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Mathematical modeling of 3D non-stationary electromagnetic fields using the vector finite element method*

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This paper is dealt with investigation of the numerical aspects concerned with using the vector finite element method for solving non-stationary electromagnetic problems. A special variational formulation and its discrete analogues are offered. Peculiarities of inputting a source current into such statements are considered. The results of some numerical experiments are presented.

1. Introduction

In the last decades, advent of a more powerful computer hardware combined with development of the numerical techniques enables carrying out complicated three-dimensional (3D) electromagnetic field computations.

The finite element method (FEM) is widespread and regarded as one of the most powerful numerical schemes. With the classical FEM, the domain of solution is subdivided into a finite number of elements, and scalar continuous trial functions are associated with each of them. The main drawback is that all the electromagnetic quantities can be discontinuous at material interfaces. Therefore, in such cases they can hardly be expanded in the terms of continuous functions [1].

The first way of overcoming this problem is introduction of vector of scalar potentials. However, the potential-based statements are related to the loss of accuracy due to the required spatial differentiation of computed potentials [1–3]. The second way is based on using the vector-shaped functions for which continuity of only one (tangential or normal) component is preserved between the adjacent elements, allowing for the discontinuity of a normal or a tangential component of the calculated field, respectively. Such a family of finite elements which use the vector basis functions was introduced in [4]. Afterwards, this theory was generalized in the vector finite element method (VFEM). Note that the wide-spread term "edge elements" coined for the vector finite elements of lowest order is explained by the form of degrees of freedom associated with edges of geometrical elements [1].

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The advantages of edge elements for solving the time-harmonic electromagnetic problems were mentioned in a number of works [5, 6]. For the time-dependent cases, this question has not been answered yet. In [7], a list of criteria to be satisfied by any numerical method for solving an electromagnetic problem in the time domain is presented:

- "good" dispersion properties,
- handling of discontinuous electric and magnetic coefficients,
- prevention of resolution of a linear system at each time step.

The edge elements completely correspond to the first and the second requirements. Unfortunately, an edge element mass matrix is not diagonal, and there is no convenient lumping technique. Nevertheless, as we will show in the sequel the computational costs of resolution of a linear system obtained from the edge element discretization of a 3D time-dependent problem are lower in comparison with a classical scalar FEM. This allows us to use the VFEM-approximations for the numerical modeling of the the 3D non-stationary electromagnetic fields.

The most prevailing numerical scheme for solving time-dependent electromagnetic problems is the explicit finite difference time-domain method (FDTD) [2]. This method is proved to be a highly efficient technique [8]. The time discretization schemes for the VFEM have also been implemented in both implicit [9] and explicit [10] versions. Note that most of implicit schemes are conditionally stable and the stability requires time steps which can be even smaller than those required for the explicit FDTD method [8]. Fortunately, unconditionally stable vector finite element implicit Newmark–Beta scheme was proposed in [8].

In this paper, we introduce a special variational formulation and construct its discrete analogue for solving a 3D electromagnetic non-stationary problem in inhomogeneous medium. The specialities of realization of the VFEM concerned with inputting a current source is investigated. The numerical properties of the Newmark–Beta scheme are analyzed. We also represent the results of a number of numerical experiments.

2. Statement of problem

A general form of Maxwell's equations in the charge-free region, where the electromagnetic quantities are smooth, is the following:

$$\nabla \times \mu^{-1}\boldsymbol{H} + \partial_t \boldsymbol{E} = 0, \tag{1}$$

$$\nabla \times \boldsymbol{H} + \partial_t(\varepsilon \boldsymbol{E}) = \boldsymbol{J}, \tag{2}$$

$$\nabla \cdot (\mu^{-1}\boldsymbol{H}) = 0, \quad \nabla \cdot (\varepsilon \boldsymbol{E}) = 0. \tag{3}$$

Here $\boldsymbol{E}$ is the electric field intensity, $\boldsymbol{H}$ is the magnetic field intensity, $\boldsymbol{J}$ is the electric current density:

$$\boldsymbol{J} = \boldsymbol{J}_0 + \sigma \boldsymbol{E}, \tag{4}$$

where $\boldsymbol{J}_0$ is the source electric current density.

