadrature formula

$$I_{ij}^{(k)} = \sum_{l=1}^{m} hb_l K(t_{ij}, t_{kl}, Y_{kl}).$$

Applying the same calculations as in the previous item, we obtain

$$J_{ij}^{(k)} - I_{ij}^{(k)} = \sum_{l=1}^{m} hb_l \bar{k}_{ijkl}(y_{kl} - Y_{kl}) + h\eta_{ijk},$$

where

$$\bar{k}_{ijkl} = \frac{\partial K(t, s, z)}{\partial z}\Big|_{(t_{ij}, t_{kl}, \xi_{kl})}, \quad \xi_{kl} \in [Y_{kl}, y_{kl}],$$

$$\eta_{ijk} = \frac{1}{h}\Big(\int_{t_k}^{t_{k+1}} K(t_{ij}, s, y(s))\, ds - \sum_{l=1}^{m} hb_l K(t_{ij}, t_{kl}, y_{kl})\Big).$$

Since $t_{kl} = t_k + \lambda_l h$ (λ_l corresponds to the nodes of the quadrature Radau formula choosed in previous item), it will be natural to suppose the quadrature coefficients b_l equal to the same ones in the Radau formula.

Using the method of finding the remainder term for the Radau formula under integration over the segment $[-1, 1]$ discribed in [1], and modifying it for an interval of arbitrary size, we obtain

$$\eta_{ijk} = P(m)\frac{h^{2m-1}}{(2m-1)!}K^{(2m-1)}(t_{ij}, z_k, y(z_k)), \tag{14}$$

where $z_k \in [t_k, t_{k+1}]$ and

$$P(m) = \frac{1}{2m^3}\Big(\frac{(m!)^2}{(2m-1)!}\Big)^2. \tag{15}$$

4. The estimation of the error $||\varepsilon_i||$

When we have obtained the approximation errors for all integrals in (3) by their quadrature analogues in two previous items, we begin to find the estimation of the error $||\varepsilon_i||$.

Subtracting equalities (4) from (3), we arrive

$$\begin{aligned}\sum_{k=0}^{i-1} J_{ij}^{(k)} + J_{ij} - \sum_{k=0}^{i-1} I_{ij}^{(k)} - I_{ij} \\ = \sum_{k=0}^{i-1}\Big(\eta_{ijk}h - \sum_{l=1}^{m} hb_l\bar{k}_{ijkl}\varepsilon_{kl}\Big) - \sum_{l=1}^{m} hb_{jl}\bar{k}_{ijl}\varepsilon_{il} + \rho_{ij}h \\ = 0, \quad j = 1,\ldots,m. \end{aligned} \tag{16}$$

Therefore, we have

$$\sum_{l=1}^{m} b_{jl}\bar{k}_{ijl}\varepsilon_{il} = r_{ij} - \sum_{k=0}^{i-1}\sum_{l=1}^{m} b_l\bar{k}_{ijkl}\varepsilon_{kl}, \tag{17}$$

where

$$r_{ij} = \rho_{ij} + \sum_{k=0}^{i-1}\eta_{ijk}. \tag{18}$$

The system of equations (17) may be written in the matrix form:

$$\bar{B}_i\varepsilon_i = R_i - \sum_{k=0}^{i-1} D_{ik}\varepsilon_k,$$

where $\bar{B}_i = (\bar{b}^i_{jl})$, $D_{ik} = (d^{ik}_{jl})$, $R_i = (r_{i1},\ldots,r_{im})^T$, and

$$\bar{b}^i_{jl} = b_{jl}\bar{k}_{ijl}, \quad d^{ik}_{jl} = b_l\bar{k}_{ijkl}. \tag{19}$$

Hence we derive the following system of inequalities:

$$\|\bar{B}_i\varepsilon_i\| \le \|R_i\| + \sum_{k=0}^{i-1}\|D_{ik}\|\cdot\|\varepsilon_k\| \le \|R_i\| + \|D_i\|\sum_{k=0}^{i-1}\|\varepsilon_k\|,$$

where $D_i = \{D_{iK} : \ \|D_{iK}\| \ge \|D_{ik}\|, \ 0 \le k \le i-1\}$, the norm of a vector $Z = (Z_1,\ldots,Z_m)^T$ is choosed as $\|Z\| = \max_l |Z_l|$, and the matrix norm is submitting to this vector norm.

