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The computing schemes of non-stationary electromagnetic fields FEM modeling in mediums with three-dimensional inhomogeneity

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Finite element non-stationary electromagnetic fields modeling technique is proposed. It permits to decrease computing expenses of three-dimensional problem solving. This technique is implemented in program complex TELMA, its efficiency is confirmed by solving significant number of both modeling tasks, and complicated practical problems.

1. Introduction

Now the finite element method (FEM) is widely applied in three-dimensional mathematical modeling of non-stationary electromagnetic fields. FEM advantages, main of which is the ability to use essentially irregular meshes with minimum number of so-called "unnecessary" nodes, have much expanded a range of electromagnetic problems, accessible for the numerical solving.

However, there is sufficiently wide class of electromagnetic problems, for which the solving of describing differential system of equations using the standard computing finite element schemes requires very large computing expenses. It results or in very large cost, or even in impossibility of the finite element solving of important practical problems. To such problems it is possible to attribute the absolute majority of three-dimensional problems of electromagnetic earth logging, and also many other problems of electromagnetic field propagation in such mediums, where influence of three-dimensional distortion is not too large in comparison with meaning of fundamental two-dimensional (axially symmetric) field, but must be calculated with sufficiently high accuracy. The proposed finite element modeling technique permits to lower computing expenses of such type problems solving in 10–100 and more times and thereby these problems become accessible for solving on not powerfull computers. This technique is implemented in program complex TELMA, its efficiency is confirmed by solving significant number of both modeling tasks, and complicated practical problems.

2. Mathematical models of investigated fields

To describe three-dimensional non-stationary electromagnetic fields we shall use Maxwell's equations system [1] in the form of:

$$\operatorname{rot}\vec{H} = \vec{J} + \sigma\vec{E} + \epsilon\frac{\partial\vec{E}}{\partial t}, \quad \operatorname{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \operatorname{div}\vec{B} = 0, \tag{1}$$

where $\vec{H}$ – magnetic field strength, $\vec{J}$ – outside current density vector (stimulating an electromagnetic field), σ – medium conductivity, $\vec{E}$ – electrical field strength, ϵ – medium dielectric constant coefficient, t – time, $\vec{B}$ – magnetic field induction (dependent on $\vec{H}$ by relation $\vec{B} = \mu\vec{H}$), μ – permeance coefficient.

Depending on electromagnetic field behavior nature, determined basically by a way of field stimulation, by investigation time range and by values of coefficients μ, σ and ϵ, the system of equations (1) can be converted to more convenient for numerical modeling form. When displacement currents $\epsilon\frac{\partial\vec{E}}{\partial t}$ can not be ignored or coefficient μ distinguished from vacuum permeance μ_0 (i.e., medium is non-homogeneity on μ) it is possible to use for electromagnetic field calculation the equation

$$\operatorname{rot}\Big(\frac{1}{\mu}\operatorname{rot}\vec{E}\Big) + \sigma\frac{\partial\vec{E}}{\partial t} + \epsilon\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{\partial\vec{J}}{\partial t}. \tag{2}$$

If μ is a function of $\vec{H}(t)$ (i.e., there are objects with nonlinear by permeance characteristics in calculation area), the system of equations (1) can be converted to the form, similar to (2):

$$\operatorname{rot}\Big(\frac{1}{\mu}\operatorname{rot}\vec{A}\Big) + \sigma\frac{\partial\vec{A}}{\partial t} + \epsilon\frac{\partial^2\vec{A}}{\partial t^2} = \vec{J}, \tag{3}$$

where $\vec{A}$ – vector-potential, determined by the relations

$$\vec{B} = \operatorname{rot}\vec{A}, \quad \vec{E} = -\frac{\partial\vec{A}}{\partial t}. \tag{4}$$

It is obvious that when equation (2) is valid (i.e., coefficients μ, σ and ϵ do not depend on time), it can be easily received from equation (3) by differentiation on t (and by replacement of $\partial\vec{A}/\partial t$ by $-\vec{E}$), i.e., equation (2) is actually a consequence of equation (3). Equations (2) and (3) practically coincide by form and therefore the numerical integration procedures for them are also identical. Thus, it would be possible to consider only more general equation (3), but in some cases equation (2) appears more convenient: for example, if electric field intensity $\vec{E}$ or time derivative of magnetic induction

$\vec{B}$ component are investigated, then it is not necessary to differentiate the received numerical solution on time.