The domain $\Omega \subset R^3$ is assumed to be simply-connected Lipschitz polyhedral domain with the boundary $\partial\Omega$. Ω may be inhomogeneous, consisting of several dielectric, magnetic and metallic regions Ω_i. Let us denote by Γ an arbitrary interface between the subdomains Ω_i. Across the interface Γ the conservation laws imply the following conditions:

$$(\boldsymbol{E}_1 - \boldsymbol{E}_2) \times \boldsymbol{n} = 0, \quad (\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \boldsymbol{n} = \boldsymbol{J}_\Gamma,$$
$$(\mu_1^{-1}\boldsymbol{H}_1 - \mu_2^{-1}\boldsymbol{H}_2) \cdot \boldsymbol{n} = 0, \quad (\varepsilon_1 \boldsymbol{E}_1 - \varepsilon_2 \boldsymbol{E}_2) \cdot \boldsymbol{n} = 0,$$

where $\boldsymbol{J}_\Gamma$ is the surface current density, $\boldsymbol{n}$ is the outward normal to Γ. This system is also supplemented by the following initials:

$$\boldsymbol{H}|_{t=0} = \boldsymbol{H}_{\mathrm{ic}}, \quad \boldsymbol{E}|_{t=0} = \boldsymbol{E}_{\mathrm{ic}},$$

and the boundary conditions

$$\boldsymbol{H} \times \boldsymbol{n} = 0, \quad \boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega.$$

We consider here the constrained second order problem which is obtained by elimination of the magnetic field intensity from (1)–(4) and introduction of the Lagrange multipliers p to impose the second constraint in (3)

$$\varepsilon\, \partial_{tt}\boldsymbol{E} + \sigma\, \partial_t \boldsymbol{E} + \nabla \times (\mu^{-1} \nabla \times \boldsymbol{E}) - \varepsilon \nabla p = -\partial_t \boldsymbol{J}_0, \tag{5}$$
$$\nabla \cdot (\varepsilon \boldsymbol{E}) = 0 \tag{6}$$

with the appropriate conditions at the interface Γ

$$(\boldsymbol{E}_1 - \boldsymbol{E}_2) \times \boldsymbol{n} = 0, \quad (\varepsilon_1 \boldsymbol{E}_1 - \varepsilon_2 \boldsymbol{E}_2) \cdot \boldsymbol{n} = 0, \tag{7}$$
$$p_1 - p_2 = 0, \quad (\mu_1^{-1} \nabla \times \boldsymbol{E}_1 - \mu_2^{-1} \nabla \times \boldsymbol{E}_2) \times \boldsymbol{n} = \partial_t \boldsymbol{J}_\Gamma, \tag{8}$$

the boundary conditions

$$p = 0, \quad \boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{9}$$

and the initial conditions

$$\boldsymbol{E}|_{t=0} = \boldsymbol{E}^0, \quad \partial_t \boldsymbol{E}|_{t=0} = \boldsymbol{E}^1. \tag{10}$$

3. Variational problem and discretization

We consider the functional space

$$H(\mathrm{rot};\Omega) = \{\boldsymbol{v} \in L^2(\Omega)^3 \mid \nabla \times \boldsymbol{v} \in L^2(\Omega)^3\}$$

equipped with the norm

$$\|\boldsymbol{u}\|^2_{H(\mathrm{rot};\Omega)} = \|\boldsymbol{u}\|^2_{0;\Omega} + \|\nabla \times \boldsymbol{u}\|^2_{0;\Omega}.$$

Here we write down $\|\cdot\|_{0;D}$ for the $L^2(D)$-norm and $(\cdot,\cdot)_{0;D}$ for the $L^2(D)$ inner product. We also denote by $H_0(\mathrm{rot};\Omega)$ the subspace of $H(\mathrm{rot};\Omega)$ with $\boldsymbol{u} \times \boldsymbol{n}$ vanishing tangential trace on $\partial\Omega$.

The mixed variational formulation of the problem (5)–(10) is set by

For given $\boldsymbol{J}_0 \in L^2(\Omega)^3$ find $\boldsymbol{E} \in H_0(\mathrm{rot};\Omega)$, $p \in H^1_0(\Omega)$ such as for any $F \in H_0(\mathrm{rot};\Omega)$, $q \in H^1_0(\Omega)$:

$$\begin{aligned}\partial_t^2(\varepsilon\boldsymbol{E},\boldsymbol{F})_{0;\Omega} + \partial_t(\sigma\boldsymbol{E},\boldsymbol{F})_{0;\Omega} + (\mu^{-1}\nabla\times\boldsymbol{E},\nabla\times\boldsymbol{F})_{0;\Omega} -{}&\\ (\varepsilon\nabla p,\boldsymbol{F})_{0;\Omega} + ((\mu^{-1}\nabla\times\boldsymbol{E})\times\boldsymbol{n},\boldsymbol{F})_{0;\partial\Omega} &= -(\partial_t\boldsymbol{J}_0,\boldsymbol{F})_{0;\Omega}, \quad (11)\\ -(\varepsilon\boldsymbol{E},\nabla q)_{0;\Omega} &= 0. \quad (12)\end{aligned}$$

