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Stable patterns formation by totalistic cellular automata*

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Abstract. Evolution of totalistic cellular automata (CA) with weighed templates are studied. As a result of simulation various stable patterns are obtained and investigated. Both synchronous and asynchronous modes of CA operation are tested. As a result of simulation by synchronous and asynchronous CA, different stable patterns emerging from one nucleation cell are obtained and classified. The influence of weight matrix entries on stable patterns is studied. The theorem about the stable patterns dependence on the ratio of positive to negative entries of a weight matrix is proved. Stable patterns formed of two nucleation cells are also investigated.

1. Introduction

Many of the phenomena and processes in physics, chemistry, biology and sociology are associated with emergence of stable patterns. Concentric, spiral spatial waves are observed in different chemical reactions. Well-known examples of stable patterns in living systems are the pigment spots and stripes on the animals' skin. Studying the stable patterns arising in self-organizing complex systems is important both from fundamental and practical points of view. For example, in chemistry, the investigation of self-organizing processes on catalysts makes possible to develop new materials with unique properties, in medicine studying self-organization of neurons has great importance to diagnostics of the brain activity.

There are many publications devoted to studying the patterns formation in dissipative non-equilibrium systems. The foundations of spatially stable patterns are laid by A. Turing in his study of reaction–diffusion morphogen systems in 1952 [1]. The first CA of the Turing pattern formation based on an activator–inhibitor interaction was proposed by D.A Young [2]. The pattern formation occurs in many well-known catalytic reactions, for instance, the carbon monoxide oxidation reaction over platinum metal [3] and the Belousov–Zabotinskii reaction [4]. Many examples of the stable patterns formation in different physical, chemical and biological systems are presented in [5, 6]. In [7, 8], the cellular automata (CA) approach to describing the self-organizing processes and patterns formation is proposed. Cellular

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automata are most suitable for describing the complex systems dynamics, because they make possible to simulate spatial heterogeneous nonlinear processes by simple local transition rules.

The objective of the paper is to investigate the stable patterns formed by evolution of cellular automata with weighted templates with a single initial nucleation cell.

Section 2 contains a formal definition of cellular automata with weighted templates. Section 3 presents the dependence of stable patterns formation on weight matrix entries. Results of computing experiments performed by synchronous and asynchronous CA are given in Section 4.

2. Definition of cellular automaton with weighted templates

The totalistic cellular automaton (CA) with weighted templates is defined by the four notions [9]:

$$\aleph = \langle A, X, \Theta, \rho \rangle, \tag{1}$$

where $A = \{0, 1\}$ is a cells *state alphabet*, $X = \{(i, j) : \ i = 1, \dots, M_i, \ j = 1, \dots, M_j\}$ is a *set of names*, Θ is a *local operator* defined by the cells transition rules, $\rho \in \{\sigma, \alpha\}$ is a *mode of operations.* A *cell* is a pair $(u, (i, j))$, where $u \in A$ is a state of the cell, (i, j) is a name of the cell. On the set of names X, the *template* $T(i, j)$ is introduced. The template defines the nearest neighbors of each cell. Further, the template $T_7(i, j)$ having the size of 7×7 cells is used:

$$T_7(i, j) = \{(i + k, j + l) : \ k, l = -3, \dots, 0, \dots, 3\}. \tag{2}$$

In the CA model, a *weight matrix* $W_{7\times7}$ is associated with the template $T_7(i, j)$. The weight matrix $W_{7\times7}$ contains positive and negative entries. The positive entries are called *activators* and are responsible for the growth of patterns. The negative entries are *inhibitors*, they impede the pattern growth. The structure of a weight matrix, used in the sequel, is the following:

$$w_{kl} = \begin{cases} p, & |k| \le 2 \wedge |l| \le 2, \\ n, & \text{otherwise}, \end{cases} \quad \Rightarrow \quad W_{7\times7} = \begin{pmatrix} n & n & \dots & n & n \\ n & p & \dots & p & n \\ \dots & \dots & \dots & \dots & \dots \\ n & p & \dots & p & n \\ n & n & \dots & n & n \end{pmatrix}, \tag{3}$$

where $n < 0$ is an inhibitor, $p > 0$ is an activator. The presence of activators and inhibitors in a system allows supporting the process in an equilibrium state, and provides conditions for the stable patterns formation [2, 5].

