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Convergence of quintic spline interpolants in terms of a local mesh ratio

A.Yu. Shadrin

In this paper we give an algorithm for finding the bounds for a ratio of two neighbouring mesh steps which provide the convergence of odd-degree spline interpolants and their derivatives. For the quintic splines numerical values are obtained which improve the estimates by S.Friedland, C.Micchelli.

1. Introduction

In this paper we study the problems of convergence of spline interpolants s_n of degree $2m-1$ and deficiency 1 to functions from the classes C^l in terms of a local mesh ratio. The interpolating nodes, which are also the spline breakpoints are given by the meshes

$$\Delta_n = \{a = t_1^{(n)} < t_2^{(n)} < \ldots < t_{N_n}^{(n)} = b\},$$

the natural boundary value condition on splines are considered:

$$s_n^{(\mu)}(f,t) = 0, \quad t = 0,1; \quad \mu = \overline{m, 2m-2}.$$

The local mesh ratio of a sequence $\Delta = \{\Delta_n\}_1^\infty$ is defined as

$$\rho_s(\Delta) = \lim_{n\to\infty} \max_{|i-j|<s} \frac{h_i^{(n)}}{h_j^{(n)}},$$

where $h_j^{(n)} = t_{i+1}^{(n)} - t_i^{(n)}$ is the local step of the n-th mesh.

We are concerned with the following

Problem 1. *For $m, s \in N$ and $0 \le l \le m-1$ given, define the exact upper bound $\rho^*_{2m-1,l,s}$ for the values $\rho_{2m-1,l,s}$, such that for any function $f \in C^l$ and arbitrary sequence Δ*

$$\rho_s(\Delta) < \rho_{2m-1,l,s} \Rightarrow \|f^{(l)} - s^{(l)}_{n,2m-1}\|_C \to 0 \quad (\overline{h}^{(n)} \to 0).$$

For $m = 2$, $l = 0$, the exact solution is

$$\rho^*_{3,0,s} = \left(\frac{3+\sqrt{5}}{2}\right)^s \sim (2.6180)^s, \tag{1}$$

as it was obtained in [9, 2, 5]. Earlier (see [1, p.29]) it was proved that for $m = 2$, $l = 1$,

$$\rho^*_{3,1,1} = \infty .$$

The case $m > 2$ was considered in [3, 8, 7] where the following bounds for the value $\rho^*_{2m-1,l,1}$ were found

$$\overline{\rho}_{2m-1,l,1} \le \rho^*_{2m-1,l,1} \le \tilde{\rho}_{2m-1,l,1}. \tag{2}$$

Moreover, the sufficient convergence conditions (the lower bound) were obtained in [3] only for $l = 0$, and divergence examples (the upper bound) were constructed in [8, 7] for $0 \le l \le m-2$. The numbers $\overline{\rho}_{2m-1,l,1}$ and $\tilde{\rho}_{2m-1,1,s}$ are defined as solutions of certain algebraic equations, their numerical values are computed in [7] for $m \le 9$.

The purpose of this paper is to improve the lower bound in (2) (i.e., the sufficient convergence conditions) for the interpolating quintic splines. Our main result is

Theorem 1. *Let $f \in C^l$, $l = 0,1,2$. Then for the quintic spline intrerpolants $s_{n,5}$ defined on $\Delta = \{\Delta_n\}_1^\infty$ the implication*

$$\rho_s(\Delta) < \hat{\rho}_{5,l,s}(M) \Rightarrow \|f^{(l)} - s^{(l)}_{n,5}\|_C \to 0 \quad (\overline{h}^{(n)} \to 0).$$

is valid.

The values $\hat{\rho}_{5,l,1}$ are given in the table along with the corresponding values $\overline{\rho}$ and $\tilde{\rho}$ from inequality (2):

l	$\overline{\rho}_{5,l,1}$	$\hat{\rho}_{5,l,1}$	$\tilde{\rho}_{5,l,1}$
1	1.1193	1.2018	1.4164
2	–	1.3584	1.8535
3	–	2.5071	–

This theorem was announced in [6], in this paper the proof is given.

2. Preliminary results

This section contains the statements proved in [6].