And, at last, let us consider a situation often occurs in practice, when the displacement currents can be ignored, permeance coefficient $\mu = \mu_0$, (μ_0 – permeance coefficient of vacuum), and the conductivity σ is equal to zero at considerable calculation area part. The elimination of displacement currents from the first equation of system (1) reduces it to the form

$$\operatorname{rot} \vec{H} = \vec{J} + \sigma \vec{E}. \tag{5}$$

When $\mu = \mu_0$, relation (5) accepts the form

$$\frac{1}{\mu_0} \operatorname{rot} \vec{B} = \vec{J} + \sigma \vec{E}. \tag{6}$$

Introduction of vector-potential $\vec{A}$ and scalar potential V using relations

$$\vec{B} = \operatorname{rot} \vec{A}, \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \operatorname{grad} V \tag{7}$$

permits to convert the system of equations (6), and system (1) of second and third equations to the form

$$-\frac{1}{\mu_0} \Delta \vec{A} + \sigma \Big(\frac{\partial \vec{A}}{\partial t} + \operatorname{grad} V \Big) = \vec{J}, \tag{8}$$

$$-\operatorname{div}(\sigma \operatorname{grad} V) - \operatorname{div}\Big(\sigma \frac{\partial \vec{A}}{\partial t}\Big) = -\operatorname{div} \vec{J}. \tag{9}$$

3. Computing scheme for case of essential displacement currents' influence or permeance inhomogeneous medium

3.1. Modeling technique

Let an electromagnetic field be described by equation (5), where coefficients μ, σ and ϵ are some three-dimensional functions of coordinates and, probably, of time. Let us choose approximation μ^0, σ^0 and ϵ^0 for original coefficients μ, σ and ϵ, so that these approximation were one- or two-dimensional functions of coordinates in any (for example, cylindrical) coordinate system and differed from original coefficients only in those calculation area Ω parts, where three-dimensional objects are given. Let us designate through $\vec{J}^0$ appropriate two-dimensional (for cylindrical system of coordinates – axially symmetric) approximation of outside currents $\vec{J}$, and through $\vec{J}^+$ –

three-dimensional part of outside currents. Note, that in most cases outside currents are axially symmetric and thereby, when cylindrical coordinate system is used for two-dimensional approximation, $\vec{J}^0 = \vec{J}$ and $\vec{J}^+ = 0$.

Let us represent vector-potential $\vec{A}$ as a sum of two vector-potentials $\vec{A} = \vec{A}^0 + \vec{A}^+$, where $\vec{A}^0$ - vector-potential, meeting the equation

$$\operatorname{rot}\Big(\frac{1}{\mu^0}\operatorname{rot}\vec{A}^0\Big) + \sigma^0\frac{\partial \vec{A}^0}{\partial t} + \epsilon^0\frac{\partial^2 \vec{A}^0}{\partial t^2} = \vec{J}^0, \tag{10}$$

and $\vec{A}^+$ - vector-potential, meeting the equation

$$\begin{aligned}\operatorname{rot}\Big(\frac{1}{\mu}\operatorname{rot}\vec{A}^+\Big) + \sigma\frac{\partial \vec{A}^+}{\partial t} + \epsilon\frac{\partial^2 \vec{A}^+}{\partial t^2}\\ = \vec{J}^+ + \operatorname{rot}\Big(\Big(\frac{1}{\mu^0} - \frac{1}{\mu}\Big)\operatorname{rot}\vec{A}^0\Big) - (\sigma - \sigma^0)\frac{\partial \vec{A}^0}{\partial t} - (\epsilon - \epsilon^0)\frac{\partial^2 \vec{A}^0}{\partial t^2}.\end{aligned} \tag{11}$$

In equation (11) $\vec{A}^0$ is considered as known vector-function, found as a solution of equation (10). It is not difficult to be convinced that then vector-potential $\vec{A} = \vec{A}^0 + \vec{A}^+$ will satisfy equation (3): for this purpose it is enough to combine equations (10) and (11).

