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On numerical implementation of the vector finite element method for electromagnetic problems*

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In this work we investigate some features of numerical implementation of the vector finite element method of lower orders for different types of elements. The comparison of data structures, computer memory requirements and application of iterative solvers for nodal and vector finite element approximations are presented.

1. Introduction

Today, the finite element method [1, 2] is the most important and widespread variational method. This method was successfully applied to analysis of various problems in electromagnetics including that of electrostatic and magnetostatic fields, eddy-current problems, high-frequency problems, electromagnetic scattering and waveguide propagation in the last thirty years.

The finite element method with degrees of freedom, associated with nodes of a previously generated mesh, has shown itself rather successfully in solution to electromagnetic problems involving scalar variables in the two- or the three-dimensional domains. Shortcomings of the above technique was discovered for problems involving vector field variables. Among them, the occurrence of non-physical or spurious solutions, inconvenience of imposing boundary conditions at material interfaces and conducting surfaces, difficulties in treating the conducting and dielectric edges and angle due to the field singularities associated with these structures [3, 4].

A new approach has been recently discovered. The vector finite elements were for the first time described by Whitney [5] forty years ago as a family of difference forms. Their importance and usefulness were not realized until recently with the work by Nédélec [6, 7] in the early 1980s. Afterwards, many authors contributed to the finite element analysis of electromagnetic problems [4, 8, 9]. In all these works, the vector finite element method has been shown to be free of all previously mentioned shortcomings.

Problems, such as long computational time and demand for a larger computer memory are very common in the today's finite element analysis due to

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increased interest to tackling high-scale, more complex and higher-dimensional problems. The vector finite elements are very promising in the 3D electromagnetic field computation for the aforementioned problems. Program codes based on the vector finite element method are often much faster and less "memory hungry" than the program codes based on ordinary nodal finite elements. In addition, the vector finite elements satisfy the continuity of only tangential or normal field components across the interfaces between two adjacent finite elements. This property is advantageous for electromagnetic field computation, because this approximation does not require any additional constraints on approximated fields apart from those prescribed from the nature of the field itself.

Although the vector finite elements are preferable for analysis of a vector field problem, they also exhibit several problems owing to their properties or numerical implementation. These problems were mentioned by many authors [3, 10], and among them an increase in degrees of freedom – in comparison with nodal approximations, requirement of postprocessor. Moreover, the numerical technique developed for the nodal-oriented finite element method was found practically inapplicable for the vector finite element analysis. These problems require further investigation and solution.

In this paper, some features of numerical implementation of vector finite element method of lower order for electromagnetic calculations are investigated. The comparison of data and matrix structures, application of an iterative solver of global systems of linear algebraic equations (SLAG), dimensionality and the machine memory requirements for nodal and vector finite element approximations is made.

2. Model problem and vector variational formulation

Let Ω be a bounded Lipschitz polyhedron in R^3 with a connected boundary $\partial\Omega$. The model problem to be considered is to compute a time-harmonic electric field **E** in a cavity Ω with a perfectly conducting boundary. Let ω denote a temporal frequency of the time-harmonic field, so that the corresponding time-dependent field $\mathcal{E}(\boldsymbol{x},t)$ at the position $\boldsymbol{x} \in \Omega$ and the time t be given by

$$\mathcal{E}(\boldsymbol{x},t) = \Re(\mathcal{E}(\boldsymbol{x})\exp(-i\omega t)).$$

Then $\boldsymbol{E}$ satisfies the Maxwell system

$$\nabla \times \mu_r^{-1} \nabla \times \boldsymbol{E} - k^2 \varepsilon_r \boldsymbol{E} = \mathcal{J} \text{ in } \Omega, \tag{1}$$

where μ_r is a relative magnetic permeability and ε_r is a relative electric permittivity in Ω. We assume μ_r and ε_r to be real, positive and piecewise constant functions of the position $\boldsymbol{x}$. In addition, we denote as $k = \omega\sqrt{\varepsilon_0\mu_0}$ a real valued wave number, where μ_0 and ε_0 are the magnetic permeability and the electric permittivity of free space, respectively. The source function $\mathcal{J}$ is related to the applied current density driving the cavity. The assumption that Ω has a perfectly conducting boundary gives the following boundary condition:

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \text{ on } \partial\Omega, \tag{2}$$

where $\boldsymbol{n}$ denotes the outward normal unit vector to $\partial\Omega$.

Throughout the paper we will assume that k^2 is not an interior Maxwell eigenvalue.