Denote the set of null-splines as

$$\dot{S}_{2m-1}(\Delta_n) = \{\sigma \in S_{2m-1}(\Delta_n) : \sigma(t_i) = 0, t_i \in \Delta_n\}.$$

Further, for a proper number $R = R(m)$ define the class $U_R[0,1]$ of functions $u \in C^m[0,1]$ with the properties

$$\begin{aligned} \text{a)}\quad & u(0) = 0, \quad u(1) = 1; \\ & u^{(\mu)}(0) = u^{(\mu)}(1) = 0, \quad \mu = \overline{1, m-1}; \\ \text{b)}\quad & \|u^{(\mu)}\|_C[0,1] \le R, \quad \mu = \overline{1,m}. \end{aligned} \tag{3}$$

For $u \in U_R[0,1]$, $M \in N$ and $i = \overline{0, N_n - M}$, set

$$u_{i,M}(t) = u\left((t - t_i)/(t_{i+M} - t_i)\right). \tag{4}$$

Lemma A. *For $M \ge m-1$ there is a number $\beta_M = \beta_M(m)$, such that for arbitrary $\sigma \in \dot{S}_{2m-1}(\Delta_n)$, $u \in U_R[0,1]$ and $i = \overline{0, N_n - M}$,*

$$\int_{t_i}^{t_{i+M}} (\sigma \cdot u_{i,M})^{(m)}(t) \cdot \sigma^{(m)}(t)\, dt \le \beta_M \int_{t_i}^{t_{i+M}} \left[\sigma^{(m)}(t)\right]^2 dt. \tag{5}$$

Set

$$\gamma_M = \gamma_M(m) = 1 + \frac{1}{\beta_M - 1}. \tag{6}$$

The following theorem is valid.

Theorem A. *Let $0 \le l \le m-1$, $f \in C^l[a,b]$. Then*

$$\|f^{(l)} - s_{n,2m-1}^{(l)}\|_C \le c(M,m) \sup_j \left\{ \sum_{i=0}^{N_n-1} \left(\frac{\overline{h}_j}{h_i}\right)^{m-l-1/2} \gamma_M^{-\frac{1}{2M}|i-j|} \right\} \|f^{(l)}\|_C,$$

where $\overline{h}_j = \max\{h_\nu : j \le \nu \le j+m-2\}$, and γ_M is defined in (5)-(6).

Set

$$\hat{\rho}_{2m-1,l,s}(M) = \left(\gamma_M^{\frac{1}{2M} \cdot \frac{1}{m-l-1/2}}\right)^s. \tag{7}$$

From Theorem A there follows

Theorem B. *Let $0 \le l \le m-1$, $f \in C^l$. Then for any $M \ge m-1$ and any sequence $\Delta = \{\Delta_n\}_1^\infty$,*

$$\rho_s(\Delta) < \hat{\rho}_{2m-1,l,s}(M) \Rightarrow \|f^{(l)} - s^{(l)}_{n,2m-1}\|_C = o(1)\,.$$

Remark. From (7) it is seen that

$$\hat{\rho}_{2m-1,0,s}(M) \le \hat{\rho}_{2m-1,1,s}(M) \le \ldots \le \hat{\rho}_{2m-1,m-1,s}(M),$$

i.e., the higher is the smoothness l of a function $f \in C^l$, the weaker are the constrains providing the convergence of spline interpolants.

Thus, Problem 1 is reduced to finding sharp estimates for the values γ_M defined in (6), i.e. to computing the exact constant β_M in inequality (5) of Lemma 1.

The first way to such a computing is to optimize the value β_M with respect to the functions $u \in U_R[0,1]$.

Another way which will be considered in the next section consists in reducing (5) to the equivalent relation between two quadratic forms

$$(Cx_i, x_i) \le \beta_M\,(Bx_i, x_i), \tag{8}$$

one of which (corresponding to the right-hand side (5)) is positive definite. In such a case the smallest value β_M providing (8) (and, therefore, (5)) is equal to the largest eigenvalue of the matrix $B^{-1}C$. The required value γ_M can be also equated to an eigenvalue of a special matrix.

