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Formation of basic type autowave processes by a cellular neural network

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In this work, a formal background for the choice of parameters of Cellular Neural Network generating basic types of autowaves, namely, a round traveling front, a round traveling pulse and a spiral wave, is presented. This background is based on investigation of phase plane properties of a CNN cell. The required conditions for emergence of the traveling front and the traveling pulse with a one-cell initial condition are presented. Simulating results are presented for the round traveling front and the round traveling pulse initiated by the one-cell initial condition and for spiral waves initiated by random initial conditions.

1. Introduction

Autowaves are objects of investigation in various disciplines and may be considered as basic processes for studying more complex ones. In spite of the maturity of the partial differential equations theory used as a background for mathematical models of autowaves, there is a number of problems concerning their computer realization. In this connection, other approaches deserve much attention particularly those ones which are initially parallel and effective for hardware and high-performance computer realization. Among such approaches, application of Cellular Neural Networks (CNN) [1] may be of considerable interest.

Though much attention is paid to investigation of various complex phenomena in CNNs, such as autowaves [2, 3], chaotic processes [4] and emergence of stable structures [5], up-to-day there are no methods for obtaining parameters of a CNN required for realization of a desirable distributed dynamic process. In this connection, the purpose of this work is to present a formal background for construction of a CNN generating basic type autowaves and to show properties of the CNN as a model for such processes.

The paper consists of four sections. Next section contains the formal representation of CNN being investigated with emphasis on its difference from usual formal models of autowaves. In Section 3, an influence of a CNN cell parameters on dynamical properties of the cell is described and formal background for a choice of the parameters to obtain useful CNN properties is presented. Section 4 describes simulation results for basic types of autowaves, namely for a traveling round front, a traveling round pulse and spiral waves, using the theoretical background presented above and two types of initial conditions – one-point and random.

2. The CNN formal representation

The most general formal representation of the Cellular Neural Network can be found in [6]. The CNN under investigation is a 2-dimensional lattice with cells placed in its nodes. To simulate autowave processes most of which are observed in two-component systems, each cell should also be such a two-component system. Therefore, each cell of the CNN has a state vector $\boldsymbol{x} = [x_1, x_2]$, a bias vector $\boldsymbol{z} = [z_1, z_2]$ and an output vector $\boldsymbol{y} = [y_1, y_2]$, where x_i and y_i are functions of time, z_i are constants. Hence, each cell may be represented as two communicating *neurons* or as a *neuron pair.* The presence of two neurons in a cell divides the CNN into *two layers* of neurons. Each neuron in a cell has weighted connections with outputs $y_{i,m}$, $m = 1, \ldots, 9$ of adjacent neurons from its neighborhood. These adjacent outputs form an output vector $\boldsymbol{y}_i$ with enumeration of its components fixed in some determined order. The configuration of the neighborhood is defined by a neighborhood template matrix

$$\boldsymbol{A}_i = \begin{pmatrix} 0 & D_i & 0 \\ D_i & -4D_i & D_i \\ 0 & D_i & 0 \end{pmatrix},$$

where $i = 1, 2$ is the number of a layer. The matrix $\boldsymbol{A}_i$ may be represented as a neighborhood vector $\boldsymbol{A}'_i$ with components, enumerated in such a way that the components in $\boldsymbol{y}_i$ correspond to the components in $\boldsymbol{A}'_i$. For simulation of autowave processes, elements D_i of the matrix $\boldsymbol{A}_i$ play the role of diffusion coefficients.

According to these notations, a state evolution of each cell may be represented by the following system of two first order differential equations:

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= F_1(x_1, y_1, y_2, z_1) + \boldsymbol{A}'_1 \boldsymbol{y}_1, \\ \frac{\partial x_2}{\partial t} &= F_2(x_2, y_1, y_2, z_2) + \boldsymbol{A}'_2 \boldsymbol{y}_2, \end{aligned} \tag{1}$$

where coordinates of a cell in the lattice are omitted for simplicity. The functions F_i represent communication between the neurons within a cell, and the second addends in the right-hand part of equations represent communications between same-layer neurons from adjacent cells. According to this, the CNN being investigated may be defined as a reaction-diffusion system.