The week formulation of problem (1) requires introduction of the Hilbert space $H(\mathbf{rot};\Omega)$, defined by

$$H(\mathbf{rot};\Omega) = \{\boldsymbol{u} \in L^2(\Omega) \mid \nabla \times \boldsymbol{u} \in (L^2(\Omega))^3\}.$$

and equipped with the following inner product and graph norm

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{rot}} := \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, d\Omega + \int_\Omega \nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v}\, d\Omega, \quad \|\boldsymbol{u}\|^2_{\mathbf{rot}} := (\boldsymbol{u}, \boldsymbol{u})_{\mathbf{rot}}.$$

A subspace of vectors in $H(\mathbf{rot};\Omega)$ with a vanishing tangential trace on $\partial\Omega$ is denoted by $H_0(\mathbf{rot};\Omega)$. Then following the usual Galerkin strategy we arrive at the problem of finding $\boldsymbol{E} \in H_0(\mathbf{rot};\boldsymbol{\Omega})$ such that

$$\int_\Omega \mu_r^{-1} \nabla \times \boldsymbol{E} \cdot \nabla \times \boldsymbol{F}\, d\Omega - \int_\Omega k^2 \varepsilon_r \boldsymbol{E} \cdot \boldsymbol{F}\, d\Omega = \int_\Omega \mathcal{J} \cdot \boldsymbol{F}\, d\Omega, \tag{3}$$

for any $\boldsymbol{F} \in H_0(\mathbf{rot};\Omega)$ and given $\mathcal{J} \in (L^2(\Omega))^3$.

3. Mesh generation and data structure

In order that a vector finite element approximation be constructed we first divide the computational domain Ω into non-overlapping finite elements T_i so as $\Omega = \cup_i T_i$. We denote by h_i the tetrahedral diameter or the triangular element T_i. Then we assume this triangulation to be shape-regular if for all T_i there is a positive constant κ such that $h_i/\rho_i < \kappa$, where ρ_i is diameter of the biggest ball contained in T_i.

Let us assume that all the tetrahedral and triangular meshes considered in this work are uniformly shape-regular. This requirement is stipulated for a higher sensibility of the vector finite element approximation for the mesh quality. As will be noted below, directions of a local coordinate system of such elements follow from the directions of edges or faces of the vector finite element, and are usually not perpendicular like in the case of nodal elements. Consequently, for triangles or tetrahedrons which disturb the

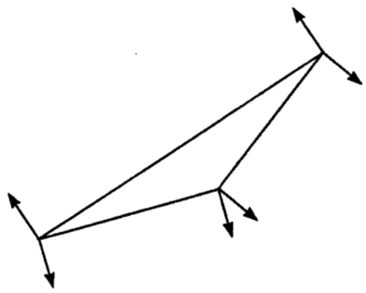

Figure 1. Local coordinate system for a faulty triangle

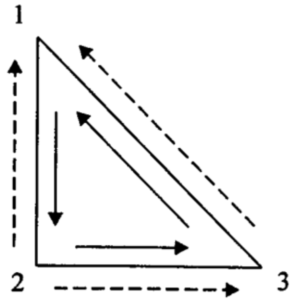

Figure 2. Example of local (solid) and global (dotted) orientation for a triangle vector finite element

shape-regularity a local coordinate system can be found nearly degenerated (Figure 1). These factors impair conditionality of SLAEs associated with such approximations. Thus, faulty finite elements must be avoided when a vector finite element approximation is constructed, or else it results in long computational time and a great loss of accuracy.

A finite element mesh, developed for the nodal finite element analysis, is node-oriented, while a mesh, constructed for the vector finite element analysis must be edge- or face- oriented depending on the type of elements. This dictates a special data structure for storage of non-regular mesh information. In contrast to the nodal finite elements, the mesh data structures constructed for the vector finite element method must contain definitions of edges and their linkage into elements. Moreover, common with a local node numeration for nodal-oriented finite element procedure, there is a local edge orientation for the vector finite elements. This orientation is accepted identical for all finite elements and can differ from the global orientation (Figure 2). It is convenient to take into account this orientation in the mesh data structure. We propose the following data structure oriented to the vector finite element method:

- The list of N nodes n_i of the mesh, prescribed by their spatial coordinates: (x_i, y_i, z_i), $i = 1, \dots, N$;
- The list of N_e edges e_i, defined through numbers of the nodes – beginnings and endings of the edges: (n_j, n_k), $j, k \in \{1, \dots, N\}$;
- The list of N_t finite elements T_i defined through numbers of the edges, contained in this element and their orientation: $(m_j e_j, m_k e_k, m_l e_l)$, $j, k, l \in \{1, \dots, N_e\}$ in the order, where orientation markers m_i are determined as

$$m_i = \begin{cases} 1, & \text{if global and local orientations of the edge are identical,} \\ -1, & \text{otherwise.} \end{cases}$$

4. Discrete problem

The basis functions for the vector finite elements are not scalar but vector functions. In this section, we describe such bases for the vector $H(\mathbf{rot};\Omega)$-conforming finite elements of lower orders and formulate a discrete problem corresponding to the variational formulation (3).