Supposing the nonsingularity of the matrix $\bar{B}_i$, we obtain

$$\|\varepsilon_i\| \le \|\bar{B}_i^{-1}\|\Big(\|R_i\| + \|D_i\|\sum_{k=0}^{i-1}\|\varepsilon_k\|\Big). \tag{20}$$

Let us find the estimation for the initial error ε_0. The system of equations (16) for this case will be as follows:

$$J_{0j} - I_{0j} = \rho_{0j} h - \sum_{l=1}^{m} h b_{jl} \bar{k}_{0jl} \varepsilon_{0l} = 0, \quad j = 1, \ldots, m.$$

From here we have

$$\left| \sum_{l=1}^{m} b_{jl} \bar{k}_{0jl} \varepsilon_{0l} \right| \leq |\rho_{0j}|,$$

and

$$\|\varepsilon_0\| \leq \|\bar{B}_0^{-1}\| \cdot \|\rho_0\|, \tag{21}$$

where $\rho_0 = (\rho_{01}, \ldots, \rho_{0m})^T$.

To obtain the resultant estimation, we use the following Henrici lemma [4]:

If $|\delta_i| \leq \mathcal{A} \sum_{k=0}^{i-1} |\delta_k| + \mathcal{B}$ *at* $i = 1, 2, \ldots$, $\mathcal{A}, \mathcal{B} > 0$ *and* $|\delta_0| \leq \eta$, *then*

$$|\delta_i| \leq (\mathcal{B} + \mathcal{A}\eta)(1 + \mathcal{A})^{i-1}, \quad i = 1, 2, \ldots .$$

Therefore, taking into account inequalities (20), (21), we obtain the following estimation:

$$\begin{aligned} \|\varepsilon_i\| \leq\; & \|\bar{B}_i^{-1}\|(\|R_i\| + \|D_i\|\,\|\bar{B}_0^{-1}\|\,\|\rho_0\|) \times \\ & (1 + \|\bar{B}_i^{-1}\|\,\|D_i\|)^{i-1}. \end{aligned} \tag{22}$$

Let us modify inequality (22) to find the factors, effecting on the decreasing of the norm of ε_i vector with increasing the stage number m. Denote

$$Q_i^m = \frac{h^m}{m!} \max_j |\max_s K^{(m)}(t_{ij}, s, y(s))|, \quad t_i \leq s \leq t_{i+1},$$

$$\bar{Q}_i^{2m-1} = \frac{h^{2m-1}}{(2m-1)!} \max_j |\max_s K^{(2m-1)}(t_{ij}, s, y(s))|, \quad t_0 \leq s \leq t_i.$$

Then, from (9) we conclude $\|\rho_i\| \leq Q_i^m \|\delta_m\|$. At the same time, from (14) and (18) we have $\|R_i\| \leq \|\rho_i\| + P(m) i \bar{Q}_i^{2m-1}$. Thus, we obtain the following inequality:

$$\begin{aligned} \|\varepsilon_i\| \leq\; & \|\bar{B}_i^{-1}\|(1 + \|\bar{B}_i^{-1}\|\,\|D_i\|)^{i-1} \times \\ & \left(P(m) i \bar{Q}_i^{2m-1} + \|\delta_m\|(Q_i^m + \|D_i\|\,\|\bar{B}_0^{-1}\| Q_0^m)\right). \end{aligned} \tag{23}$$

The analysis of this inequality will be carried out below.

5. Stability of the method for perturbation of kernel and right-hand side of equation

Let us consider a case, when $\bar{K}(t, s, y(s))$ and $\bar{f}(t)$ are used in (1) instead of $K(t, s, y(s))$ and $f(t)$ respectively. Here

$$\|K(t,s,y(s)) - \bar{K}(t,s,y(s))\|_{C([t_0,T]\times[t_0,T])} < \theta,$$
$$\|f(t) - \bar{f}(t)\|_{C([t_0,T]} < \gamma.$$

Let us suppose also that the smoothness restrictions onto the function $\bar{K}(t,s,y(s))$ are the same as for the original function $K(t,s,y(s))$.