In this investigation, the following functions F_i were chosen as follows:

$$\begin{aligned} F_1(\cdot) &= \alpha x_1 + a y_1 + b y_2 + z_1, \\ F_2(\cdot) &= \beta x_2 + c y_1 + d y_2 + z_2, \end{aligned} \tag{2}$$

where the output y_i of each neuron is defined as a piece-wise linear function of corresponding state x_i and described by the following equation:

$$y_i = \frac{1}{2}(|x_i + 1| - |x_i - 1|). \tag{3}$$

It is necessary to note that, in comparison with usual mathematical models of autowaves, the simple type of the function F_i is used, but the nonlinearity of the function is "closed". In other words, in contrast with the usual models of autowaves, which have the following type of the functions F_i for a two-component system:

$$F_1 = f_1(x_1, x_2), \qquad F_2 = f_2(x_1, x_2),$$

where the functions f_i are nonlinear, in the CNN model being investigated the functions F_i are the following:

$$F_1 = g_1(x_1, f_1(x_1), f_1(x_2)), \qquad F_2 = g_2(x_2, f_2(x_1), f_2(x_2)),$$

where the functions g_i are simple polinoms while the functions f_i are defined as (3).

Application of the output function (3) allows to divide the phase plane of system (1) into nine linear regions, where standard methods [8] of analysis may be applied. In this work, the following definitions of the linear regions will be used:

$$\begin{aligned}
D^{-+} &= \{(x_1, x_2) : \ x_1 \le 1, \ x_2 \ge 1\}, \\
D^{++} &= \{(x_1, x_2) : \ x_1 \ge 1, \ x_2 \ge 1\}, \\
D^{--} &= \{(x_1, x_2) : \ x_1 \le 1, \ x_2 \le 1\}, \\
D^{+-} &= \{(x_1, x_2) : \ x_1 \ge 1, \ x_2 \le 1\}, \\
D^{l+} &= \{(x_1, x_2) : \ -1 < x_1 < 1, \ x_2 \ge 1\}, \\
D^{+l} &= \{(x_1, x_2) : \ x_1 \ge 1, \ -1 < x_2 < 1\}, \\
D^{l-} &= \{(x_1, x_2) : \ -1 < x_1 < 1, \ x_2 \le -1\}, \\
D^{-l} &= \{(x_1, x_2) : \ x_1 \le -1, \ -1 < x_2 < 1\}, \\
D^{ll} &= \{(x_1, x_2) : \ -1 < x_1 < 1, \ -1 < x_2 < 1\}
\end{aligned} \tag{4}$$

and also the following unions will be useful: $D_s = \{D^{++}, D^{+-}, D^{-+}, D^{--}\}$ and $D_p = \{D^{l+}, D^{+l}, D^{l-}, D^{-l}\}$.

3. Phase plane of a neuron pair

An emergence and propagation of any distributed process in the CNN is determined by the phase plane properties of a neuron pair and their transformation as a result of intercell communications. A topological structure of the phase plane is determined by the functions F_i, while the addends $\boldsymbol{A}_i'\boldsymbol{y}_i$ force the shift of stable points and limit trajectories during a CNN functioning. Therefore, it is necessary to determine the functions F_i required for realization of a round traveling front, a round traveling pulse and spiral waves.