Representation of the solution of (3) as a sum of the solutions $\vec{A}^0$ of (10), defined in two-dimensional (as a rule, axially symmetric) area Ω^0, and of the solution $\vec{A}^+$ of (11), defined in the original three-dimensional area Ω, in many cases permits to increase significantly the accuracy of numerical calculation and simultaneously decrease the computing expenses in comparison with numerical solving of equation (3) directly. It is reached because vector-potential $\vec{A}^0$ can be calculated with much higher accuracy and with considerably small computing resource expenses by reduction of its determination dimension. If the main part of a total field $\vec{A}$ is concentrated in two-dimensional field $\vec{A}^0$, then the field $\vec{A}^+$ (which can be interpreted as a field of three-dimensional objects' influence) can be calculated with high accuracy on three-dimensional mesh with much smaller number of nodes in comparison with number of nodes in three-dimensional mesh, that is necessary to solve boundary value problem for original equation (3) with the same accuracy.

Let us consider the following peculiarity of system of equations (10)–(11) more closely. As stated above, the main advantage of the numerical solution of system of equations (10)–(11) in comparison with numerical solution of (3) is that dimension of vector-potential $\vec{A}^0$ domain of definition for 10) in comparison with dimension of vector-potential $\vec{A}$ domain of definition for (3) can be reduced. But all functions forming equation (11) (and $\vec{A}^0$ also) should be defined in three-dimensional area Ω. Thus, in equations (10) and

(11) vector-potential $\vec{A}^0$, in general, has different domain of definition. The solution of (10) is a vector-function or even scalar (when vector $\vec{A}^0$ has only one non-zero component) function of two coordinates (as a rule, coordinates z and r of cylindrical coordinate system), defined in area Ω^0. But the part of (11), vector-function $\vec{A}^0$, is a function of three coordinates, defined in area Ω. It should be received from the solution of two-dimensional equation (10) by recalculation on conformity of two-dimensional area Ω^0 points to original three-dimensional area Ω points and on conformity of vector-functions' components in various coordinate systems. Similarly, coefficients μ^0, σ^0 and ϵ^0 and currents $\vec{J}^0$ have different domains of definition in equations (10) and (11), but the recalculation procedure for them is not necessary. Exception can be only coefficient μ^0, if μ^0 is a function of $\vec{B}^0 = \operatorname{rot}\vec{A}^0$, then recalculation of μ^0 is similar to recalculation of one $\vec{A}^0$ component.

Let us note another very important peculiarity of equation (11). The basic source members in equation (11) differ from zero only in those subareas of area Ω, where the medium characteristics μ^0, σ^0 and ϵ^0, defined the problem for vector-potential $\vec{A}^0$ (see (10)), differ from correspondent medium characteristics μ, σ and ϵ of original problem. It is the cardinal advantages of considered modeling technique. Actually the sources of a three-dimensional field $\vec{A}^+$ are three-dimensional objects in original area Ω, and difficultly approximated abrupt changes derivative is absent in the solution $\vec{A}^+$ of (11).

3.2. Time approximation and equivalent variational formulation of three-dimensional problem

Before receiving equivalent variational formulation of boundary value problem for (11), let us transform it. Taking into account identity

$$\operatorname{rot}_\xi(\lambda \operatorname{rot}\vec{G}) \equiv -\operatorname{div}(\lambda \operatorname{grad}\vec{G}) + \operatorname{div}\Big(\lambda\frac{\partial\vec{G}}{\partial\xi}\Big),$$

the vector equation (11) can be converted to the form:

$$-\operatorname{div}\Big(\frac{1}{\mu}\operatorname{grad}A_\xi^+\Big) + \operatorname{div}\Big(\frac{1}{\mu}\frac{\partial\vec{A}^+}{\partial\xi}\Big) + \sigma\frac{\partial A_\xi^+}{\partial t} + \epsilon\frac{\partial^2 A_\xi^+}{\partial t^2}$$
$$= J_\xi^+ - \operatorname{div}\Big(\Big(\frac{1}{\mu^0}-\frac{1}{\mu}\Big)\operatorname{grad}A_\xi^0\Big) + \operatorname{div}\Big(\Big(\frac{1}{\mu^0}-\frac{1}{\mu}\Big)\frac{\partial\vec{A}^0}{\partial\xi}\Big) -$$
$$(\sigma-\sigma^0)\frac{\partial A_\xi^0}{\partial t} - (\epsilon-\epsilon^0)\frac{\partial^2 A_\xi^0}{\partial t^2}, \qquad (12)$$

where ξ – one of variables x, y or z.

