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Solution of two-dimensional Prandtl equations by Monte Carlo method

N.A. Simonov

New numerical method for approximating two-dimensional flow field for the viscous incompressible fluid in the vicinity of the flat boundary is introduced. Using the vorticity formulation of the Prandtl equation we come to the heat equation with nonlinear right-hand side. We consider various boundary value problems for this equation and represent its solution in the sum of three heat potentials. System of nonlinear integral equations for the solution and its derivatives is constructed. Difference approaches to approximating the derivatives in the direction along the boundary lead to the different structures of the system. Randomization of the iterative method of solving this system makes it possible to construct unbiased Monte Carlo estimate, its variance is proved to be finite.

Introduction

Two-dimensional Prandtl equations are considered in this paper. It is well-known [8] that they govern viscous incompressible flows in the boundary layer. In order to simplify our considerations we use its vorticity form thus having the only function w which defines the whole of the flow field. Since the vorticity equation from the mathematical point of view is a parabolic equation with a nonlinear right-hand side, the initial value of w has to be known and on the solid walls a classical boundary value problem has to be set.

In [2] stochastic algorithm for solving the Prandtl boundary layer equations has been proposed by Chorin. Vorticity in this paper has been introduced on the boundary in a way which ensured that zero boundary conditions on components of the velocity field were approximately satisfied at every time step. Many other different techniques for overcoming this difficulty have been employed in various numerical methods. One possible way of solving this problem is the method used by Anderson in [1].

Let the boundary value problem be posed. As a consequence we can consider the governing equation as the heat equation with the nonlinear integral-differential right-hand side. Solution of this problem can be represented as a sum of three potentials. We consider this relation as the equation for the unknown vorticity, obtain the analogous relations for its derivatives and construct the iterative method for solving this system. Next we randomize this iterative method and derive the desired stochastic algorithm.

1. Integral equations

Consider a two-dimensional viscous incompressible flow in the boundary layer. It is well-known [8] that the motion of the fluid near the solid walls is governed by the Prandtl equations. We use them written in terms of vorticity

$$\begin{aligned} w_t &= \nu w_{yy} - u w_x - v w_y, \\ u_x + v_y &= 0. \end{aligned} \tag{1.1}$$

Here u, v are components of the velocity vector and

$$w = -u_y \tag{1.2}$$

is vorticity. The coordinate system is selected so that the flat part of the body surface coincides with the X, Z plane. We suppose that velocity distribution does not depend on z coordinate and that the third component of the velocity is constantly equal to zero. As a consequence vorticity is governed by one scalar function which does not depend on z. So we can consider all functions as defined only in the region $x > 0$, $y > 0$. By the physical reasons we have that velocity of the fluid is equal to zero on the surface of the solid body. In our particular case it means that

$$u = 0, \quad v = 0, \tag{1.3}$$

when $y = 0$, $x > 0$. We suppose that the flow is undisturbed at the infinity, its velocity is constant and parallel to the plane X, Z

$$\lim_{n \to \infty} (u, v) = (u_\infty, 0),$$

and, also,

$$(u, v) = (u_\infty, 0),$$

when $x < 0$.

Relation (1.2) follows from the definition of vorticity and originates in the fact that $v \ll u$ in the boundary layer. Taking into account (1.3) and the equation of continuity for two-dimensional incompressible fluid this equality makes it possible to determine the velocity vector using the following formulas. Let us suppose that the vorticity is already known. Then integrating (1.2) over y from y to infinity we have

$$u(x,y,t) = u_\infty + \int_y^\infty w(x,s,t)ds. \tag{1.4}$$

Next we differentiate this equality with respect to x and then integrate it one more time over y. Then from the equation of continuity we have

$$v(x,y,t) = -\int_0^y ds \int_s^\infty w_x(x,s',t)ds'. \tag{1.5}$$

Suppose now that the initial value of vorticity is known

$$w(x,y,0) = w_0(x,y) \tag{1.6}$$

and that $w_0(x,y)$ is a continuous function of y. We consider (1.1) as a one-dimensional heat equation with a nonlinear right-hand side. From mathematical point of view we have to define some boundary condition at $y=0$ in order to complete formulation of the problem.

One of the possible ways is to use difference approximation of the relation (1.2) in order to define vorticity at $y=0$ [10]. So we have

$$w(x,0,t) = \psi_1(x,t), \tag{1.7}$$

where

$$\psi_1(x,t) = -\frac{u(x,0,t)}{\Delta y} = -\frac{1}{\Delta y}\Big(u_\infty + \int_0^\infty w(x,s,t)ds\Big)$$

with an accuracy of $O(\Delta y)^2$, when Δy tends to zero. Then we can write down the solution of problem (1.1), (1.6), (1.7) in the following form: (see, for example [12])

$$\begin{aligned} w(x,y,t) &= -\frac{2\nu y}{\sqrt{2\pi}} \int_0^t \frac{1}{\sigma^3}\psi_1(x,\tau)\exp\Big(-\frac{y^2}{2\sigma^2}\Big)d\tau \\ &+ \frac{1}{\sqrt{2\pi}\sigma_0}\int_0^\infty w_0(x,s)\Big[\exp\Big(-\frac{y_-^2}{2\sigma_0^2}\Big) - \exp\Big(-\frac{y_+^2}{2\sigma_0^2}\Big)\Big]ds \\ &+ \int_0^t d\tau \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma}\Big[\exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) - \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big] f(x,s,t)ds \\ &\equiv K_0^1\psi_1 + K_2^1 w_0 + K_3^1 f. \end{aligned} \tag{1.8}$$

Here

$$f = -uw_x - vw_y \tag{1.9}$$

is the bilinear functional of w and we denote for brevity $y_+ = y + s$, $y_- = y - s$, $\sigma^2 = 2\nu(t-\tau)$, $\sigma_0^2 = 2\nu t$.