In this work, the phase plane with a stable limit cycle is used as a background for construction of CNN generating round traveling front, round traveling pulse and vortexes. Because of peculiarities of the functions $F_i(\cdot)$, the well-known criteria [8] cannot be applied to determine the existence of a stable limit cycle on the phase plane. However, continuity of trajectories on the neuron pair phase plane and some qualitative criteria of a limit cycle presence allow to use the following

Proposition 1. *If the closed simply connected region G bounded by a cycle C of intersection multiplicity 1 may be drawn on the phase plane around a single unstable node or focus and all trajectories crossing C pass into G, then there is a stable limit cycle inside G.*

3.1. Equilibrium points

To determine the coefficients of polinoms in (2), it is necessary to clarify their influence on the types and coordinates of equilibrium points on the phase plane. In accordance with (2), an isolated neuron pair is described by the following system of equations

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha x_1 + ay_1 + by_2 + z_1, \\ \frac{dx_2}{dt} &= \beta x_2 + cy_1 + dy_2 + z_2. \end{aligned} \tag{5}$$

Hence, each linear region D contains a *single* equilibrium point P with the following coordinates:

$$\begin{aligned} P^{-+} &: \left(-\frac{1}{\alpha}(b-a+z_1), -\frac{1}{\beta}(d-c+z_2)\right), \\ P^{++} &: \left(-\frac{1}{\alpha}(a+b+z_1), -\frac{1}{\beta}(c+d+z_2)\right), \\ P^{--} &: \left(-\frac{1}{\alpha}(z_1-a-b), -\frac{1}{\beta}(z_2-c-d)\right), \end{aligned}$$

$$
\begin{aligned}
P^{+-} &: \left(-\frac{1}{\alpha}(a-b+z_1), -\frac{1}{\beta}(c-d+z_2)\right), \\
P^{l+} &: \left(-\frac{b+z_1}{\alpha+a}, \frac{c(b+z_1)}{\beta(\alpha+a)} - \frac{d+z_2}{\beta}\right), \\
P^{+l} &: \left(\frac{b(c+z_2)}{\alpha(\beta+d)} - \frac{a+z_1}{\alpha}, -\frac{c+z_2}{\beta+d}\right), \\
P^{l-} &: \left(-\frac{z_1-b}{\alpha+a}, \frac{c(z_1-b)}{\beta(\alpha+a)} - \frac{z_2-d}{\beta}\right), \\
P^{-l} &: \left(-\frac{b(z_2-c)}{\alpha(\beta+d)} - \frac{z_1-a}{\alpha}, -\frac{z_2-c}{\beta+d}\right), \\
P^{ll} &: \left(-\frac{b}{\alpha+a}\cdot\frac{cz_1-(\alpha+a)z_2}{(\alpha+a)(\beta+d)-bc} - \frac{z_1}{\alpha+a}, \frac{cz_1-(\alpha+a)z_2}{(\alpha+a)(\beta+d)-bc}\right).
\end{aligned}
\tag{6}
$$

Having certain set of parameters α, β, a, b, c, d, z_1, and z_2, each equilibrium point may be characterized either as real or as virtual [7] one. An equilibrium point may be virtual either by both coordinates X_i or by one of them.

On the base of the well-known methods (see e.g., [9]), the type and stability of equilibrium points for system (5) are defined by the roots of a characteristic equation:

$$
\begin{aligned}
&\lambda^2 - \lambda\left[\alpha+\beta+a\frac{dy_1}{dx_1}+d\frac{dy_2}{dx_2}\right]+ \\
&\left[\alpha\beta+\alpha d\frac{dy_2}{dx_2}+\beta a\frac{dy_1}{dx_1}+(ad-bc)\frac{dy_1}{dx_1}\frac{dy_2}{dx_2}\right]=0.
\end{aligned}
\tag{7}
$$

Taking into account the fact that $\frac{dy_i}{dx_i} = \{0,1\}$, $i = 1,2$ in accordance with (3), it is easy to obtain characteristic roots for each linear region, e.g., $\lambda_1 = \alpha$, $\lambda_2 = \beta$ for all regions D_s.

Following Proposition 1, the subarea containing an unstable node or focus should be determined by

Proposition 2. *The stable limit trajectory exists on the phase plane of a neuron pair iff the unstable focus or node belongs to the region* D^{ll}.