The local operator Θ changes a cell state (i, j) depending on the states of the neighboring cells allocated by the template $T_7(i, j)$:

$$\Theta(i,j):\ (a,(i,j)) \to (a',(i,j)), \tag{4}$$

where

$$a' = \begin{cases} 1, & s > 0, \\ 0, & \text{otherwise}, \end{cases} \qquad s = \sum_{k,l=-3,\dots,3} w_{kl} \cdot a_{i+k,j+l}.$$

Here w_{kl} are defined by (3) and $a_{i+k,j+l}$ are states of neighbor cells covered by the template $T_7(i,j)$. The term *totalistic CA* means that a new cell state is a function of the sum or a weighted sum of states of the cells belonging to the template.

There are two modes of application of a local operator to the cells of a CA: synchronous σ and asynchronous α modes. In the synchronous mode, a local operator is applied to all the cells of the CA, all being updated simultaneously. Whereas the asynchronous mode of CA prescribes a local operator to be applied to a randomly chosen cell, changing its state immediately. Application of a local operator to all the cells of the CA is named *iteration*. An iteration transfers the cellular array $\Omega(t)$ into $\Omega(t+1)$, where t is an iteration number. The sequence $\aleph(\Omega) = \Omega(0), \Omega(1), \dots, \Omega(t_{\text{fin}})$ is named *evolution*.

Simulation is performed for the initial cellular array $\Omega(0)$ with one and two nucleation cells. A nucleation cell is a cell in a state $a = 1$. A nucleation cell is situated in the center of the cellular array, other elements of the array being equal to zero.

3. The stable patterns formation by CA evolution

The above-defined CA exhibits self-organization properties. This means that after a number of iterations, the CA evolves to a steady state, notably, such that does not change or changes in no more than some number of cells [10, 11]. If the states $a = 1$ in $\Omega(t)$ are marked with black color and $a = 0$— with white one, then $\Omega(t)$ is displayed as a black-and-white pattern. Some patterns of $\aleph$ evolution are shown in Figure 1.

The main parameters affecting the CA evolution are the initial state $\Omega(0)$ and the weight matrix. The number of initial nucleation cells essentially affects the patterns formation. Further, the patterns formed with a single initial nucleation cell are investigated, although some patterns formed of two nucleation cells are presented in Section 4. It is revealed that for a fixed array size and initial state, the pattern is uniquely determined by the ratio of activator to inhibitor:

Theorem. *Evolution of the totalistic CA with weighted templates for a specified initial state and a cellular array size is uniquely determined by the ratio $\frac{p}{n}$. Identical patterns are formed for the same value of $\frac{p}{n}$.*