Substituting ψ_1 in (1.8) we come to the integral equation for w

$$w = -K_1^1 w - K_3^1(uw_x) - K_3^1(vw_y) + K_2^1 w_0 - K_0^1(u_\infty(\Delta y)^{-1}), \tag{1.10}$$

which defines vorticity with an accuracy of $C(\Delta y)^2$. Here we denote the integral operator

$$K_1^1 w(x,y,t) = \int_0^t d\tau \int_0^\infty ds \quad \frac{2\nu y}{\sqrt{2\pi\sigma^3}\Delta y} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) w(x,s,\tau).$$

Another, to somewhat extent more productive approach is to pose boundary value problem of the Neumann type. Let us suppose that the first derivative w_y is known at the boundary

$$w_y(x,0,t) = \psi(x,t). \tag{1.11}$$

Then, (see, for example [12]) the solution of problem (1.1), (1.6), (1.11) may be written in the following form:

$$\begin{aligned} w(x,y,t) = &-\frac{2\nu}{\sqrt{2\pi}} \int_0^t \frac{1}{\sigma}\psi(x,\tau)\exp\Big(-\frac{y^2}{2\sigma^2}\Big)d\tau \\ &+ \frac{1}{\sqrt{2\pi}\sigma_0} \int_0^\infty w_0(x,s)\Big[\exp\Big(-\frac{y_-^2}{2\sigma_0^2}\Big) + \exp\Big(-\frac{y_+^2}{2\sigma_0^2}\Big)\Big]ds \\ &+ \int_0^t d\tau \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma}\Big[\exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) + \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big] f(x,s,t)ds. \end{aligned} \tag{1.12}$$

In order to determine the value of ψ we proceed as follows (see [1]). Let $u_0(x,0) = 0$, where

$$u_0(x,y) = u_\infty + \int_y^\infty w_0(x,s)ds.$$

It means that the initial vorticity distribution satisfies zero boundary condition for X component of the velocity. Next we use (1.4) to define $u(x,0,t)$ for $t > 0$ and take the derivative of this expression with respect to time. If we set $\frac{\partial}{\partial t}u(x,0,t)$ equal to zero, then this requirement will ensure that zero boundary condition holds for u for an arbitrary time t. So we have

$$\frac{\partial}{\partial t}\int_0^\infty w(x,s,t)ds = 0, \qquad \int_0^\infty \frac{\partial w}{\partial t}(x,s,t)ds = 0.$$

It will be recalled now that vorticity satisfies the Prandtl equation (1.1). As a consequence the latter equality is equivalent to

$$\int_0^\infty [\nu w_{ss} - uw_x - vw_s](x,s,t)ds = 0$$

and after integrating the first term we have

$$w_y(x,0,t) = \frac{1}{\nu}\int_0^\infty [-uw_x - vw_s](x,s,t)ds. \tag{1.13}$$

Now the desired boundary condition can be written in the following form:

$$\psi(x,t) = \frac{1}{\nu}\int_0^\infty f(x,s,t)ds. \tag{1.14}$$

Thus, after substituting (1.14) in (1.11) we can consider it as the integral equation for w

$$w = K_0 w_0 + K_1(uw_x + vw_y). \tag{1.15}$$

We use the following notations for the integral operators here:

$$\begin{aligned} K_0 w_0(x,y,t) &= \frac{1}{\sqrt{2\pi}\sigma_0}\int_0^\infty w_0(x,s)\Big[\exp\Big(-\frac{y_-^2}{\sigma_0^2}\Big) + \exp\Big(-\frac{y_+^2}{\sigma_0^2}\Big)\Big]ds, \\ K_1 f(x,y,t) &= \int_0^t d\tau \int_0^\infty ds \frac{1}{\sqrt{2\pi}\sigma} f(x,s,\tau) \\ &\quad \times\Big[2\exp\Big(-\frac{y^2}{2\sigma^2}\Big) - \exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) - \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big]. \end{aligned} \tag{1.16}$$

We see now that the right-hand sides of equations (1.10) and (1.15) depend on w, w_x and w_y. So we have to construct closing integral equations for w_x and w_y. In order to do that we proceed as follows.

Let us denote the heat differential operator

$$H = \frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial y^2}$$

and suppose that w is differentiable to the required extent. Then we have from (1.1)

$$H w_y = -(u w_x + v w_y)_y = f_y.$$

Differentiating (1.6) with respect to y we obtain the initial value for w_y

$$w_y(x,y,0) = \frac{\partial}{\partial y} w_0(x,y)$$

and (1.13) determines the boundary value for this function. As a result a mixed problem of the Dirichlet type is posed for w_y and its solution can be written in the following form [12]:

$$\begin{aligned} w_y(x,y,t) = & -\frac{2\nu y}{\sqrt{2\pi}} \int_0^t \frac{1}{\sigma^3} \psi(x,\tau) \exp\Big(-\frac{y^2}{2\sigma^2}\Big) d\tau \\ & + \frac{1}{\sqrt{2\pi}\sigma_0} \int_0^\infty w_{0y}(x,s) \Big[\exp\Big(-\frac{y_-^2}{2\sigma_0^2}\Big) - \exp\Big(-\frac{y_+^2}{2\sigma_0^2}\Big)\Big] ds \qquad (1.17) \\ & + \int_0^t d\tau \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} \Big[\exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) - \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big] f_s(x,s,t) ds. \end{aligned}$$

Next we substitute (1.14) in this equality and integrate the latter term by parts. As a consequence we have the integral equation for w_y

$$w_y = K_0^{(1)} w_{0y} + K_2(u w_x + v w_y), \qquad (1.18)$$

where the integral operators are denoted as follows:

$$\begin{aligned} K_0^{(1)} w_{0y}(x,y,t) &= \frac{1}{\sqrt{2\pi}\sigma_0} \int_0^\infty w_{0y}(x,s) \Big[\exp\Big(-\frac{y_-^2}{\sigma_0^2}\Big) - \exp\Big(-\frac{y_+^2}{\sigma_0^2}\Big)\Big] ds, \\ K_2 f(x,y,t) &= -\int_0^t d\tau \int_0^\infty ds \frac{1}{\sqrt{2\pi}\sigma^3} f(x,s,\tau) \qquad (1.19) \\ & \times \Big[2y \exp\Big(-\frac{y^2}{2\sigma^2}\Big) + y_- \exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) + y_+ \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big]. \end{aligned}$$