For the parallelepiped and the rectangular finite elements of lower order ($k = 1$), the basis functions arise from a simple tensor-product-based construction. The basis function $\boldsymbol{w}_{e_i}$ corresponding to the edge e_i belongs to

$$P_k^3 = \{\boldsymbol{u} \mid u_x \in Q_{k-1,k,k};\ u_y \in Q_{k,k-1,k};\ u_z \in Q_{k,k,k-1}\}$$

in a 3D case and the space

$$P_k^2 = \{\boldsymbol{u} \mid u_x \in Q_{k-1,k};\ u_y \in Q_{k,k-1,}\}$$

in a 2D case, where $Q_{l,m,n}$ is the space of polynomials of the three variables (x, y, z), whose maximum degrees are, respectively, l in x, m in y, n in z; $Q_{l,m}$ is the space of polynomials of the two variables (x, y) whose maximum degrees are, respectively, l in x, m in y. These functions are selected that the following properties of $H(\mathbf{rot};\Omega)$-conforming finite elements of lower order be satisfied:

1. A tangential component of the shape function at all edges of the element is equal to 0, except the edge, associated with this function (the tangential component is equal to 1 at this edge);
2. Basis functions have zero divergence and non-zero but constant **rot**.

We next introduce a vector basis function for the triangular and the tetrahedral vector basis functions. For a certain edge e_i of a triangular or a tetrahedral element with the nodes n_j and n_k are the beginning and the end of the edge, respectively, we define the basis function

$$\boldsymbol{w}_{e_i} = (\lambda_{n_j}\nabla\lambda_{n_k} - \lambda_{n_k}\nabla\lambda_{n_j})|e_i|,$$

where λ_{n_k} is a nodal shape function (2D or 3D for a triangular or a tetrahedral elements, respectively) associated with the node n_k, $|e_i|$ is a length of the edge e_i. These vector basis functions also satisfy conditions 1, 2.

In [9], it is shown that the basis functions defined in this way are actually the basis functions for vector $H(\mathbf{rot})$-conforming finite elements of lower order. Then we have the following approximation of the field:

$$\boldsymbol{E}^h = \sum_{\{e\}} E_e \boldsymbol{w}_e,$$

where $\{e\}$ is a set of the edge indices of an element; E_e are degrees of freedom and equal to a tangential component of the vector $\boldsymbol{E}$ at the edge e; $\boldsymbol{w}_e$ is the basis function corresponding to this edge. Such an approximation ensures only the continuity of the tangential components of the sought for field due to special properties of the vector $H(\mathbf{rot})$-conforming finite elements – exactly what we need in electromagnetic field computations.

Note that the form of degrees of freedom of the vector $H(\mathbf{rot})$-conforming finite elements of lower order accounts for the prevailing term *edge* whose elements are used for denotation of these elements. In the case of a higher order, moments of tangential components on edges and faces as well as moments over the whole elements occur as degrees of freedom [9].

Using the aforesaid, we formulate a discrete analogue to problem (3):

For given $\mathcal{J}^h \in (L^2(\Omega))^3$ find $\boldsymbol{E}^h \in V^h$ such that for any $\boldsymbol{F}^h \in V^h$

$$\int_\Omega \mu_r^{-1} \nabla \times \boldsymbol{E}^h \cdot \nabla \times \boldsymbol{F}^h \, d\Omega - \int_\Omega k^2 \varepsilon_r \boldsymbol{E}^h \cdot \boldsymbol{F}^h \, d\Omega = \int_\Omega \mathcal{J}^h \cdot \boldsymbol{F}^h \, d\Omega, \quad (4)$$

where V^h is a finite dimensional subspace of $H(\mathbf{rot};\Omega)$.

The procedure of assembling a global SLAE is similar to a standard one for a nodal finite element method. The consideration of boundary conditions (2) is analogous to accounting the Dirichlet boundary conditions for the nodal approximation.

5. Dimensionality and global matrix structure

The most frequently mentioned drawback of the vector finite elements is larger dimensionality of global SLAEs in comparison with the nodal finite element schemes. For the regular tetrahedral domain discretization there are approximately five to six times as many edges as nodes. In the 3D computations, to each node correspond three scalar variables for the nodal finite element approximation. Whereas to each edge corresponds one scalar variable for the vector finite element approximation of lower order. Therefore, dimensionality of global SLAEs for the vector finite element method is approximately twice as large as for a nodal one. It is an unattractive fact, but we notice that using the parallelepiped edge elements does not lead to increasing the number of degrees of freedom. Increasing N dimensionality of global SLAEs for the regular discretization of a cube (n_h is the number of steps of discretization in each axial direction) for nodal, tetrahedral edge and parallelepiped (hexahedral) edge elements is shown in Figure 3.