The pair of functional spaces $(H(\mathrm{rot};\Omega),\ H^1_0(\Omega))$ is chosen according to the Ladyzhenskaya–Babuškha–Brezzi (LBB) constraint [11].

For the spatial discretization of (11)–(12) we construct the following approximations for $\boldsymbol{E}$ and p:

$$\begin{aligned}\boldsymbol{E}_h &= \sum_{i=1}^{N_E} e_i\boldsymbol{F}_i \in W^h = \mathrm{span}\{\boldsymbol{F}_1,\boldsymbol{F}_2,\ldots,\boldsymbol{F}_{N_E}\} \subset H_0(\mathrm{rot};\Omega),\\ p_h &= \sum_{i=1}^{N_N} p_i q_i \in V^h = \mathrm{span}\{q_1,\, q_2,\ldots,\, q_{N_N}\} \subset H^1_0(\Omega),\end{aligned}$$

where W^h is a discrete space with the basis of Nédélec's vector shape functions [4], V^h is a discrete space with the basis of a standard scalar first order shape function, N_E and N_N being the number of edges and nodes of the finite element grid, respectively.

The above-said brings about to the matrix system of ordinary differential equations:

$$\begin{aligned}M_\varepsilon\partial_t^2\boldsymbol{e} + M_\sigma\partial_t\boldsymbol{e} + G\boldsymbol{e} - C\boldsymbol{p} &= \partial_t\boldsymbol{j}, \quad (13)\\ -C^T\boldsymbol{e} &= 0, \quad (14)\end{aligned}$$

where $\boldsymbol{e} = [e_1,e_2,\ldots,e_{N_e}]^T$, $\boldsymbol{p} = [p_1,p_2,\ldots,p_{N_N}]^T$, $\boldsymbol{j}$ is the discretization of the right-hand side in (11), and the matrices are given by

$$M_{\varepsilon,ij} = (\varepsilon \boldsymbol{F}_i, \boldsymbol{F}_j)_{0;\Omega}, \qquad M_{\sigma,ij} = (\sigma \boldsymbol{F}_i, \boldsymbol{F}_j)_{0;\Omega},$$
$$G_{ij} = (\mu^{-1}\nabla \times \boldsymbol{F}_i, \nabla \times \boldsymbol{F}_j)_{0;\Omega} + ((\mu^{-1}\nabla \times \boldsymbol{F}_i) \times \boldsymbol{n}, \boldsymbol{F}_j)_{0;\partial\Omega},$$
$$C_{ij} = (\varepsilon \nabla q_i, \boldsymbol{F}_j)_{0;\Omega}.$$

Using the Newmark–Beta approximation at each time step k, we obtain the following discrete-time analogue to problem (13), (14):

$$\begin{aligned} &M_\varepsilon(\Delta t)^{-2}(\boldsymbol{e}^k - 2\boldsymbol{e}^{k-1} + \boldsymbol{e}^{k-2}) + M_\sigma(\Delta t)^{-1}(\boldsymbol{e}^k - \boldsymbol{e}^{k-1}) + \\ &\quad G(\beta \boldsymbol{e}^k + (1-2\beta)\boldsymbol{e}^{k-1} + \beta \boldsymbol{e}^{k-2}) - C(\beta \boldsymbol{p}^k + (1-2\beta)\boldsymbol{p}^{k-1} + \beta \boldsymbol{p}^{k-2}) \\ &\quad = (\Delta t)^{-1}(\boldsymbol{j}^k - \boldsymbol{j}^{k-1}), \qquad -C^T \boldsymbol{e}^k = 0, \end{aligned}$$

where $\boldsymbol{e}^k$, $\boldsymbol{p}^k$, $\boldsymbol{j}^k$ are the discrete-time representation of $\boldsymbol{e}$, $\boldsymbol{p}$, $\boldsymbol{j}$ at the time step k, namely, $\boldsymbol{e}^k = \boldsymbol{e}(k\Delta t)$.