It is obvious enough that we can integrate $K_0^{(1)} w_{0y}$ by parts and substitute it by the resulting integral

$$K_0^{(2)} w_0(x,y,t) = \frac{1}{\sqrt{2\pi\sigma_0^3}} \int_0^\infty w_0(x,s) \times \Big[y_- \exp\Big(-\frac{y_-^2}{2\sigma_0^2}\Big) + y_+ \exp\Big(-\frac{y_+^2}{2\sigma_0^2}\Big)\Big] ds.$$

Remark 1. In [10] another presentation of Hw_y was used, the first term of the function f was differentiated and so the second derivative w_{xy} was involved in consideration. In order to pose boundary value problem of the Dirichlet type the expression for Hu was used (see [8]) and equality $w_y = -u_{yy}$. As a consequence boundary values of w_y and w_{xy} have been set equal to zero that lead to the following integral equation:

$$w_y = -K_3^1(u_y w_x + u w_{xy}) + K_4^1(v w_y) + K_2^1 w_{0y},$$

where $K_4^{\prime 1}$ is the integral operator, emerging after integrating K_3^1 by parts

$$K_4^1(f) = \int_0^t d\tau \int_0^\infty ds \frac{1}{\sqrt{2\pi\sigma^3}} f(x,s,t) \times \Big[y_- \exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) + y_+ \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big].$$

Differentiating Hw_y with respect to x we come to the following expression:

$$Hw_{xy} = w_x w_x - u_x w_{xy} - (u w_{xx} + v_x w_y + v w_{xy})_y,$$

which contains w_{xx} in the right-hand side. As a consequence we have the following integral equation for w_{xy}:

$$w_{xy} = -K_3^1(u_{yx} w_x + u_x w_{xy} + u_y w_{xx}) + K_4^1(v_x w_y + v w_{xy}) + K_2^1 w_{0xy}.$$

We pass on to the equation for w_x now. In order to obtain the integral representation of this function we take (1.18) and differentiate it with respect to x. The right-hand side of the resulting equality contains $f_x = -(u w_x + v w_y)_x$ which depends on the second derivatives of w, so our aim is to rearrange it.

Remark 2. In [10] integral equations were constructed not only for function w and its first derivatives w_x, w_y but to the second derivatives w_{xy} and w_{xx} also. So it is obvious enough that the equation for w_x has no need to be rearranged and it was used in its natural form

$$w_x = -K_1^1 w_x - K_3^1(u_x w_x + v_x w_y + v w_{xy} + u w_{xx}).$$

Let us differentiate the continuity equation with respect to y and (1.2) with respect to x. From here we have

$$w_x = v_{yy}.$$

In the boundary layer v satisfies the following differential equation [8]:

$$Hv = -(uv_x + vv_y),$$

and so

$$f_x = Hw_x = Hv_{yy} = -(uv_x + vv_y)_{yy}. \tag{1.20}$$

From here we have

$$w_x = K_0 w_{0x} + K_1[(uv_x + vv_y)_{yy}].$$

The latter integral can be integrated by parts thus leading to the following expression:

$$\begin{aligned} K_1[(uv_x + vv_y)_{yy}] &= \int_0^t d\tau \int_0^\infty ds \frac{1}{\sqrt{2\pi}\sigma^3}(uv_x + vv_s)_s \\ &\quad \times \Big[- y_- \exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) + y_+ \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big] \\ &= \int_0^t d\tau \int_0^\infty ds \frac{1}{\sqrt{2\pi}\sigma^3}(uv_x + vv_s) \\ &\quad \times \Big[\Big(1 - \frac{y_-^2}{\sigma^2}\Big)\exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) + \Big(1 - \frac{y_+^2}{\sigma^2}\Big)\exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big] \\ &\equiv K_3(uv_x + vv_y), \end{aligned}$$

and so (since $v_y = -u_x$)

$$w_x = K_0 w_{0x} + K_3(uv_x - vu_x). \tag{1.21}$$

From (1.5) we see that v_x depends on w_{xx} and so we have to define this function somehow.

In order to do that we can use different approaches. One of them is to use some integral equation of the first kind. For example, we can consider the integral equation for w_x as the equation for w_{xx}

$$K_3^1(uw_{xx}) = -w_x - K_1^1 w_x - K_3^1(u_x w_x + v_x w_y + v w_{xy})$$

and use an iterative method for solving such system of four integral equations of the second kind for the functions w, w_y, w_x, w_{xy} and one integral equation of the first kind for w_{xx}.

Another method of computing w_{xx} was introduced in [10], where some suppositions about the smallness of third derivatives of function w in the boundary layer were introduced. As a consequence the exact formula for Hw_{xx} was simplified in such a way that lead to the closed system of five integral equations of the second kind.

The third approach is to use some difference approximation for w_{xx}. Suppose that w and its derivatives are to be computed at some point (x, y, t). We divide the interval $[0, x]$ into m intervals of the equal length $\Delta x = x/m$ and then substitute different approximation

$$\tilde{w}_{xx}(x, y, t) = \frac{w_x(x, y, t) - w_x(x - \Delta x, y, t)}{\Delta x} \tag{1.22}$$

for w_{xx} in (1.21).

Taking into account this substitution we can rewrite now our system of the integral equations for $\boldsymbol{W} = (w, w_y, w_x)'$ in the following form:

$$\begin{aligned} w &= K_0 w_0 + K_1\big((u_\infty + L_1 w) w_x - w_y L_2(L_1 w_x)\big), \\ w_y &= K_0^{(2)} + K_2\big((u_\infty + L_1 w) w_x - w_y L_2(L_1 w_x)\big), \\ w_x &= K_0 w_{0x} + K_3\Big(- \frac{1}{\Delta x}(u_\infty + L_1 w) \\ &\quad \times L_2\big(L_1(w_x(x, \cdot, \cdot) - w_x(x - \Delta x, \cdot, \cdot))\big) + L_1 w_x L_2(L_1 w_x)\Big), \end{aligned} \tag{1.23}$$

where, bearing in mind (1.4), (1.5), we denote the integral operators L_1 and L_2 as follows:

$$L_1 w(y) = \int_0^\infty w(s)ds, \qquad L_2 w(y) = \int_0^y w(s)ds,$$

$$L_2(L_1 w)(y) = \int_0^y ds \int_s^\infty w(s')ds'.$$

It follows from the third equation in (1.23) that we have to define $\boldsymbol{W}$ at $x = 0$. Since u and v are constant for $x \leq 0$ then we can set $w = w_x = w_y = 0$, when $x = 0$, $y > 0$.