The system of quadrature equations, approximating new integral equation with perturbations, is the following:

$$\sum_{k=0}^{i-1}\sum_{l=1}^{m} hb_l\bar{K}(t_{ij},t_{kl},\bar{Y}_{kl}) + \sum_{l=1}^{m} hb_{jl}\bar{K}(t_{ij},t_{il},\bar{Y}_{il}) = \bar{f}_{ij}. \tag{24}$$

To find the estimation error $\|\bar{\varepsilon}_i\|$, $i = 0,\ldots,I-1$, where $\bar{\varepsilon}_{ij} = \bar{Y}_{ij} - y_{ij}$, $\bar{\varepsilon}_i = (\bar{\varepsilon}_{i1},\ldots,\bar{\varepsilon}_{im})^T$, we consider the approximation of the separate integral terms. Denote

$$\bar{I}_{ij} = \sum_{l=1}^{m} hb_{jl}\bar{K}(t_{ij},t_{il},\bar{Y}_{il}).$$

Then

$$\begin{aligned}
J_{ij} - \bar{I}_{ij} &= \int_{t_i}^{t_{ij}} K(t_{ij},s,y(s))\,ds - \sum_{l=1}^{m} hb_{jl}K(t_{ij},t_{il},y_{il}) + \\
&\quad \sum_{l=1}^{m} hb_{jl}(K(t_{ij},t_{il},y_{il}) - \bar{K}(t_{ij},t_{il},y_{il})) + \\
&\quad \sum_{l=1}^{m} hb_{jl}(\bar{K}(t_{ij},t_{il},y_{il}) - \bar{K}(t_{ij},t_{il},\bar{Y}_{il})) \\
&= \rho_{ij}h + \sum_{l=1}^{m} hb_{jl}(\theta_{ijl} + \bar{\bar{k}}_{ijl}(y_{il} - \bar{Y}_{il})),
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\theta_{ijl} &= K(t_{ij},t_{il},y_{il}) - \bar{K}(t_{ij},t_{il},y_{il}),\\
\bar{\bar{k}}_{ijl} &= \frac{\partial \bar{K}(t,s,z)}{\partial z}\Big|_{(t_{ij},t_{il},\xi_{il})}, \quad \xi_{il} \in [\bar{Y}_{il}, y_{il}].
\end{aligned}$$

Analogously, denoting

$$\bar{I}_{ij}^{(k)} = \sum_{l=1}^{m} hb_l\bar{K}(t_{ij},t_{kl},\bar{Y}_{kl}),$$

we have

$$J_{ij}^{(k)} - \bar{I}_{ij}^{(k)} = \eta_{ijk}h + \sum_{l=1}^{m} hb_l(\theta_{ijkl} + \bar{\bar{k}}_{ijkl}(y_{kl} - \bar{Y}_{kl})), \tag{26}$$

where

$$\theta_{ijkl} = K(t_{ij}, t_{kl}, y_{kl}) - \bar{K}(t_{ij}, t_{kl}, y_{kl}),$$
$$\bar{\bar{k}}_{ijkl} = \frac{\partial \bar{K}(t,s,z)}{\partial z}\Big|_{(t_{ij},t_{kl},\xi_{kl})}, \quad \xi_{kl} \in [\bar{Y}_{kl}, y_{kl}].$$

Subtructing (24) from (3), we obtain

$$\sum_{k=0}^{i-1} J_{ij}^{(k)} + J_{ij} - \sum_{k=0}^{i-1} \bar{I}_{ij}^{(k)} - \bar{I}_{ij} = f_{ij} - \bar{f}_{ij}.$$