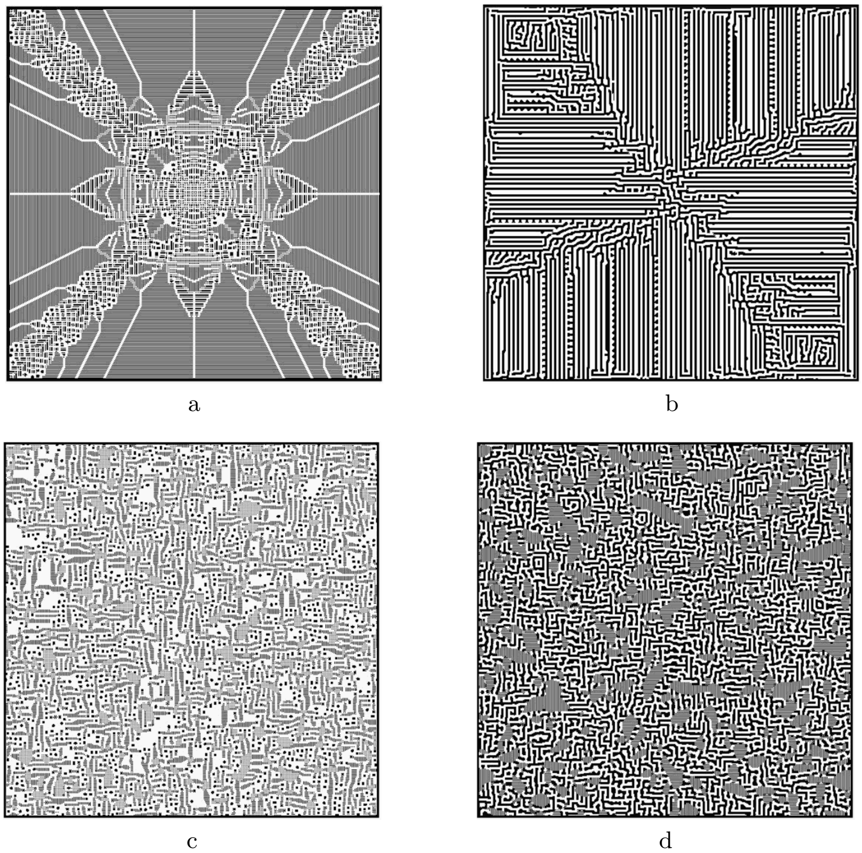


Figure 1. Patterns formed by ℵ evolution from one nucleation cell for the synchronous mode: a) $n = -1$, $p = 0.56$, b) $n = -1$, $p = 0.9$ and asynchronous mode: c) $n = -1$, $p = 0.5$, d) $n = -1$, $p = 1$

Proof. Let us consider application of $\Theta(i, j)$ to a cell $(i, j) \in X$. According to (4), a new state of the cell (i, j) depends on the sign of s:

$$s = n \sum_{k=-3}^{3} \left(a_{i-3,j+k} + a_{i+3,j+k} + a_{i+k,j-3} + a_{i+k,j+3}\right) + p \sum_{k=-2}^{2} \sum_{l=-2}^{2} a_{i+k,j+l} = n\,\Sigma_{\text{out}} + p\,\Sigma_{\text{in}}. \tag{5}$$

Then a new state of the cell (i, j) is as follows:

$$a' = \begin{cases} 1, & s > 0, \\ 0, & \text{otherwise}, \end{cases} \quad \Rightarrow \quad a' = \begin{cases} 1, & \Sigma_{\text{out}} + \dfrac{p}{n}\Sigma_{\text{in}} < 0, \\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

Consequently, a new value of the cell state depends on the ratio $\frac{p}{n}$ and does not depend on the specific values of activator and inhibitor. □

4. Results of computational experiments

Computational experiments were performed for a cellular array of the size of $M_i \times M_j = 500 \times 500$ cells. The initial state is a single nucleation cell in the center of cellular array. Periodic boundary condition is used. Both synchronous and asynchronous modes were tested.

According to the theorem, the key parameter of the patterns formation process is the absolute value of the coefficients ratio $\frac{p}{n}$. Identical ratios form identical patterns. Therefore, it is sufficient to investigate the dependence of CA evolution on values of p assuming $n = -1$.

Stable patterns formed as a result of the evolution of the *synchronous CA* $\aleph_\sigma$ for different values of p are shown in Figure 2. A steady state of the synchronous CA is alternation of two patterns on each iteration. For example, alternation of two geometric figures — a square and a cross — arises at $p \in (0.4, 0.5]$. In this case evolution is as follows. On first iteration, a square is formed of a nucleation cell. On the next iteration, a cross is formed of the square. Then the square is formed again, etc. Such an alternation occurs also for fancy figures and strips, though in this case a few cells states change, but not the whole pattern (Figure 3). Many different crosses are formed by the evolution of $\aleph_\sigma$. These crosses are named as it is shown in

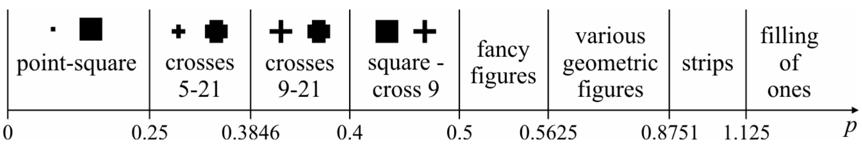
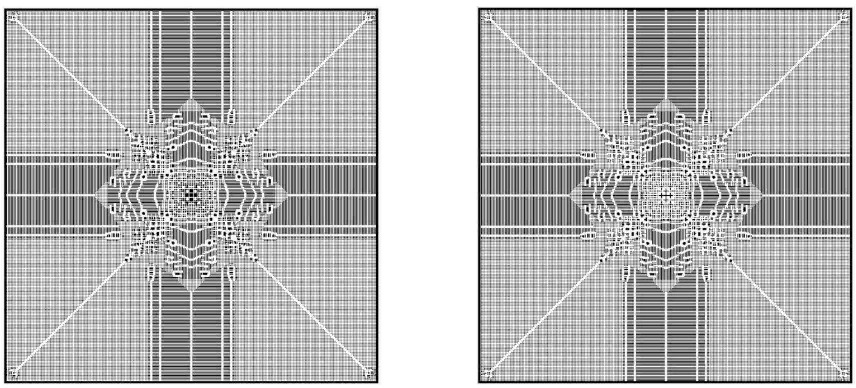