Remark 3. From now and on we consider only the system (1.23) and construct a stochastic method of its solution.

2. Iterative method and its convergence

For brevity we rewrite (1.23) as a single integral equation

$$W = \mathcal{K}(W, W) + F \tag{2.1}$$

and consider the following iterative process:

$$W^{(n+1)} = (1-\alpha)W^{(n)} + \alpha\mathcal{K}(W^{(n)}, W^{(n)}) + \alpha F \equiv PW^{(n)}, \tag{2.2}$$

where $0 < \alpha \leq 1$ is a real parameter.

Suppose now that all the components of the vector function W considered as functions depending only on y lie in the closed subdomain $S(C)$ of the function space $L(\varepsilon)$. We define this space as follows. Integrale function w belongs to $L(\varepsilon)$ if there exists such positive constant ε that the integral

$$J(\varepsilon, x, t) = \int_0^{+\infty} |w(x, y, t)| \exp(\varepsilon y) dy \tag{2.3}$$

is finite for all values of parameters x and t. Supposing that J is bounded uniformly over x and t we can define the norm in $L(\varepsilon)$

$$\|w\|_{(\varepsilon)} = J(\varepsilon, x, t).$$

Note that finitary functions belong to this space with arbitrary ε.

The following definition will be also in use:

$$\|w\|^t = \sup_{0<\tau<t} \|w\|,$$

where the norm in the right-hand side is considered in one of the spaces $L(\varepsilon)$, L_1 or L_∞, and

$$\|W\|_{(\varepsilon)} = \max_i \|W_i\|_{(\varepsilon)},$$

where $W_1 = w$, $W_2 = w_y$, $W_3 = w_x$ in our particular case.

We can define domain $S(C)$ now as the set of such functions w that

$$\|w\|^t_{(\varepsilon)} \leq C$$

uniformly over x for one positive constant C.

Let w_0, w_{0y}, w_{0x} lie in $S(C/3)$. Then the direct evaluation gives us the upper bound for the norm of the free term components of equation (2.1)

$$\begin{aligned}
\|\boldsymbol{F}_i\|_{(\varepsilon)} \le & \int_0^{+\infty} ds|\boldsymbol{W}_{0i}(x,s)| \int_0^{+\infty} dy \frac{\exp(\varepsilon y)}{\sqrt{2\pi}\sigma_0} \\
& \times \Big[\exp\Big(-\frac{y_-^2}{2\sigma_0^2}\Big) + \exp\Big(-\frac{y_+^2}{2\sigma_0^2}\Big)\Big] \\
= & \exp(\varepsilon^2 \nu t) \int_0^{+\infty} |\boldsymbol{W}_{0i}(x,s)| \Big[\exp(\varepsilon s)\Phi\Big(\frac{s+\sigma_0^2\varepsilon}{\sigma_0}\Big) \\
& + \exp(-\varepsilon s)\Phi\Big(\frac{-s+\sigma_0^2\varepsilon}{\sigma_0}\Big)\Big] \\
\le & \exp(\varepsilon^2\nu t)\Big[\|\boldsymbol{W}_{0i}\|_{(\varepsilon)} + \|\boldsymbol{W}_{0i}\|_{L_1}\Phi(\sigma_0\varepsilon)\Big] \le \frac{2}{3}C,
\end{aligned} \tag{2.4}$$

where the latter inequality can be made true by the corresponding choice of ε for particular t. Here $\Phi(x)$ is the distribution function of the standard normal distribution.

Next we evaluate components of the term $\mathcal{K}$ depending on $\boldsymbol{W}$. Our objective is to prove that the operator P maps the set of vector-functions $\boldsymbol{W}$ with components in $S(C)$ into itself.

In order to do that we integrate values of the integral operators K_1, K_2, K_3 multiplied by $\exp(\varepsilon y)$ over y from zero to infinity. So we have (in analogy with (2.4))

$$\begin{aligned}
\|K_1(uw_x + vw_y)\|_{(\varepsilon)} \le & \int_0^t d\tau \int_0^{+\infty} ds|uw_x + vw_s|(x,s,\tau)\exp(\varepsilon^2\nu(t-\tau)) \times \\
& \Big[2\Phi(\varepsilon\sigma) + \exp(\varepsilon s)\Phi\Big(\varepsilon\sigma + \frac{s}{\sigma}\Big) + \exp(-\varepsilon s)\Phi\Big(\varepsilon\sigma - \frac{s}{\sigma}\Big)\Big] \\
\le & \big(\|u\|^t_{L_\infty}\|w_x\|^t_{(\varepsilon)} + \|v\|^t_{L_\infty}\|w_y\|^t_{(\varepsilon)}\big)\frac{4}{\varepsilon^2\nu}\big(\exp(\varepsilon^2\nu t) - 1\big),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
\|K_2(uw_x + vw_y)\|_{(\varepsilon)} \le & \int_0^t d\tau \int_0^{+\infty} ds|uw_x + vw_s|(x,s,\tau) \times \\
& \Big[\frac{2}{\sqrt{2\pi}\sigma}(1 + \exp(\varepsilon s)) + \varepsilon\exp(\varepsilon^2\nu(t-\tau))\Big(2\Phi(\varepsilon\sigma) + \\
& \exp(\varepsilon s)\Big(2\Phi(\varepsilon\sigma) - \Phi\Big(\varepsilon\sigma + \frac{s}{\sigma}\Big)\Big) + \exp(-\varepsilon s)\Phi\Big(\varepsilon\sigma - \frac{s}{\sigma}\Big)\Big)\Big] \\
\le & 4\Big[\Big(\frac{t}{\pi\nu}\Big)^{1/2} + \frac{\exp(\nu\varepsilon^2 t) - 1}{\nu\varepsilon}\Big]\big(\|u\|^t_{L_\infty}\|w_x\|^t_{(\varepsilon)} + \|v\|^t_{L_\infty}\|w_y\|^t_{(\varepsilon)}\big),
\end{aligned} \tag{2.6}$$