Taking into account (25) and (26), we have

$$\sum_{l=1}^{m} b_{jl}\bar{\bar{k}}_{ijl}\bar{\varepsilon}_{il} = \bar{r}_{ij} - \sum_{k=0}^{i-1}\sum_{l=1}^{m} b_l \bar{\bar{k}}_{ijkl}\bar{\varepsilon}_{kl},$$

where

$$\bar{r}_{ij} = r_{ij} + \sum_{l=1}^{m} b_{jl}\theta_{ijl} + \sum_{k=0}^{i-1}\sum_{l=1}^{m} b_l\theta_{ijkl} - \gamma_{ij}/h, \quad \gamma_{ij} = f_{ij} - \bar{f}_{ij}.$$

Now using the same technique as in the previous item, we obtain the following system of inequalities for the norms of errors:

$$\|\bar{\varepsilon}_i\| \le \|\bar{\bar{B}}_i^{-1}\| \Big(\|\bar{R}_i\| + \|\bar{D}_i\| \sum_{k=0}^{i-1} \|\bar{\varepsilon}_k\| \Big), \tag{27}$$

where

$$\bar{D}_i = \{\bar{D}_{iK},\ \|\bar{D}_{iK}\| \ge \|\bar{D}_{ik}\|,\ 0 \le k \le i-1\},$$
$$\bar{\bar{B}}_i = (\bar{\bar{b}}_{jl}^{i}), \quad \bar{D}_{ik} = (\bar{d}_{jl}^{ik}), \quad \bar{R}_i = (\bar{r}_{i1}, \ldots, \bar{r}_{im})^T,$$
$$\bar{\bar{b}}_{jl}^{i} = b_{jl}\bar{\bar{k}}_{ijl}, \quad \bar{d}_{jl}^{ik} = b_l\bar{\bar{k}}_{ijkl}.$$

Find the estimation for components of $\bar{R}_i$ vector. Here we take into account that $\sum_{l=1}^{m} b_l = 1$ and $\sum_{l=1}^{m} b_{jl} = \lambda_j$. The second equality followed from (7) with $k = 1$. Then we have

$$|\bar{r}_{ij}| \le |r_{ij}| + \lambda_j \sum_{l=1}^{m} |\theta_{ijl}| + \sum_{k=0}^{i-1}\sum_{l=1}^{m} |\theta_{ijkl}| + |\gamma_{ij}|/h$$

and

$$\|\bar{R}_i\| \le \|R_i\| + m(i+1)\theta + \gamma/h. \tag{28}$$

For norms of the initial errors we have the estimation

$$\Big|\sum_{l=1}^{m} b_{jl}\bar{\bar{k}}_{0jl}\bar{\varepsilon}_{0l}\Big| \le |\rho_{0j}| + \Big|\sum_{l=1}^{m} b_{jl}\theta_{0jl}\Big| + \frac{|\gamma_{0j}|}{h}$$
$$\le |\rho_{0j}| + \lambda_j\Big|\sum_{l=1}^{m} \theta_{0jl}\Big| + \frac{|\gamma_{0j}|}{h}.$$

Hence

$$\|\bar{\varepsilon}_0\| \le \|\bar{\bar{B}}_0^{-1}\|(\|\rho_0\| + \theta m + \gamma/h). \qquad (29)$$

Applying the Henrici lemma to inequalities (27), (29) and accounting (28), we have

$$\|\bar{\varepsilon}_i\| \le q_i\Big(\theta m(i+1+g_i) + \gamma(1+g_i)/h + \|R_i\| + g_i\|\rho_0\|\Big),$$

where

$$g_i = \|\bar{D}_i\|\,\|\bar{\bar{B}}_0^{-1}\|, \quad q_i = \|\bar{\bar{B}}_i^{-1}\|(1 + \|\bar{\bar{B}}_i^{-1}\|\,\|\bar{D}_i\|)^{i-1}.$$

Finally, we conclude that at small perturbations of the right-hand side of the equation and the integral kernel with a fixed time step h, the error of the solution to equation with perturbations will be close to estimation (22).