Figure 2. Stable patterns formed as a result of the synchronous CA evolutions for different values of p



a b

Figure 3. Alternation of two fancy figures for $p = 0.501$: a) the pattern formed on even iteration and b) the pattern formed on odd iteration

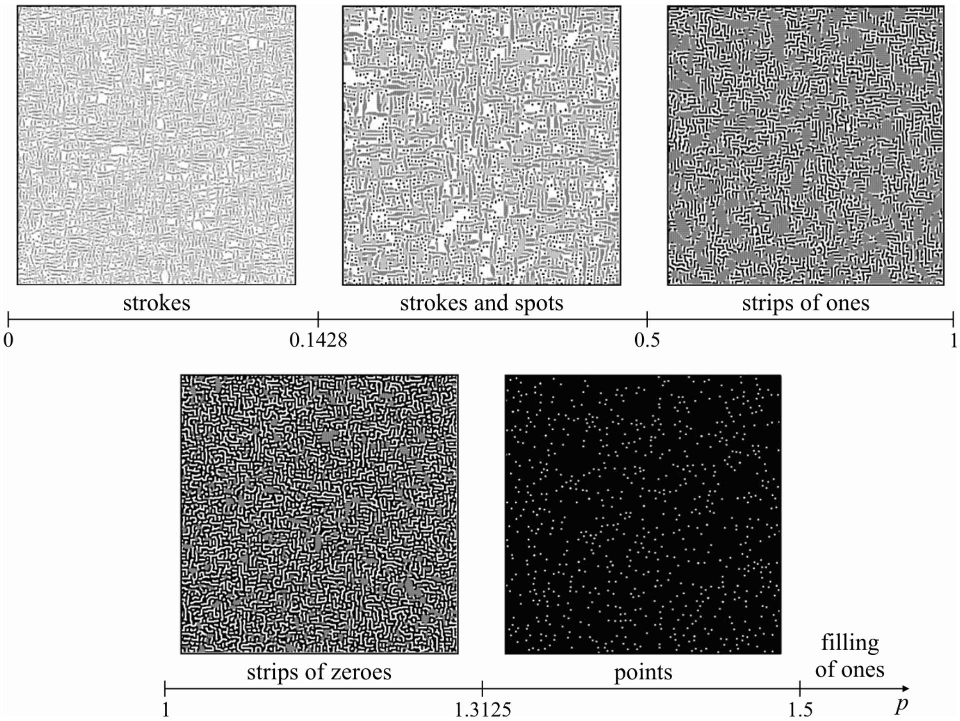


Figure 4. Stable patterns formed as a result of the asynchronous CA evolutions for different values of p

Figure 2. For example, the pattern name "crosses 5–21" denotes alternation of the cross of 5 ones and the cross of 21 ones. The range $p \in (0.5625, 0.8751)$ comprises some small ranges of various geometric figures alternation.

Stable patterns formed by the evolution of the *asynchronous CA* $\aleph_\alpha$ for different values of p are shown in Figure 4. These patterns consist of strokes, which are formed as a result of application of the local operator to randomly chosen cells. The steady state of $\aleph_\alpha$ is a coverage of the whole array with patterns such as figures formed of strokes, strips and points against the black background. For example, fancy figures consisting of strokes and points are formed for $p < 0.1429$.

To analyze and to classify all patterns, the following quantitative characteristics are introduced:

- The number of ones N_1 and the number of zeros N_0.
- Convergence C is the number of iterations which is needed for the evolution to become stable.
- The number of connected components for ones L_1 and zeros L_0. A connected component is a maximal subset of cells in the state "one"

("zero"), all having a path to each cell in this subset. Here the path is a sequence of cells in the state "one" ("zero") such that each next cell is neighboring for a previous one.

- Percolation along ones P_1 and zeros P_0 in the vertical and horizontal directions. Percolation is the number of connected components containing two cells belonging to different borders of a cellular array.
- Tortuosity Tor is the number of angles on the borders of the connected components.

Analyzing the evolutions of synchronous CA $\aleph_\sigma$, three modes of behavior can be identified:

Mode 1. Alternation of two geometric figures ($p \in [0, 0.5] \cup (0.5625, 0.8751)$);

Mode 2. Formation of various fancy figures and strips spreading over the whole cellular array ($p \in (0.5, 0.5625] \cup [0.8751, 1.125]$);

Mode 3. Filling the whole array with ones ($p > 1.125$).