$$\|K_3(u\tilde{v}_x - vu_x)\|_{(\varepsilon)} = \int\limits_0^{+\infty} dy \exp(\varepsilon y)\Big|\int\limits_0^t d\tau \int\limits_0^\infty ds \frac{1}{\sqrt{2\pi\sigma^3}} \times$$
$$\Big[-y_- \exp\Big(-\frac{y_-^2}{2\sigma^2}\Big) + y_+ \exp\Big(-\frac{y_+^2}{2\sigma^2}\Big)\Big](u\tilde{v}_x + vv_s)_s\Big|$$
$$\le \int\limits_0^t d\tau \int\limits_0^\infty ds |-w\tilde{v}_x - u\tilde{u}_{xx} + u_x^2 + vw_x|(x,s,\tau) \times$$
$$\Big[\frac{2\exp(\varepsilon s)}{\sqrt{2\pi}\sigma} + \varepsilon \exp(\varepsilon^2\nu(t-\tau))\Big(\exp(\varepsilon s) \times$$
$$\Big(2\Phi(\varepsilon\sigma) - \Phi\Big(\varepsilon\sigma + \frac{s}{\sigma}\Big)\Big) + \exp(-\varepsilon s)\Phi\Big(\varepsilon\sigma - \frac{s}{\sigma}\Big)\Big)\Big]$$
$$\le 2\Big[\Big(\frac{t}{\pi\nu}\Big)^{1/2} + \frac{\exp(\nu\varepsilon^2 t) - 1}{\nu\varepsilon}\Big] \times \tag{2.7}$$
$$\Big(\|\tilde{v}_x\|_{L_\infty}^t \|w\|_{(\varepsilon)}^t + \|u\|_{L_\infty}^t \|\tilde{u}_{xx}\|_{(\varepsilon)}^t + \|u\|_{L_\infty}^t \|u_x\|_{(\varepsilon)}^t + \|v\|_{L_\infty}^t \|w_x\|_{(\varepsilon)}^t\Big),$$

where obvious inequality $\|w\|_{L_1} \le \|w\|_{(\varepsilon)}$ is used.

It will be recalled now that

$$\|w\|_{(\varepsilon)}^t \le C, \quad \|w_y\|_{(\varepsilon)}^t \le C, \quad \|w_x\|_{(\varepsilon)}^t \le C,$$

and so we have to evaluate only the norms of u, v and their derivatives that are used in the right-hand sides of (2.5)–(2.7). It follows from (1.4) that:

$$\|u\|_{L_\infty} \le u_\infty + \|w\|_{L_1} \le u_\infty + C,$$
$$\|u_x\|_{L_\infty} \le \|w_x\|_{L_1} \le C,$$
$$\|u_x\|_{(\varepsilon)} \le \int\limits_0^\infty dy \exp(\varepsilon y) \int\limits_y^\infty |w_x| ds = \frac{1}{\varepsilon}(\|w_x\|_{(\varepsilon)} - \|w_x\|_{L_1}) \le \frac{C}{\varepsilon},$$
$$\|\tilde{u}_{xx}\|_{(\varepsilon)} = \int\limits_0^\infty dy \exp(\varepsilon y)\Big|\int\limits_y^\infty \tilde{w}_{xx} ds\Big| \tag{2.8}$$
$$\le \frac{1}{\varepsilon\Delta x}(\|w_x\|_{(\varepsilon)}(x) + \|w_x\|_{(\varepsilon)}(x - \Delta x) -$$
$$\|w_x\|_{L_1}(x) - \|w_x\|_{L_1}(x - \Delta x)) \le \frac{2C}{\varepsilon\Delta x}.$$

Using the definition of v from (1.5) we have

$$\|v\|_{L\infty} \le \int_0^y ds \exp(-\varepsilon s) \int_s^\infty |w_x| \exp(\varepsilon s') ds' \le \frac{1}{\varepsilon}\|w_x\|_{(\varepsilon)} \le \frac{C}{\varepsilon},$$
$$\|\tilde{v}_x\|_{L_\infty} \le \frac{1}{\varepsilon \Delta x}\big(\|w_x\|_{(\varepsilon)}(x) + \|w_x\|_{(\varepsilon)}(x - \Delta x)\big) \le \frac{2C}{\varepsilon\Delta x}. \tag{2.9}$$

Summing up (2.4)–(2.9) we come to the following inequalities:

$$\|(PW)_1\|_{(\varepsilon)}(x,t) \le C\Big(1 - \frac{\alpha}{3} + 4\alpha\Big(u_\infty + C + \frac{C}{\varepsilon}\Big)r_1(t)\Big),$$
$$\|(PW)_2\|_{(\varepsilon)}(x,t) \le C\Big(1 - \frac{\alpha}{3} + 4\alpha\Big(u_\infty + C + \frac{C}{\varepsilon}\Big)\big(r_1(t) + r_2(t)\big)\Big), \tag{2.10}$$
$$\|(PW)_3\|_{(\varepsilon)}(x,t) \le C\Big(1 - \frac{\alpha}{3} + \frac{2\alpha}{\varepsilon}\Big(\frac{u_\infty + 2C}{\Delta x} + 2C\Big)\big(r_1(t) + r_2(t)\big)\Big),$$

where

$$r_1(t) = \frac{1}{\varepsilon^2\nu}\big(\exp(\varepsilon^2 \nu t) - 1\big), \qquad r_2(t) = (\pi\nu)^{-1/2}t^{1/2}.$$

We see that $r_1(t) = O(t)$ when t tends to zero and so

$$\|PW\|^t_{(\varepsilon)} \le C \tag{2.11}$$

for sufficiently small t.