3. Scheme of γ_M computation

Integrate the both sides of inequality (5) by parts. Using definitions (3)-(4) of the function $u_{i,M}$ and the fact that σ is a null-spline of degree $2m-1$, we obtain

$$\sum_{\nu=1}^{m-1}(-1)^{\nu+1}\,\sigma^{(m-\nu)}(\cdot)\,\sigma^{(m-1+\nu)}(\cdot)\,\Big|^{t_{i+m}}$$
$$\le \beta_M \sum_{\nu=1}^{m-1}(-1)^{\nu+1}\,\sigma^{(m-\nu)}(\cdot)\,\sigma^{(m-1+\nu)}(\cdot)\,\Big|_{t_i}^{t_{i+m}}\,.$$

Rewrite this relation in a vector form, for this pupose introduce the vectors $x_i = (x_i^1, \ldots x_i^{2m-2})$ with the components

$$x_i^\mu = \frac{1}{\mu!}\sigma^{(\mu)}(t_i), \quad i = \overline{0, N_n}, \quad \mu = \overline{1, 2m-2}.$$

It is known (see [3]) that these vectors are connected by the linear transformation

$$x_{i+1} = A_i x_i$$

with the matrices A_i of the form

$$A_i = -D(h_i^{-1})\, A\, D(h_i), \tag{9}$$

where, in turn,

$$D(h) = \operatorname{diag}(h, h^2, \ldots, h^{2m-2}), \tag{10}$$

$$A = \left\{C^{\mu}_{2m-1} - C^{\mu}_{\nu}\right\}^{2m-2}_{\mu,\nu=1}, \quad C^q_p = \tfrac{p!}{q!(p-q)!}\,. \tag{11}$$

If we define a symmetric matrix T by the relation

$$(Tx, x) = \sum_{\nu=1}^{m-1} (-1)^{\nu+1}(m-\nu)!\,(m-1+\nu)!\, x^{m-\nu} x^{m-1+\nu}, \tag{12}$$

and set

$$A_{i,M} = A_{i+M-1} \times A_{i+M-2} \times \cdots \times A_i,$$

then the vector form of (5) will look like

$$(TA_{i,M}\, x_i,\, A_{i,M}\, x_i) \le \beta_M \left[(TA_{i,M}\, x_i,\, A_{i,M}\, x_i) - (Tx_i,\, x_i)\right],$$

or

$$(A^*_{i,M} T A_{i,M}\, x_i,\, x_i) \le \beta_M \left([A^*_{i,M} T A_{i,M} - T]\, x_i,\, x_i\right).$$

Finally, substituting $x_i = A^{-1}_{i,K} x_{i+K}$, $K = \overline{0, M}$, to the latter formula we obtain the following expression, which is equivalent to (5):

$$\begin{aligned}(A^*_{i+K,M-K} & T A_{i+K,M-K}\, x_{i+K},\, x_{i+K}) \\ & \le \beta_M \left(\left[A^*_{i+K,M-K} T A_{i+K,M-K} - (A^{-1}_{i,K})^* T A^{-1}_{i,K}\right] x_{i+K},\, x_{i+K}\right),\end{aligned} \tag{13}$$

where it is implied $A_{i,0} = E$, $A_{i,1} = A_i$, $K = \overline{0, M}$.

The quadratic form standing in the right-hand side of (13) corresponds to the right-hand side of (5), hence, the symmetric matrix

$$B = A^*_{i+K,M-K} T A_{i+K,M-K} - (A^{-1}_{i,K})^* T A^{-1}_{i,K}$$

is a positive definite one. Thus, for any symmetric matrix C the smallest value β_M provided the inequality

$$(Cx, x) \le \beta_M (Bx, x)$$

is equal [4, p.290] to the largest eigenvalue of the matrix $B^{-1}C$. The value (see (6))

$$\beta_M^{-1} - 1 = \gamma_M^{-1}$$

coincides in this case with the smallest among those roots of the characteristic equation

$$\| (C^{-1}B - E) - \lambda E \| = 0$$

for the matrix $C^{-1}B - E$ which are greater than -1.