For asynchronous CA $\aleph_\alpha$, the following modes of behavior are identified:

Mode 1. Formation of black patterns against the white background, $N_0 > N_1$, $p \in [0, 1]$;

Mode 2. Formation of white patterns against the black background, $N_0 < N_1$, $p \in (1, 1.5]$;

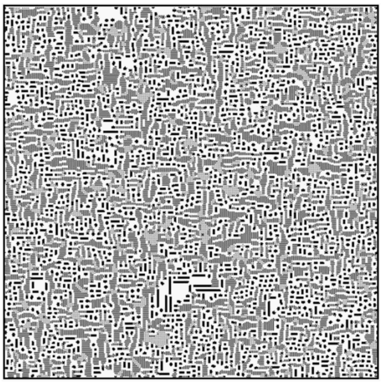
Mode 3. Filling the whole cellular array with ones, $p > 1.5$.

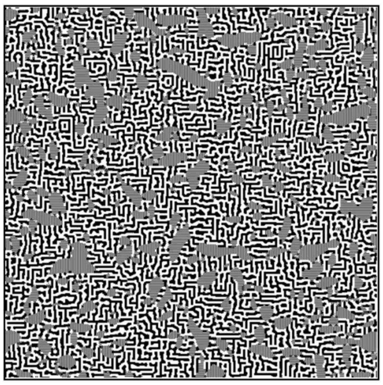
In Table 1, some quantitative characteristics of the stable patterns are presented. For the synchronous CA, characteristics of fancy figures and strips are considered. These patterns are shown in Figures 1b and 3a. For the asynchronous CA, characteristics of patterns formed for $p = 0.6$ and 1 are given. These patterns are shown in Figure 5.

The fancy figures formed by $\aleph_\sigma$ for $p = 0.501$ and the pattern formed by $\aleph_\alpha$ for $p = 0.6$ have similar values of N_1. But other their characteristics are very different. As well as pattern formed by $\aleph_\sigma$ for $p = 0.9$ considerably differs from that formed by $\aleph_\alpha$ for $p = 1$. As a result of comparing characteristics of stable patterns formed by the evolutions of synchronous and asynchronous CA (see Table 1), the following conclusions are made:

Table 1. Quantitative characteristics of the stable patterns formed by the evolution of CAs

CA	Stable patterns	p	N_1	C	P_0	P_1	L_1	L_0	Tor
$\aleph_\sigma$	fancy figures	0.501	76025	293	206	0	40637	25	162117
$\aleph_\sigma$	strips	0.9	116202	524	3	51	105	64	5102
$\aleph_\alpha$	strokes and spots	0.6	76312	80	386	0	13547	108	21035
$\aleph_\alpha$	strips of ones	1.0	121056	78	220	211	1317	582	18568





$p = 0.6$ $p = 1$

Figure 5. Some stable patterns formed by asynchronous CA

1. The convergence rate for an asynchronous CA is less than those for a synchronous one;
2. The number of connected components and the tortuosity for fancy figures are greater than those for other patterns;
3. The percolation and the number of connected components for strips formed by an asynchronous CA are greater than those for a synchronous one.

All the above-considered stable patterns are formed of one nucleation cell. More than one nucleation cell allows us to create patterns that are absolutely different from the above-considered patterns. Depending on a distance between nucleation cells, various patterns are formed. Computer experiments have shown that nucleation cells, located at any distance, affect the formation of patterns spreading over the whole array (fancy figures, stripes formed by $\aleph_\sigma$ and all patterns formed by $\aleph_\alpha$).

For example, fancy figures formed from two nucleation cells located at the distance along a horizontal axis $S_x = 100$ and along a vertical axis $S_y = 100$ are shown in Figure 6a. The neighboring nucleation cells affect the geometric figures formation (Mode 1 of $\aleph_\sigma$) if the distance between them is less than the template size ($S \leq 7$). Therefore, more complex patterns are formed.

In Figure 6b and 6c, the stable patterns formed for $p = 0.45$ of $\aleph_\sigma$ of the two neighboring nucleation cells are shown. In Figure 6b, the stable state of the CA evolution for the nucleation cells situated at $S_x = 2$ and $S_y = 3$ is given. Alternation of two CA states on even and odd iterations is observed. In Figure 6c, alternation of geometric figures, formed of nucleation cells located at the distance along a horizontal axis $S_x = 3$ and along a vertical axis $S_y = 4$, is presented.