In order to complete the proof of convergence of the iterative process (2.2) it remains to show that operator P is contracting on $S(C)$. We have

$$\begin{aligned} W^{(n+1)} - W^{(n)} &= PW^{(n)} - PW^{(n-1)} \\ &= (1-\alpha)\big(W^{(n)} - W^{(n-1)}\big) \\ &\quad + \alpha\big(\mathcal{K}(W^{(n)}, W^{(n)}) - \mathcal{K}(W^{(n-1)}, W^{(n-1)})\big) \\ &= (1-\alpha)\big(W^{(n)} - W^{(n-1)}\big) + \alpha\big(\mathcal{K}(W^{(n)} - W^{(n-1)}, W^{(n)}) \\ &\quad + \mathcal{K}(W^{(n-1)}, W^{(n)} - W^{(n-1)})\big), \end{aligned} \tag{2.12}$$

where the latter equality is the consequence of bilinearity of operator $\mathcal{K}$. Applying inequalities (2.5)–(2.9) to (2.12) we arrive at

$$\begin{aligned} \|w^{(n+1)} - w^{(n)}\|_{(\varepsilon)} &\le (1-\alpha)\|w^{(n)} - w^{(n-1)}\|^t_{(\varepsilon)} + \\ &\quad \alpha \cdot 4Cr_1(t)\Big[\|w^{(n)} - w^{(n-1)}\|^t_{L_1} + \\ &\quad \Big(\frac{1}{\varepsilon} + 1 + \frac{u_\infty}{C}\Big)\|w^{(n)}_x - w^{(n-1)}_x\|^t_{(\varepsilon)} + \frac{1}{\varepsilon}\|w^{(n)}_y - w^{(n-1)}_y\|^t_{(\varepsilon)}\Big], \end{aligned}$$

$$\|w_y^{(n+1)} - w_y^{(n)}\|_{(\varepsilon)} \le (1-\alpha)\|w_y^{(n)} - w_y^{(n-1)}\|^t_{(\varepsilon)} + \\ \alpha \cdot 4C\big(r_1(t) + r_2(t)\big)\Big[\|w^{(n)} - w^{(n-1)}\|^t_{L_1} + \\ \Big(\frac{1}{\varepsilon} + 1 + \frac{u_\infty}{C}\Big)\|w_x^{(n)} - w_x^{(n-1)}\|^t_{(\varepsilon)} + \frac{1}{\varepsilon}\|w_y^{(n)} - w_y^{(n-1)}\|^t_{(\varepsilon)}\Big],$$

$$\|w_x^{(n+1)} - w_x^{(n)}\|_{(\varepsilon)} \le (1-\alpha)\|w_x^{(n)} - w_x^{(n-1)}\|^t_{(\varepsilon)} + \\ \alpha \cdot 2C\big(r_1(t) + r_2(t)\big)\Big[\frac{4}{\varepsilon\Delta x}\|w^{(n)} - w^{(n-1)}\|^t_{(\varepsilon)} + \\ \frac{1}{\varepsilon}\Big(4 + \frac{2}{\Delta x} + \frac{u_\infty}{C\Delta x}\Big)\|w_x^{(n)} - w_x^{(n-1)}\|^t_{(\varepsilon)}(x) + \\ \frac{1}{\varepsilon\Delta x}\Big(2 + \frac{u_\infty}{C}\Big)\|w_x^{(n)} - w_x^{(n-1)}\|^t_{(\varepsilon)}(x - \Delta x)\Big].$$

So, for sufficiently small t we have

$$\|PW^{(n)} - PW^{(n-1)}\|_{(\varepsilon)} \le (1 - \alpha + \alpha R(t))\|W^{(n)} - W^{(n-1)}\|^t_{(\varepsilon)}, \quad (2.13)$$

where $R(t) = O(t^{1/2})$, when t tends to zero.

Thus, conditions of the fixed-point theorem [5] are fulfilled and so the iterative process (2.2) converges to the unique solution W of the integral equation (2.1) with components in $S(C)$.

3. Random estimate

We construct a stochastic algorithm now for solving the system of nonlinear integral equations (2.1). Randomizing iterative algorithm (2.2) we come to the need of modelling of the branching Markov chain.

There exist different approaches to defining a branching process (see, for example [3, 4, 6]). Starting with the constructive convenience considerations we choose the phase space of the chain states to be $[0, +\infty) \times [0, T]$ with the natural Lebesque σ-algebra of subsets.

We turn to the computation of the solution of system (2.1) now. In order to do that we construct, as it is called in the theory of the Monte Carlo methods, a conjugate estimate and construct it in accordance with the recurrence relation emerging from the definition of iterative process (2.2). We set $\alpha = 1$ in order to simplify formula manipulations. We can do it since this parameter does not affect the convergence of the algorithm. Denote by $z = (y, t)$, $\xi_i(z)$ – random estimate for the function $w_i(z)$. This estimate and the branching Markov chain are constructed simultaneously. Let z_k^n

be the k-th point in the n-th generation of the process. We have to choose the law of branching now. First, we set that with probability $g(z_k^n)$ this particular branch of the chain terminates independently of other branches. Or, alternatively, we terminate all branches of the chain simultaneously with probability g. We require that our branching process is degenerating so the mean value of the number of branches that are given rise at every point must be less or equal to one. Let $p_1(z,z')$ be distribution density of z' subject to condition that z is fixed. We choose it in accordance with the kernels of the integral operators K_1, K_2 and K_3. Denote by γ the random variable which is equal to zero with probability g and $\gamma = 1$ with probability $1-g$.

According to the double randomization principle [3] we can use in (1.23) instead of the integral operators $L_1 w$, $L_2(L_1 w)$ their conditionally independent unbiased estimates. Let $p_2(y,s)$, $p_3(y,s)$ be distribution densities on $[y,+\infty)$ and $[0,y]$ correspondingly. Then

$$\begin{aligned} L_1 w(y) &= \mathbf{E}\Big(\frac{w(s)}{p_2(y,s)}\Big), \\ L_2(L_1 w)(y) &= \mathbf{E}\Big(\frac{w(s')}{p_3(y,s)p_2(s,s')}\Big), \end{aligned} \tag{3.1}$$