It is shown in [8] that the VFEM–Newmark–Beta approximation is unconditionally stable for $\beta \geq 1/4$. In this paper, we investigate this numerical scheme with the object of finding the optimal parameter β.

4. The features of implementation

One of the difficulties arising in the edge finite element analysis of electromagnetic problems is the necessity of inputting source current in a manner satisfying the solenoidal condition $\nabla \cdot \boldsymbol{J}_0 = 0$. This constraint strongly influences the convergence of the iterative process for solving systems of linear algebraic equations (SLAE) generated during the edge finite element analysis [12]. This problem is rather hard to solve on account of properties of the vector finite elements, an appropriate mesh structure and the form of vector shape functions.

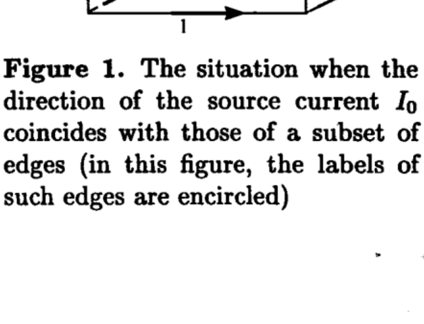

Figure 1. The situation when the direction of the source current I_0 coincides with those of a subset of edges (in this figure, the labels of such edges are encircled)

Consider a simple situation, when the direction of a source current coincides with the directions of some edges of the finite element grid (Figure 1). In this case, the value of the source current must be uniformly distributed on a closed-loop subset of edges. Then the module of the current source density at these edges can be evaluated in the following form:

$$|\boldsymbol{J}_{0\,\mathrm{edge}}| = \frac{|I_0|}{|S| n_e},$$

where $|I_0|$ is the source current module, $|S|$ is the area of the conductor cross-section, n_e is the number of the edges coinciding with the direction of a source current. For example, if $|I_0| = 1$ A, and we consider the case, shown in Figure 1, the source current density for each of the edges with numbers 5, 6, 7, 8 will be $|J_{0\,\mathrm{edge}}| = (0.25\ \mathrm{A})/|\mathrm{S}|$.

Note that for the case, when the direction of the source current does not coincide with any of edges of the grid, it is not possible to obtain exact representation of the current. The algorithm of inputting the source current in this situation is very complicated and leads to unjustified outlay. Therefore, the way out is in constructing a finite element grid whose edges are known to coincide with source current direction.

As we mentioned before, the VFEM requires the total revision of the classical FEM technique. In this connection, we will dwell on the structure of the program complex which realizes the VFEM. This program complex consists of the following modules:

- the builder of the finite element meshes oriented to the VFEM,
- the generator of the matrix portraits in some sparse format,
- the module, responsible for evaluating and assembling the VFEM matrices,
- the iterative solver of SLAEs,
- the post-processor which prepares the data for interpretation and visualization.

The VFEM imposes strict demands for the quality of a grid. Moreover, if there is an electric current source in the solution domain, the mesh generator has to properly take into account its configuration. The outcome of the mesh generator module is mesh data oriented to the edge finite elements, namely, those containing information on the edge orientation and linkage.

The structure of the global VFEM SLAE's is highly sparse in comparison with a classical FEM, therefore the special matrix portrait generator is required for efficient solution of the large-scale problems. The evaluation and assembling of local VFEM matrices into the global ones do not essentially differ from this step in the FEM analysis except for the procedure of imposing the boundary conditions, which are associated with tangential components of the sought for fields.

Note that in this work we use the conjugate gradient iterative solver (CG), which is proved to be an efficient tool in the VFEM analysis [3].

It is obligatory that a program complex oriented to the VFEM should include a post-processor. This is concerned with the form of degrees of freedom which are not the values of the sought for fields at mesh points as in the classical FEM, but the values of tangential components at each edge. The majority of the available visualization tools deal with the nodal-oriented data. Consequently, it is necessary to construct a procedure for the translation of the edge-oriented data to the generally accepted format.

5. Numerical experiments

In this section, the results of investigation of properties of the proposed numerical schemes are presented.