Let us approximate equation (12) on time. Let t_k be a value of time t, defined current time layer. Through $\vec{U}^j$ we designate value of vector-potential $\vec{A}^+$ on j time layer, through $\vec{W}^j$ – value of vector-potential $\vec{A}^0$ on j time layer, i.e.:

$$\vec{U}^j(x,y,z) = \vec{A}^+(x,y,z,t_j), \quad \vec{W}^j(x,y,z) = \vec{A}^0(x,y,z,t_j). \tag{13}$$

We consider only completely implicit schemes, i.e., when approximate of equation (12) on time on k time layer, in all members of this equation, not containing time derivative, we take component of vector-functions' $\vec{A}^+$, $\vec{A}^0$, $\vec{J}^+$ values at $t = t_k$. For approximation of equation (12) items, containing time derivative, besides values $\vec{A}^+$ and $\vec{A}^0$ on current (k) time layer we use their values on two or three previous time layers.

The time approximation using two previous time layers t_{k-1} and t_{k-2}, we carry out as follows. Let us designate through $\eta_1^k(t)$, $\eta_2^k(t)$ and $\eta_3^k(t)$ quadratic polynomial of variable t on interval (t_{k-2}, t_k):

$$\eta_i^k(t) = a_i^k + b_i^k t + c_i^k t^2. \tag{14}$$

These polynomial factor $a_i^k(t)$, $b_i^k(t)$ and $c_i^k(t)$ values should be calculated to satisfy the following equalities:

$$\eta_i^k(t_{k-2}) = \delta_{i1}, \quad \eta_i^k(t_{k-1}) = \delta_{i2}, \quad \eta_i^k(t_k) = \delta_{i3}, \quad i = 1,2,3, \tag{15}$$

where δ_{il} – Kronecker symbol. Polynomial will be used to replace approximated functions $u(x,y,z,t)$ on time interval (t_{k-2}, t_k) by their quadratic interpolater

$$\begin{aligned} \tilde{u}(x,y,z,t) &= u(x,y,z,t_{k-2})\eta_1^k(t) + u(x,y,z,t_{k-1})\eta_2^k(t) + \\ &\quad u(x,y,z,t_k)\eta_3^k(t). \end{aligned} \tag{16}$$

Really, the function $\tilde{u}(x,y,z,t)$ is quadratic polynomial on time and coincides with approximated function $u(x,y,z,t)$ at $t = t_{k-2}$, $t = t_{k-1}$ and $t = t_k$ (it follows from definition of functions $\eta_i^k(t)$, see (14)–(15)).

Replacing $\vec{A}^+$ and $\vec{A}^0$ values in (12) by their quadratic time interpolaters in all members, containing $\vec{A}^+$ and $\vec{A}^0$ time derivative, and taking into account designations (13), we receive equation for calculation of vector-function $\vec{A}^+$ component values on k time layer:

$$\begin{aligned} &-\operatorname{div}\Big(\frac{1}{\mu}\operatorname{grad} U_\xi^k\Big) + \operatorname{div}\Big(\frac{1}{\mu}\frac{\partial \vec{U}^k}{\partial \xi}\Big) + \alpha U_\xi^k \\ &\quad = J_\xi^+ - \beta U_\xi^{k-1} - \gamma U_\xi^{k-2} - \operatorname{div}\Big(\Big(\frac{1}{\mu^0} - \frac{1}{\mu}\Big)\operatorname{grad} W_\xi^k\Big) + \\ &\quad \operatorname{div}\Big(\Big(\frac{1}{\mu^0} - \frac{1}{\mu}^0\Big)\frac{\partial \vec{W}^k}{\partial \xi}\Big) + \tilde{\alpha} W_\xi^k + \tilde{\beta} W_\xi^{k-1} + \tilde{\gamma} W_\xi^{k-2}, \end{aligned} \tag{17}$$

where ξ – one of the variables x, y or z, and α, β, $gamma$, $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ – some constants, which are the values of polynomial $\eta_i^k(t)$ derivative at $t = t_k$.