Applied to (13) this means that for computing γ_M we need to find the roots of equation

$$\| -[A^*_{i+K,M-K} T A_{i+K,M-K}]^{-1} [A^*_{i,K} T A_{i,K}] - \lambda E \| = 0$$

or of the equivalent one

$$\| [A^*_{i,K} T A_{i,K}] + \lambda [A^*_{i+K,M-K} T A_{i+K,M-K}] \| = 0, \tag{14}$$

and choosing among them the root λ_M, closest to -1 from the right, equalize

$$\gamma_M = -\lambda_M^{-1}. \tag{15}$$

4. Results for quintic splines

Here we will show how the above given scheme works for quintic splines, but beforehand for more clarity we consider the case of cubic spline interpolation.

4.1. Cubic splines

Set in (9)-(12) $m = 2$, $M = 1$, i.e., we are concerned with one interval $[t_i, t_{i+1}]$, which can be considered as unit one. We have

$$A_i = A_{i,1} = -\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

$$A^*_{i,1} T A_{i,1} = -\begin{pmatrix} 2 & 7/2 \\ 7/2 & 6 \end{pmatrix}.$$

Equation (14) where we consider $K = 0$, $A_{i,0} = E$, takes the form

$$\left\| \begin{matrix} 2\lambda & \frac{7}{2}\lambda + \frac{1}{2} \\ \frac{7}{2}\lambda + \frac{1}{2} & 6\lambda \end{matrix} \right\| = 12\lambda^2 - \frac{1}{4}(7\lambda + 1)^2 = 0,$$

whence

$$\lambda_{1,2} = -(7 \pm 4\sqrt{3}) = -(2 \pm \sqrt{3})^2,$$
$$\lambda_M(m) = \lambda_1(2) = -(2 - \sqrt{3})^2,$$

and basing on (15)

$$\gamma_M(m) = \gamma_1(2) = (2 + \sqrt{3})^2.$$

Therefore, by virtue of (7) we can take in Theorem B the value

$$\hat{\rho}_{3,0,s}(1) = (2 + \sqrt{3})^{\frac{2}{3}\cdot s} \sim (2.4060)^s,$$

which is somewhat worse than (1).

4.2. Quintic splines

Set in (9)-(12) $m = 3$, $M = 2$, i.e., consider two neighbouring intervals $[t_i, t_{i+1}]$, $[t_{i+1}, t_{i+2}]$ with lengthes h_i, h_{i+1}. In this case

$$A_{i+1,1} = \begin{pmatrix} 4 & 3h_{i+1} & 2h_{i+1}^2 & h_{i+1}^3 \\ 10h_{i+1}^{-1} & 9 & 7h_{i+1} & 4h_{i+1}^2 \\ 10h_{i+1}^{-2} & 10h_{i+1}^{-1} & 9 & 6h_{i+1} \\ 5h_{i+1}^{-3} & 5h_{i+1}^{-2} & 5h_{i+1}^{-1} & 4 \end{pmatrix},$$

$$A_{i,1}^{-1} = \begin{pmatrix} 4 & -3h_i & 2h_i^2 & -h_i^3 \\ -10h_i^{-1} & 9 & -7h_i & 4h_i^2 \\ 10h_i^{-2} & -10h_i^{-1} & 9 & -6h_i \\ -5h_i^{-3} & 5h_i^{-2} & -5h_i^{-1} & 4 \end{pmatrix},$$

$$T = 12 \begin{pmatrix} 0 & & & -1 \\ & & 1/2 & \\ & 1/2 & & \\ -1 & & & 0 \end{pmatrix}.$$

The further scheme goes like that. Let

$$\frac{h_{i+1}}{h_i} = \alpha > 0 \tag{16}$$

Solving equation (14) we obtain the values $\beta_M(\alpha)$ and $\gamma_M(\alpha)$, such that the inequality (5) is valid with the intervals $[t_i, t_{i+1}]$, $[t_{i+1}, t_{i+2}]$, satisfying (16). Now, if we take $\beta_M = \max_\alpha \beta_M(\alpha)$, then (5) will be valid for two interavals with arbitrary length ratio. By (6), the value $\beta_M = \max_\alpha \beta_M(\alpha)$ corresponds to the value

$$\gamma_M = \min_\alpha \gamma_M(\alpha),$$

which we start now to compute.