So we have the following recurrence formula:

$$\begin{aligned} \xi_1(x,z_k^n) = F_1(z_k^n) + \gamma(z_k^n)&\frac{k_1(z_k^n, z_{3k}^{n+1})}{(1-g(z_k^n))p_1(z_k^n, z_{3k}^{n+1})} \times \\ &\Big[\Big(u_\infty + \frac{\xi_1(x, z_{3k-1}^{n+1})}{p_2(y_{3k}^{n+1}, y_{3k-1}^{n+1})}\Big)\xi_3(x, z_{3k}^{n+1}) - \\ &\quad \xi_2(x, z_{3k}^{n+1})\frac{\xi_3(x, z_{3k-2}^{n+1})}{p_3(y_{3k}^{n+1}, y')p_2(y', y_{3k-2}^{n+1})}\Big], \\ \xi_2(x,z_k^n) = F_2(z_k^n) + \gamma(z_k^n)&\frac{k_2(z_k^n, z_{3k}^{n+1})}{(1-g(z_k^n))p_1(z_k^n, z_{3k}^{n+1})} \times \\ &\Big[\Big(u_\infty + \frac{\xi_1(x, z_{3k-1}^{n+1})}{p_2(y_{3k}^{n+1}, y_{3k-1}^{n+1})}\Big)\xi_3(x, z_{3k}^{n+1}) - \\ &\quad \xi_2(x, z_{3k}^{n+1})\frac{\xi_3(x, z_{3k-2}^{n+1})}{p_3(y_{3k}^{n+1}, y')p_2(y', y_{3k-2}^{n+1})}\Big], \end{aligned} \tag{3.2}$$

$$\xi_3(x,z_k^n) = F_3(z_k^n) + \gamma(z_k^n)\frac{k_3(z_k^n, z_{3k}^{n+1})}{(1-g(z_k^n))p_1(z_k^n, z_{3k}^{n+1})} \times$$
$$\left[-\left(u_\infty + \frac{\xi_1(x, z_{3k-1}^{n+1})}{p_2(y_{3k}^{n+1}, y_{3k-1}^{n+1})}\right) \times\right.$$
$$\frac{\xi_3(x, z_{3k-2}^{n+1}) - \xi_3(x-\Delta x, z_{3k-2}^{n+1})}{\Delta x \cdot p_3(y_{3k}^{n+1}, y')p_2(y', y_{3k-2}^{n+1})} +$$
$$\left.\frac{\xi_3(x, z_{3k-1}^{n+1})}{p_2(y_{3k}^{n+1}, y_{3k-1}^{n+1})} \cdot \frac{\xi_3(x, z_{3k-2}^{n+1})}{p_3(y_{3k}^{n+1}, y')p_2(y', y_{3k-2}^{n+1})}\right].$$

Thus every point z_k^n with probability $1-g$ gives birth to three points in the next generation for given x and one point for $x - \Delta x$, y' is an auxiliary point and does not belong to our branching chain.

We pass to the problem of the choice of distribution densities p_1, p_2, p_3 now. First, p_1 must be taken in accordance with the kernels of the integral operators K_1, K_2 and K_3. This condition is not so strict as it may seem since $K_i(y,t;s,\tau)$, $i = 1,2,3$ are continuous and bounded functions for all $(s,\tau) \in [0,+\infty) \times [0,t)$. Exponential type of these kernels suggests that we can use γ distribution in the process of random sampling of τ for given t, y and s. This procedure, however, seems to be ineffective for small t as it has been noted in [7]. Requiring that our estimates have finite variance, we come to conclusion that integrals of the following type must be finite also.

$$I_1 = \int_0^t d\tau \int_0^\infty ds \frac{k_i^2(y,t;s,\tau)}{p_1(y,t;s,\tau)} w_j^2(s,\tau),$$
$$I_2 = \int_0^\infty \frac{w_j^2(s)}{p_2(y,s)} ds, \tag{3.3}$$
$$I_3 = \int_0^y ds' \int_{s'}^\infty \frac{w_j^2(s)}{p_3(y,s')p_2(s',s)} ds.$$

It will be recalled now that $w_j \in L(\varepsilon)$ so it is sufficient to require that $w_j \in L_\infty$ also. From here we arrive at the following consequence: we can use exponential densities

$$p_1(y,t;s,\tau) = \varepsilon_1 p_0(t,\tau)\exp(-\varepsilon_1 s),$$
$$p_2(y,s) = \varepsilon_2 \exp(-\varepsilon_2(s-y)),$$
$$p_3(y,s) = \varepsilon_3 \exp(-\varepsilon_3 s)(1-\exp(-\varepsilon_3 y))^{-1},$$

with some positive constants ε_1, ε_2, ε_3 and the choice of p_0 is rather arbitrary. For example, we can set τ to be uniformly distributed on $(0,t)$. As a result we come to the following simulation formulas:

$$\begin{aligned} y_{3k}^{n+1} &= -\varepsilon_1 \ln \alpha_1, & t_{3k}^{n+1} &= t_k^n \cdot \alpha_0, \\ y_{3k-1}^{n+1} &= y_{3k}^{n+1} - \varepsilon_2 \ln \alpha_2, & t_{3k-1}^{n+1} &= t_{3k}^{n+1}, \\ y_{3k-2}^{n+1} &= y' - \varepsilon_2 \ln \alpha_4, & t_{3k-2}^{n+1} &= t_{3k}^{n+1}, \\ y' &= -\frac{1}{\varepsilon_3} \ln \big(1 - \alpha_3(1 - \exp(-\varepsilon_3 y^{n+1} 3k))\big). \end{aligned}$$

Note that the distribution of y_{3k}^{n+1} does not depend here on y_k^n, so we can evaluate the solution in the set of points $y_0^{(j)}$ for given t_0 on the same trajectories of the Markov chain.

We prove now that the estimates constructed in accordance with (3.2) are unbiased estimates for the solution of the system of integral equations (1.20). Applying the operator of conditional mathematical expectation to (3.2) we come to conclusion that $\mathbf{E}\xi_i$, $i = 1,2,3$ satisfy (1.23). It is the consequence of independence of $\xi_i(z_k^n)$ and $\xi_j(z_l^n)$ for all i, j, n when $k \neq l$. In order to conclude the proof we have to show that $\mathbf{E}\xi_i$ are finite. From clarity considerations we take the particular law of branching in which we terminate all branches of the Markov chain in one generation of points simultaneously with the constant probability g. Then we have

$$\xi(z_1^1) = \boldsymbol{F}(z_1^1) + \Phi(\boldsymbol{F}(z^2)) + \ldots + \underbrace{\Phi(\ldots(\Phi}_{N}(\boldsymbol{F}(z^{N-1})))\ldots), \tag{3.4}$$

where $\xi = (\xi_1, \xi_2, \xi_3)'$, z^n denotes all points in n-th generation, Φ is the functional from (3.2) if we rewrite it as

$$\xi = \boldsymbol{F} + \Phi(\xi),$$

and N is a random number of generations.