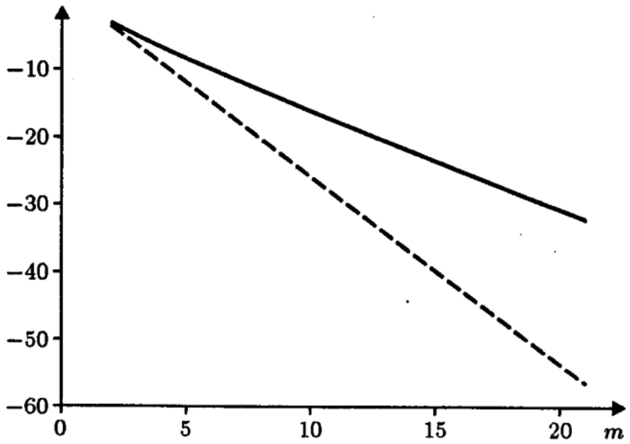
6. The numerical analysis of estimation for the error $\|\varepsilon_i\|$

Carry out the numerical analysis of estimation for the error defined by (23). Because the values $\|\bar{B}_i^{-1}\|$, $\|D_i\|$, Q_i^m, and $\bar{Q}_i^{2m-1}$ do not depend on the kernel of integral operator, we will demand their boundedness only. Let us analyse the behavior of two remained functions from (23), namely $P(m)$ and $\|\delta_m\|$. Since the nodes λ_j, as it was pointed above, were choosed equal to the nodes of the quadrature Radau formula, then the values of the required functions may be obtained numerically, if we have found the values of the nodes λ_j at $j = 1, \dots, m$ and have calculated δ_{jm} and $P(m)$ by (13) and (15) respectively. Let us consider the graphs of logarithms for calculated values of these functions, depending on the stage number m (the figure).

These functions are approximated with the sufficient accuracy by the following curves:

$$\|\delta_m\| \approx \exp(-1.5m - 0.7), \quad P(m) \approx \exp(-2.77m + 1.9).$$

From here we conclude that the decreasing of the third factor on the right-hand side of inequality (23) under increasing of the stages number m has the exponential character, and the decreasing of the whole norm of $\|\varepsilon_i\|$ error must be analogous, despite of the possible growth of the norms of some matrices in (23) in view of their possible bad conditions, which are compensated by the quick decreasing of third factor.



Graphs of functions $\ln \|\delta_m\|$ (solid line) and $\ln P(m)$ (dashed line)

7. Numerical experiments and discussion

To illustrate the above-mentioned results, we consider the numerical solution to the following integral equation of the first kind [5] by the implicit Runge–Kutta method, stated in this work:

$$\int_0^t \sin(t-s)y(s)\,ds = \exp(t^2/2) - 1, \tag{30}$$

with the kernel having the form $k(t,t) = 0$.

.This equation is convenient for analysis, because there exists the equivalent Volterra equation of the first kind with kernel's form $k(t,t) \neq 0$:

$$\int_0^t \cos(t-s)y(s)\,ds = t\exp(t^2/2), \tag{31}$$

and the equivalent Volterra equation of the second kind:

$$y(t) - \int_0^t \sin(t-s)y(s)\,ds = (1+t^2)\exp(t^2/2) \tag{32}$$

that will be also solved by the same method. The solution to the Volterra integral equation of the second kind was considered in [6].

The exact solution to equations (30)–(32) has the form

$$y(t) = (2+t^2)\exp(t^2/2) - 1.$$

Because all the kernels for problems (30)–(32) are linear with respect to the required function, then each of them is a particular case of the linear equation

$$\int_0^t k(t,s)y(s)\,ds = f(t) + \alpha y(t),$$

where $\alpha = 0$ for (30), (31), and $\alpha = 1$ for (32). Then the elements of matrices $\bar{B}_i$ and D_{ik}, defined by equalities (19), are the following:

$$\bar{b}^i_{jl} = b_{jl}k(t_{ij}, t_{il}), \quad d^{ik}_{jl} = b_l k(t_{ij}, t_{kl}).$$

That allows us to find $\|\bar{B}_i^{-1}\|$ and $\|D_i\|$ in the explicit form.

Let us analyse the results of numerical solution under solving each of equations (30)–(32) by stages (number of subgrid points), assuming the values $m = 6, 10, 20$, and also for the one-step ($I = 1$) and the multistep ($I = 4$) algorithms. The obtained solutions are presented in the table for $T = 3.2$.