This proposition may be represented by the system of inequalities:

$$
\begin{aligned}
&\alpha+\beta+a+d>0, \\
&(\alpha-\beta+a-d)^2+4bc\le 0.
\end{aligned}
\tag{8}
$$

Note, that the second inequality is met iff $bc < 0$.

Types and coordinates of all other equilibrium points required to obtain a stable limit cycle are determined by the following

Lemma 1. *If* 1) *all equilibrium points in subareas D_p are saddles and* 2) *all equilibrium points in subareas D_s are stable nodes, then the following inequality*

$$\alpha + a - |b| + |z_1| < 0,$$
$$\beta - |c| + d + |z_2| < 0$$

determines virtuality of all equilibrium points in subareas D_p and D_s.

3.2. Limit trajectories

The type and coordinates of the equilibrium point in each subarea of the phase plane determine the directions of trajectories in this subarea. The following lemmas determine the properties of trajectories in subareas D_s and D_p which lead to the existance of the limit cycle on the whole phase plane.

Lemma 2. *If all subareas from D_s contain only virtual stable nodes, then in each subarea from D_p it is possible to draw an arc L', which connects opposite boundaries of this subarea and forms a closed region G' so, that all trajectories in the subarea which intersect L' are pass into G'.*

Lemma 3. *If all subareas from D_s contain only virtual stable nodes, then in each subarea from D_s it is possible to draw an arc L', which connects opposite boundaries of this subarea and forms a closed region G' so, that all trajectories in the subarea which cross L' are pass into G'.*

The following theorem defines a required condition for existence of stable limit cycle on the phase plane of system (5).

Theorem. *If the phase plane of a neuron pair described by system* (5) *contains only virtual saddles in subareas from D_p and only virtual stable nodes in subareas from D_s, then a stable limit cycle exists on the phase plane.*

The theorem leads to the system of inequalities which determines the existence of a stable limit cycle on the phase plane of system (5):

$$\begin{gathered} \alpha < 0, \qquad \beta < 0, \\ a > -\alpha, \qquad d > -\beta, \\ (\alpha - \beta + a - d)^2 + 4bc \leq 0, \\ \alpha + a - |b| + |z_1| < 0, \\ \beta - |c| + d + |z_2| < 0. \end{gathered} \tag{9}$$

4. Formation of autowaves

Generation of a traveling front or a traveling pulse requires to have one or two real stable equilibrium points on the phase plane respectively. Such points may be obtained by choosing appropriate values of variables z_i in (5). For the CNN under investigation it is possible to have only one or two such points being a real stable nodes superposed on virtual saddles and placed on boundaries between two adjacent regions.

An autowave emergence process is considered for two types of initial conditions: 1) a one-cell initial condition which determines the certain values of state variables of a single cell while all other cells are characterized by a stable equilibria; and a whole-network random initial condition which determines a random initialization of state variables of all CNN cells.

4.1. Determined one-cell initial condition

The one-cell initial condition allows to consider only one direction of an autowave emergence and only two neuron pairs placed on this direction due to the symmetry of $\boldsymbol{A}_i$ matrix. Let the cell with the certain initial values be defined as C_a and its adjacent cell as C_N. Then the *required* initial condition for a round traveling front emergence in the CNN with one real stable node placed on the boundary between regions D^{l+} and D^{++} on the phase plane may be defined as the following inequality:

$$y_{2,a} < 1 - \frac{c}{D_2}\left(1 - y_{1,N}\right). \tag{10}$$

Such inequality may be obtained for any other combinations of regions from D_s and D_p.