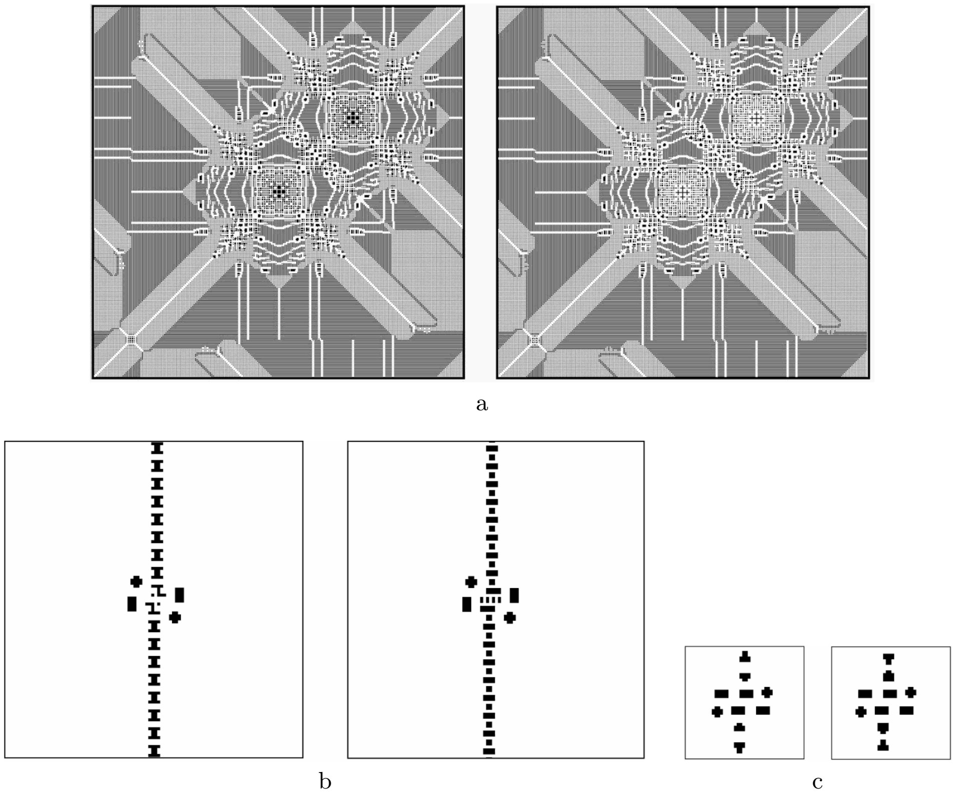


Figure 6. Stable patterns formed of the two neighboring nucleation cells located at the distance a) $S_x = 100$ and $S_y = 100$ for $p = 0.501$, b) $S_x = 2$ and $S_y = 3$ for $p = 0.45$, and c) $S_x = 3$ and $S_y = 4$ for $p = 0.45$

5. Conclusion

In this paper, the evolution of the totalistic CA with weighted templates for one nucleation cell in the initial condition is investigated. The variety of stable patterns such as fancy figures, spots, strips, diamonds, crosses and other geometric figures is obtained by computer experiments. The dependence of stable patterns on values of weight matrix coefficients is studied. It has been revealed and proved that for the fixed array size and initial state, the pattern formation is uniquely determined by the ratio $\frac{p}{n}$.

To analyze stable patterns, the quantitative characteristics such as convergence, the number of connected components, the percolation and the tortuosity were calculated for various values of p. As a result of studying evolutions of $\aleph$, three modes of behavior for $\aleph_\sigma$ and $\aleph_\alpha$ have been defined.

The stable patterns formed by $\aleph_\sigma$ and $\aleph_\alpha$ essentially differ. A distinguishing feature of stable patterns formed by the asynchronous mode is the pattern spreading over the whole cellular array for any value of p. Whereas in terms of the synchronous mode, for the majority of values p, formation of bounded geometric figures is observed.

The stable patterns formation for two neighboring nucleation cells is studied as well. It is obtained that stable patterns formed of two nucleation cells are essentially different from those formed of one cell. The dependence of the nucleation cells location on the patterns formation is investigated.

Studying patterns formed by the evolution of CA with weighted templates is useful for understanding the basic mechanisms of the formation of spatial structures in complex non-equilibrium systems.

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