Unknowns in equation (17) are only the components U_x^k, U_y^k and U_z^k of vector-function $\vec{U}^k$, all vector-functions, belonging to the right-hand side of this equation, are considered given (vector-functions $\vec{U}^{k-1}$ and $\vec{U}^{k-2}$ should be received by solving similar equations on previous time layers or from initial conditions, and vector-functions $\vec{W}^k$, $\vec{W}^{k-1}$ and $\vec{W}^{k-2}$ – by recalculating of boundary value problem for equation (10) solution).

Similarly to the above-stated time approximation of equation (12) using three-layered scheme, approximation of this equation using four-layered scheme can be carried out. For construction of four-layered scheme on k time layer 4 functions $\eta_1^k(t)$, $\eta_2^k(t)$, $\eta_3^k(t)$ and $\eta_4^k(t)$ are introduced, which are cubic polynomial of variable t on interval (t_{k-3}, t_k).

Note, that the above-stated completely implicit three- and four-layered schemes have very good time approximation accuracy and thus do not give oscillation of the numerical solution at transition from one time layer to another (the same as completely implicit two-layered scheme, used for the solving of parabolic type equations). It makes their more preferable in comparison with other implicit two- and three-layered schemes (including the frequently used Crank–Nicholson scheme [2, 3]), used for finite difference approximation of differential equations of parabolic and hyperbolic types.

To receive the equivalent variational formulation of boundary value problem for equation (12), let us multiply it by sampling function $\Psi(x, y, z)$, integrate over area Ω and apply to items with operation div Green's formula (integration by parts). As a principle of large volume is used, the sampling function Ψ is equal to zero on external boundary S of calculation area Ω. As a result for equation (17) we receive equivalent variational equality

$$\int_\Omega \frac{1}{\mu} \operatorname{grad} U_\xi^k \cdot \operatorname{grad} \Psi \, d\Omega - \int_\Omega \frac{1}{\mu} \frac{\partial U_\xi^k}{\partial \xi} \cdot \operatorname{grad} \Psi \, d\Omega + \int_\Omega \alpha U_\xi^k \Psi \, d\Omega$$
$$= \int_\Omega \left(\frac{1}{\mu^0} - \frac{1}{\mu}\right) \operatorname{grad} W_\xi^k \cdot \operatorname{grad} \Psi \, d\Omega - \int_\Omega \left(\frac{1}{\mu^0} - \frac{1}{\mu}\right) \frac{\partial W_\xi^k}{\partial \xi} \cdot \operatorname{grad} \Psi \, d\Omega +$$
$$\int_\Omega (J_\xi - \beta U_\xi^{k-1} - \gamma U_\xi^{k-2} + \tilde{\alpha} W_\xi^k + \tilde{\beta} W_\xi^{k-1} + \tilde{\gamma} W_\xi^{k-2}) \Psi \, d\Omega, \qquad (18)$$

where sampling function Ψ – any element of Hilbert's space H_0^l of functions, defined in three-dimensional area Ω, having square summable first derivative and became zero on boundary S of area Ω.

3.3. Calculation of matrices and right part vectors of finite element SLAE

Let us consider that the area Ω is decomposed into tetrahedral finite elements. Let us receive the contributions of one finite element Ω_m to components of matrix and right part vector of finite element SLAE. When tetrahedral finite elements and piecewise linear basic functions used, on one finite element only four basic functions Ψ_l are differ from zero, each of which is equal to 1 in one of tetrahedron Ω_m vertex, which is i node of finite element mesh, and to zero in all other vertexes of Ω_m. Let us introduce local vertex numbering on Ω_m. Let us designate through ψ_1, ψ_2, ψ_3 and ψ_4 local basic functions on finite element Ω_m.