Nevertheless, even the tetrahedral vector finite elements occur computationally effective in terms of computer memory and time in spite of the larger dimensionality. It is known that the amount of computer work required to solve a SLAE by means of iterative solvers is proportional to the number of non-zero entries in the matrix. The number of non-zero entries

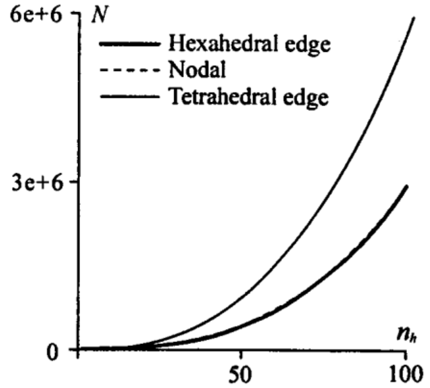

Figure 3. Dimensionality of global SLAEs for different types of elements

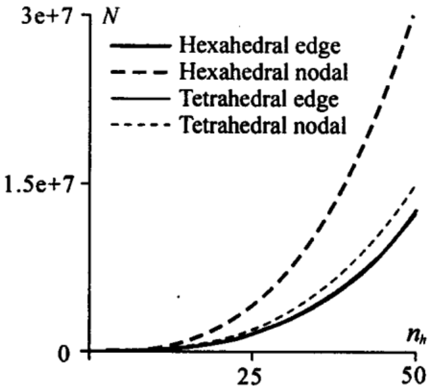

Figure 4. The number of non-zero entries in global SLAE for different types of elements

is crucial for direct methods, too [3]. If we evaluate this number of non-zero entries NN for vector and nodal approximations on the same mesh, we will find that a rigidity matrix for vector approximation is sparser (Figure 4). Therefore it is the comparison in favor of the edge elements. Computational efficiency of using the vector finite elements is also corroborated by numerical experiments.

6. Numerical results

Problems concerned with solving SLAEs generated by the vector finite element method have been already discussed by many authors [3, 11]. Our investigations demonstrate that using a restarted CG for solving SLAEs with symmetrical matrices gives a good effect. The methods BiCG and restarted BiCGStab prove their efficiency for solving SLAEs with non-symmetrical matrices.

In the table, the results of solving problem (4) for $\mu_r = 1$, $\varepsilon_r = 1$ and $k^2 = 1$ on different meshes are given. The problem was solved in the unit cube (n_h is the number of discretization steps in each axial direction) known analytical solution

$$\boldsymbol{E}_0 = \begin{pmatrix} -2\cosh(\pi x)\sinh(\pi y)\sinh(\pi z) \\ \sinh(\pi x)\cosh(\pi y)\sinh(\pi z) \\ \sinh(\pi x)\sinh(\pi y)\cosh(\pi z) \end{pmatrix}$$

which is matched to the boundary conditions and the right-hand side. The precision of solvers is fixed at 10^{-10}. In the table, N_{CG} is the number of CG iterations, N_{rCG} is the number of restarted CG iterations, n are dimensionalities of the SLAEs.

Element type	$n_h = 10$			$n_h = 20$		
	n	N_{CG}	N_{rCG}	n	N_{CG}	N_{rCG}
Tetrahedral edge	6930	276	97	51660	477	289
Tetrahedral node	3993	228	84	27783	1035	—
Parallelepiped edge	3630	241	113	26460	362	163
Parallelepiped node	3993	252	94	27783	1284	—

When the SLAEs of not a large dimensionality are solved, the number of iterations for nodal and vector finite element approximations is comparable, and the restarted CG yields better results than a standard CG for both approximations. But when the dimensionality of SLAEs increases, the number of iterations of a standard CG is notably smaller for the vector finite element method. This confirms the computational effectiveness of vector approximations for solving high-dimensional 3D problems. Moreover, for nodal approximation we failed to find a restart parameter for a restarted CG to improve the convergence of the method in comparison with standard CG.

Investigation of the edge element approximations for solution of the practical electromagnetic problems prove their advantage, too. In this case, if we use the vector approximation, we obtain more physical solutions than for a nodal one. In addition, these approximations are found to be more effective in regard to computer memory and time.

Conclusion

In this work, we have carried out the analysis of features of numerical implementation of the vector finite element method of lower orders. The special data structure for storage of the edge-oriented meshes is proposed. The comparison of the dimensionalities and numbers of non-zero entries in matrices of discrete analogues is presented. This comparison have shown effectiveness of the vector approximations with respect to the memory requirement. Numerical experiments, realized for the model problem, have proved the advantages of such approximations in regard to computer time, too.

Peculiar properties of the vector finite element approximations for solving electromagnetic problems in domains with discontinuous physical parameters call for further analysis.

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