Substituting in (14) with $K = 1$, $M = 2$ the calculated matrices $A_{i+1,1}$, $A_{i,1}^{-1}$, T and omitting the factor 12 from the latter one, we find

$$\begin{Vmatrix} 60\,(\lambda-\alpha^3)\,h_{i+1}^{-3} & 60\,(\lambda+\alpha^2)\,h_{i+1}^{-2} & 50\,(\lambda-\alpha)\,h_{i+1}^{-1} & 29\,(\lambda+1) \\ 60\,(\lambda+\alpha^2)\,h_{i+1}^{-2} & 60\,(\lambda-\alpha)\,h_{i+1}^{-1} & \frac{101}{2}\,(\lambda+1) & 30\,(\lambda-\alpha^{-1})\,h_{i+1} \\ 50\,(\lambda-\alpha)\,h_{i+1}^{-1} & \frac{101}{2}\,(\lambda+1) & 43\,(\lambda-\alpha^{-1})\,h_{i+1} & 26\,(\lambda-\alpha^{-2})\,h_{i+1}^{2} \\ 29\,(\lambda+1) & 30\,(\lambda-\alpha^{-1})\,h_{i+1} & 26\,(\lambda-\alpha^{-2})\,h_{i+1}^{2} & 16\,(\lambda-\alpha^{-3})\,h_{i+1}^{3} \end{Vmatrix}$$

$$= \frac{1}{4}\,\lambda^4 + a(\alpha)\,\lambda^3 + [b(\alpha) + 1/2]\,\lambda^2 + a(\alpha)\,\lambda + \frac{1}{4} = 0, \qquad (17)$$

where the functions $a(\alpha)$, $b(\alpha)$ have the following form:

$$a(\alpha) = 120\left\{2\,(\alpha^3 + \alpha^{-3}) + 13\,(\alpha^2 + \alpha^{-2}) + 34\,(\alpha + \alpha^{-1}) + 46\tfrac{1}{120}\right\},$$

$$b(\alpha) = 240\left\{7\,(\alpha^3 + \alpha^{-3}) + 43\,(\alpha^2 + \alpha^{-2}) + 109\,(\alpha + \alpha^{-1}) + 146\tfrac{1}{240}\right\}.$$

The change $y = \lambda + \lambda^{-1}$ reduces (17) into the quadratic equation

$$\frac{1}{4}\,y^2 + a(\alpha)\,y + b(\alpha) = 0. \qquad (18)$$

One can be convinced that the value $|y_{\min}(\alpha)|$ of the smallest absolute root of the equation (18) takes the least (with respect to α) value for $\alpha = 1$. For such α equation (18) takes the form:

$$\frac{1}{4}\,y^2 + 17\,281\,y + 111\,361 = 0,$$

whence

$$y_{\min}(1) = -2\,(17281 - 2080\sqrt{69}) = -2\,\frac{49 + \sqrt{81 - 1/48}}{9 + \sqrt{81 - 1/48}}.$$

Since,

$$y_{\min}(1) = \lambda_M(m) + \lambda_M^{-1}(m), \quad \gamma_M(m) = -\lambda_M^{-1}(m),$$

finally we obtain

$$\gamma_2(3) = \frac{\sqrt{x}+1}{\sqrt{x}-1}, \quad x = \frac{29+\sqrt{81-1/48}}{20},$$

and numerical value is the following:

$$\gamma_2(3) \sim 6.2856.$$

The values $\tilde{\rho}_{5,l,1}$ in Theorem 1 are derived from Theorem B and (7). Now the comparison with known estimates is to our favour (see Table 1).

The numerical experiments show that the values $\hat{\rho}_{2m-1,l,1}(M)$ increase monotonically with respect to M. For example, for $\hat{\rho}_{5,0,1}(M)$ we have such a numerical computation:

M	3	6	9	12
$\hat{\rho}_{5,0,1}(M)$	1.2653	1.3341	1.3541	1.3656

Since

$$\sup_M \hat{\rho}_{2m-1,l,s}(M) \leq \rho^*_{2m-1,l,s},$$

these experiments make evidence to the conjecture [3] that

$$\rho^*_{2m-1,l,s} = \left(\tilde{\rho}_{2m-1,l,1}\right)^s.$$

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