Consider (2.2) and set $\alpha = 1$, $\boldsymbol{W}^{(0)} = \boldsymbol{F}$. Then we have

$$\mathbf{E}\big(\xi(z_1^1) \mid N = n\big) = \boldsymbol{W}^{(n)},$$

and $\boldsymbol{W}^{(n)}$ can be represented as a finite segment of series (3.4), where Φ has to be substituted by $\mathcal{K}$, and N is equal to n. It follows from (2.13) that series $\boldsymbol{W}^{(n)}$ converges absolutely at a geometric progression speed. This makes possible to evaluate the right-hand side in the expression

$$\mathbf{E}\xi(z_1^1) = \sum_{n=0}^{\infty} \mathbf{E}\big(\xi(z_1^1) \mid N = n\big)\mathbf{P}(N = n) = \sum_{n=0}^{\infty} \boldsymbol{W}^{(n)}(z_1^1)g(1-g)^n,$$

using an arbitrary convenient norm, and so we have

$$|\mathbf{E}\xi(z_1^1)| \leq \text{const},$$

thus accomplishing our proof of unbiasedness. The analogous but more cumbersome expressions and considerations lead to the same result in the case of other branching laws.

It remains to prove that the estimate constructed in accordance with (3.2) has finite variance or finite second moments, which is equivalent. In order to do that we proceed as follows. We take (3.2), multiply these expressions term by term and apply the operator of mathematical expectation. As a result we come to the system of integral equations satisfied by components of the second moments matrix $R_{ij}(z) = \mathbf{E}\xi_i(z)\xi_j(z)$, $i,j = 1,2,3$

$$\begin{aligned}
R_{11} &= F_1(2W_1 - F_1) + K_{11}^{(p)}\big[R_{33}(2uu_\infty - u_\infty^2 + L_1^{(p)}(R_{11})) - \\
&\qquad R_{22}L_2^{(p)}(L_1^{(p)}(R_{33}) + 2uvR_{23})\big],\\
R_{13}(x,\cdot) &= F_1W_3 + F_3W_1 - F_1F_3 + \\
&\quad K_{13}^{(p)}\Big[-W_3(2uu_\infty - u_\infty^2 + L_1^{(p)}(R_{11}))\frac{1}{\Delta x}\times\\
&\quad L_2\big(L_1(W_3(x,\cdot) - W_3(x-\Delta x,\cdot))\big) + \\
&\quad W_2u\cdot\frac{1}{\Delta x}L_2^{(p)}\big(L_1^{(p)}(R_{33}(x,\cdot) - S(x,x-\Delta x))\big) + \\
&\quad W_3\cdot L_2(L_1(W_3))\cdot\big(u_\infty W_1 + L_1^{(p)}(R_{13}(x,\cdot))\big) - \qquad (3.5)\\
&\quad W_2\cdot L_1(W_3)\cdot L_2^{(p)}\big(L_1^{(p)}(R_{33}(x,\cdot))\big)\Big],\\
R_{33}(x,\cdot) &= F_3(2W_3 - F_3) + \\
&\quad K_{33}^{(p)}\Big[(2uu_\infty - u_\infty^2 + L_1^{(p)}(R_{11}))\frac{1}{(\Delta x)^2}\times\\
&\quad L_2^{(p)}\big(L_1^{(p)}(R_{33}(x,\cdot)) + R_{33}(x-\Delta x,\cdot) - 2S(x,x-\Delta x)\big) + \\
&\quad L_1^{(p)}(R_{33}(x,\cdot))\cdot L_2^{(p)}\big(L_1^{(p)}(R_{33}(x,\cdot))\big) - \\
&\quad 2\big(u_\infty W_1 + L_1^{(p)}(R_{33}(x,\cdot))\big)\frac{1}{\Delta x}\times\\
&\quad L_2^{(p)}\big(L_1^{(p)}(R_{33}(x,\cdot) - S(x,x-\Delta x))\big)\Big],
\end{aligned}$$

R_{12}, R_{22} look like as R_{11} and R_{23} is similar to R_{13}. Here

$$K_{ij}^{(p)}[w](z) = \int \frac{k_i(z,z')k_j(z,z')}{(1-g(z))P_1(z,z')} \cdot w(z')dz,$$

$$L_1^{(p)}(w) = \int_y^\infty \frac{w(s)}{p_2(y,s)}ds, \qquad L_2^{(p)}(L_1^{(p)}(w)) = \int_0^y ds' \int_{s'}^\infty \frac{w(s)}{p_3(y,s')p_2(s',s)}ds,$$

and $S(x, x-\Delta x) = \mathbf{E}(\xi_3(x,\cdot)\xi_3(x-\Delta x,\cdot))$.

Next in the full analogy with the case of K_i, $i = 1,2,3$, we can prove that $K_{ij}^{(p)}$ are contracting operators on $S(C)$ and $L_1^{(p)}(w)$, $L_2^{(p)}(L_1^{(p)}(w))$ are finite on that set of functions for the appropriate choice of densities p_1, p_2, p_3. The same arguing leads to the fact that for sufficiently small t the analog of the Neumann series for the system of nonlinear integral equations (3.5) converges to the finite solution R_{ij} which, considered as a function of y, belongs to $S(C)$.

4. Conclusion

In this paper we have introduced new stochastic algorithm for solving two-dimensional boundary layer equations. It is based on the representation of a solution of heat equation as a sum of three potentials. Having desired to obtain a closed system of integral equations we have come to the need of using difference approximation for w_{xx}. As a result the solution of this system is equal to w, w_y, w_x with an accuracy of $O(\Delta x)^2$. It should be noted that Δx must not be constant.

Another characteristic feature of the algorithm is that we have to use a sufficiently large probability of terminating of the Markov chain and thus only its first terms are effectively computed. One of the possible ways of overcoming this difficulty is to simulate several initial steps without termination.

The idea to write this paper have been provoked by works of Chorin [2], and there exist some ways of combining the algorithms. It means that we can track the motion of vortex sheets with the help of the method described in this paper. Thus, the stochastic algorithm may be considered as a new mathematical model for the two-dimensional fluid motion.

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