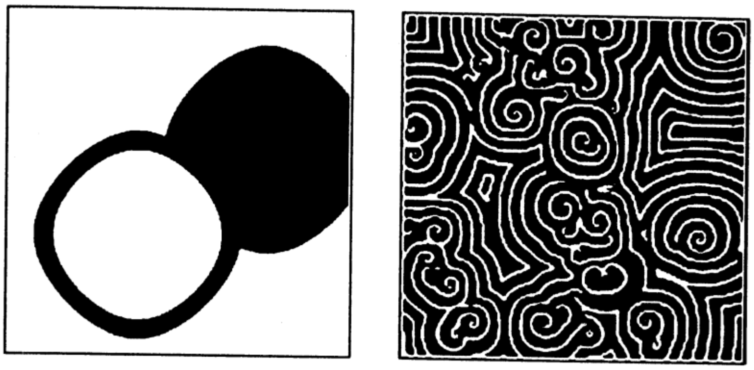
The required initial condition for a round traveling front emergence in the CNN with two stable nodes is similar to the previous case. The required condition for emergence of a round traveling pulse contains one more inequality, e.g., $y_{1,a} < 1$ for the same placement of one stable node, while the second one being placed on the boundary between regions D^{l-} and D^{+-} and may be obtained for all other valid cases.

Simulation of these two types of CNNs initiated by a determined one-cell initial condition was done for the CNN with 149×149 cells with the following values of parameters

$$\alpha = -1, \quad \beta = -1, \quad a = 1.5, \quad d = 1.5, \quad b = 1, \quad c = -1$$

with $z_1 = 0.4$, $z_2 = 0.5$ and $z_1 = 0.5$, $z_2 = 0.5$ for the CNN with one and two stable nodes respectively.

Propagation of a traveling pulse and a traveling front initiated according to the one-cell initial conditions is illustrated by the figure (a).



a b

Autowave processes obtained in the Cellular Neural Network being investigated: a – with one-cell initial conditions, b – with random initial conditions

Based on simulation results, the following conclusions can be made:

1. The bigger is the diffusion coefficient D_i the wider are fronts of autowaves.
2. A dependence of the autowave speed on the diffusion coefficients has a nonlinear character and differs for the two types of CNN.
3. An annihilation of two colliding autowaves and propagation of autowaves through the CNN boundaries without reflection occurs.

So, the autowaves in CNN behave similarly to autowaves in physical or chemical systems.

4.2. Random initial condition

While the determined one-cell initial condition allows to observe the emergence of one autowave and to understand a possibility to control this emergence process, the random initial condition is aimed to reveal self-organization properties of the CNN.

At the random initial condition, all cell states of the 149×149 CNN were set up to a random value from the range $[-0.5, 0.5]$. All parameters of cells were the same as in the case of a one-cell initial condition. Both types of the CNN, namely with one real stable node and with two real stable nodes on the phase plane of a neuron pair were simulated using the neuron pair parameters stated above. Spiral waves obtained using the random initial conditions are illustrated by the figure (b).

Based on simulation results, the following conclusions can be made:

1. Three types of autowave processes can be observed: a traveling round front, a traveling round pulse and a spiral wave.

2. In the CNN with one stable node, increasing of the diffusion coefficient values decreases the number of spiral wave sources, which are finally transformed into round traveling pulse sources and disappear.
3. In the CNN with two stable nodes, increasing of the diffusion coefficient values also decreases the number of spiral wave sources, which are mostly coupled, more stable and are not transformed into traveling pulse sources.

5. Conclusion

In this work, a formal background for the choice of parameters of CNN generating basic types of autowaves, namely, a round traveling front, a round traveling pulse and spiral waves, is presented. The required conditions for emergence of the traveling front and the traveling pulse are presented for two types of CNNs: with one stable node and with two stable nodes on the phase plane of a neuron pair. Simulating results show a similarity of the autowaves in the CNN being investigated with autowaves in real physical or chemical systems. The structure of the CNN presented allows to apply the CNN for the investigation not only basic autowave processes but also more complex ones. More over, the CNN properties allow to consider this model as a "direct-simulation" method and to apply it as an alternative to existing methods of investigation of complex self-organizing media. An inherent parallelism of the CNN model allows to achieve high efficiency in investigation and computational modeling of many complex phenomena.

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