To validate the accuracy of the Newmark–Beta scheme, we solve the model problem for which a smooth analytical solution $\boldsymbol{u}_0(x, y, z, t)$ is known. The solution domain is cube with the lateral length of 0.2 m. The mesh has the cell size $h = 10^{-2}$ m, the number of unknowns being 26560. The time step is fixed $\Delta t = 2.5 \cdot 10^{-6}$ s. The computations are performed over the time period $t = 0$ to 0.25 ms. For the solution to the resulting SLAE, the iterative method CG with precision 10^{-10} is used. The average number of CG iterations at each time step is 64. As expected, the solution error is increased in time. In Figure 2, the behavior of $H(\text{rot}; \Omega)$-norm of the difference δ between analytical and computed solutions for a family of parameters β is shown. The performed investigations show that the choice of the parameter $\beta = 1/4$ minimizes the solution error and dissipation of the scheme.

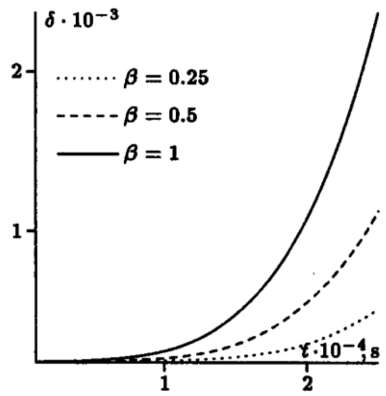

Figure 2. The behavior of the solution error of the Newmark–Beta scheme for different β

Characteristics of the proposed numerical scheme in view of solution of the electromagnetic problems in the domains with discontinuous magnetic and electric coefficients are investigated. The model problem of simulation of the electromagnetic field caused by the coil in vacuum (the diameter is 0.025 m), which is located above the steel plate (the thickness is 0.02 m, $\sigma = 1.3 \cdot 10^{-3}$ Sm/m, $\mu = 50\mu_0$, $\varepsilon = 0$). There is a transparent hole with the square cross-section (the lateral length is 0.02 m) in the plate. The amplitude of the source current is $|I_0| = 1$ A. At the time $t_0 = 3 \cdot 10^{-5}$ s the source current is switched off. The computations are performed over the time period $t = 0$ to 6 ms. The solution domain is the cube, the number of unknowns is 270641. The times step is $\Delta t = 3 \cdot 10^{-5}$ s. We use the iterative method CG with precision 10^{-10}, the average number of CG iterations at each time step being 208.

The classical FEM-approximation for solving such a problem requires, on average, 2600 CG iterations for the dimensionality of the SLAE 270763. In addition, the SLAE, generated from the classical FEM has a noticeably more

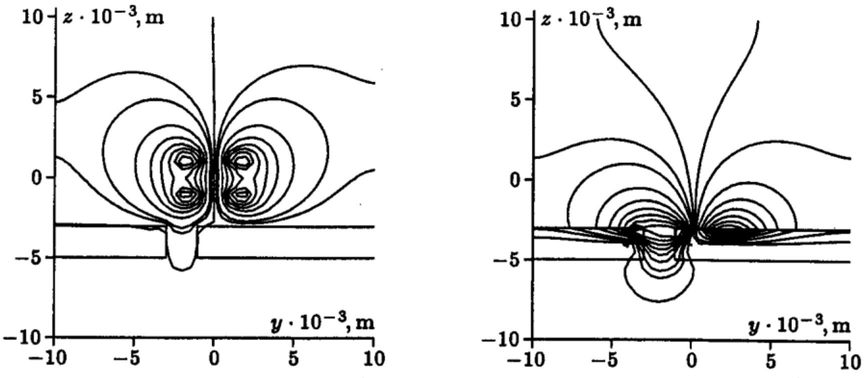

Figure 3. The field component E_y for $t = 0.51$ ms (left) and 0.9 ms (right)

denser matrix structure, i.e., imposes higher demands for computer memory. This confirms the efficiency of the VFEM for solving 3D electromagnetic problems. In Figure 3 the distribution of the field component E_y in the cross-section passing through the middle of the plate hole at the times $t = 0.51$ and 0.9 ms is presented.

Conclusion

The investigation of the VFEM approximation for solving 3D electromagnetic problems in the inhomogeneous media was performed. The special vector mixed variational formulations oriented to the VFEM were constructed. The peculiarities of inputting the source current for the VFEM approximations and the program complex structure were analyzed.

A number of numerical experiments were carried out. It was experimentally shown that choosing the parameter $\beta = 1/4$ in the Newmark-Beta method minimizes the solution error. The efficiency of the VFEM schemes in comparison with the classical FEM was shown.

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