The type of integral equation	m	$I = 1$		$I = 4$	
		$\bar{B}_I^{-1}$	eps	$\bar{B}_I^{-1}$	eps
Volterra I kernel $k(t,t) = 0$	6	333.6	1095.6	4353.7	6.54
	10	2685.7	36.53	46208	8.2e-03
	20	57104.8	2.38e-04	706075.9	5.04e-09
Volterra I kernel $k(t,t) \neq 0$	6	45.8	73.3	105.6	0.3
	10	115.9	4.23	304.4	1.65e-04
	20	425.4	7.48e-06	1285.5	6.64e-09
Volterra II	6	6.12	0.26	1.32	4.66e-06
	10	6.12	1.5e-05	1.32	4.55e-13
	20	6.12	8.8e-09	1.32	0

The exact solution to problems (30)–(32) in this point is equal to 2047.18492416. The values of error **eps** for quadrature methods, obtained in [5], are $\text{eps}_1 = 68.4$, $\text{eps}_2 = 1.56$, $\text{eps}_3 = 0.5$ respectively. These methods are similar to the trapezium method, but with the using of the information about separateness of the kernels $k(t,s)$ for problems (30)–(32).

Analysing the table data we may note that the matrix $\bar{B}_I$ is bad-posed for the Volterra equations of the first kind, and especially for those with kernel's form $k(t,t) = 0$. As a result, we have a quick growth of the norm of inverse matrix to this case with an increasing of the number of the subgrid nodes m. Perhaps this circumstance hinders to applying of many other quadrature methods of high accuracy, for which the analog of this matrix posesses the same properties.

From the other hand, this matrix is well-posed for the Volterra equations of the second kind, so the norm of inverse matrix is practically permanent under the increasing of the number of subgrid nodes, and is small by its value.

This leads to a reason of the necessity of reduction of the Volterra integral equation of the first kind to the second kind for its further numerical solving, if it is possible. For equations (2) this transformation is obtained by differentiation of its both sides with respect to t under the condition that the kernel and the right-hand side have these derivatives and $k(t,t) \neq 0$.

Recall to the values **eps** in the table that are equal to

$$\mathsf{eps} = |\varepsilon_{Im}| = |y(t_{Im}) - Y_{Im}|, \quad t_{Im} = 3.2.$$

The values **eps** at $m = 6$ for the Volterra equation of the first kind are too large. That indicates an insufficient number of stages for its solving. Under the increasing of the number of stages, despite of the above-mentioned growth of the norm of $\bar{B}_I^{-1}$ matrix, the error **eps** decreases until sufficiently small value. That circumstance confirms numerically the results at the end of the previous item on the character of changing of the error $\|\varepsilon_i\|$ in (22).

Under the increasing of the number of time steps, the decreasing of the error **eps** also takes place. That is naturally, if we take into account the decreasing of the step value h for ρ_{ij} and η_{ijk} in formulas (9) and (14). From here for choosed one-step and four-steps algorithms the ratio of norms of these values will be 4^m and 4^{2m-1} respectively. From the other hand, for the Volterra equation of the first kind in this case the increasing of $\|\bar{B}_I^{-1}\|$ takes place and, hence, the third factor on the right-hand side of inequality (22) also increases. This circumstance lowers the effect of increasing of the number of time steps, though the obtained values **eps** are decreased in a few exponents.

The norm of $\bar{B}_I^{-1}$ matrix for the Volterra equation of the second kind, conversely, is decreased under increasing of the number of time steps. By virtue of that, the changing of time step value h here will be more important factor, as for the Volterra equation of the first kind, i.e., the value of the error **eps** is decreased more quickly under increasing of the number of steps I. That is also observed by the calculated values in the table.

In conclusion we note the advantage of using the implicit Runge–Kutta method of high accuracy, especially for the Volterra equations of the first kind. So, for example, the value of the error **eps** in the table under using of 20th-stages one-step method is less than the same one under using of 10th-stages four-steps method.

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