Through $\vec{q}_j = (q_j^x, q_j^y, q_j^z)$ we designate local basic function's weights in representation of desired vector-function $\vec{U}^k = (U_x^k, U_y^k, U_z^k)$ as a linear combination of basic functions ψ_j on finite element Ω_m, i.e.,

$$\vec{U}^k\Big|_{\Omega_m} = \sum_{j=1}^{4} \vec{q}_j \psi_j.$$

To calculate finite element Ω_m local matrix, let us replace integration area Ω in equation (18) with finite element Ω_m, desired vector-function $\vec{U}$ with its local basic functions decomposition on Ω_m, and sampling function – with one of local basic functions $\psi_l, l = 1, \ldots, 4$. Then let us transform left-hand side of equation (18) for $\xi = x$:

$$\int_{\Omega_m} \frac{1}{\mu} \operatorname{grad}\Big(\sum_{j=1}^{4} q_j^x \psi_j\Big) \cdot \operatorname{grad} \psi_l \, d\Omega -$$
$$\int_{\Omega_m} \frac{1}{\mu} \frac{\partial}{\partial x} \Big(\sum_{j=1}^{4} q_j^x \psi_j\Big) \cdot \operatorname{grad} \psi_l \, d\Omega + \int_{\Omega_m} \alpha \Big(\sum_{j=1}^{4} q_j^x \psi_j\Big) \psi_l \, d\Omega$$
$$= \int_{\Omega_m} \sum_{j=1}^{4} \Big\{ \Big[\frac{1}{\mu} \Big(\frac{\partial \psi_l}{\partial x} \frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_l}{\partial z} \frac{\partial \psi_j}{\partial z} \Big) + \alpha \psi_j \psi_l \Big] q_j^x -$$
$$\frac{1}{\mu} \frac{\partial \psi_l}{\partial y} \frac{\partial \psi_j}{\partial x} q_j^y - \frac{\partial \psi_l}{\partial z} \frac{\partial \psi_j}{\partial x} q_j^z \Big\} d\Omega. \tag{19}$$

The similar transformations are carried out for y and z.

Let us designate

$$d_{lj} = \int_{\Omega_m} \psi_l \psi_j \, d\Omega, \qquad c_{lj}^{xx} = \int_{\Omega_m} \frac{1}{\mu} \Big(\frac{\partial \psi_l}{\partial y} \frac{\partial \psi_j}{\partial y} + \frac{\partial \psi_l}{\partial z} \frac{\partial \psi_j}{\partial z} \Big) d\Omega,$$
$$c_{lj}^{xy} = - \int_{\Omega_m} \frac{1}{\mu} \frac{\partial \psi_l}{\partial y} \frac{\partial \psi_j}{\partial x} d\Omega, \qquad c_{lj}^{xz} = - \int_{\Omega_m} \frac{1}{\mu} \frac{\partial \psi_l}{\partial z} \frac{\partial \psi_j}{\partial x} d\Omega. \tag{20}$$

Let us define c_{lj}^{yx}, c_{lj}^{yy}, c_{lj}^{yz}, c_{lj}^{zx}, c_{lj}^{zy}, c_{lj}^{zz} similarly. Through blocks-elements $\bar{c}_{lj}$ and $\bar{d}_{lj}$ we designate matrices, consisting of elements (20):

$$\bar{c}_{lj} = \begin{pmatrix} c_{lj}^{xx} & c_{lj}^{xy} & c_{lj}^{xz} \\ c_{lj}^{yx} & c_{lj}^{yy} & c_{lj}^{yz} \\ c_{lj}^{zx} & c_{lj}^{zy} & c_{lj}^{zz} \end{pmatrix}, \qquad \bar{d}_{lj} = \begin{pmatrix} d_{lj} & 0 & 0 \\ 0 & d_{lj} & 0 \\ 0 & 0 & d_{lj} \end{pmatrix}. \tag{21}$$

Let us consider that coefficient α is a constant on finite element Ω_m (since practically in all applied problems coefficients σ and ϵ are really piecewise constant functions of space variables). Then, taking into account designations (20), the contributions (19) in finite element SLAE from finite element Ω_m can be written as follows:

$$\sum_{j=1}^{4} (\bar{c}_{lj} + \alpha \bar{d}_{lj})\, \vec{q}_j, \qquad l = 1, \ldots, 4. \tag{22}$$

Thus, the finite element Ω_m local matrix is a sum of local matrices of rigidity $\bar{c}$ and of mass $\bar{d}$, consisting of block-component as (21) with elements, determined by formulas (20).

When piecewise linear basic functions used, it is possible to consider permeance coefficient μ as constant inside each finite element Ω_m. Therefore the value of $1/\mu$ in relation (20) can be exported from integral and the local rigidity matrix $\bar{c}$ can be represented as $\bar{c} = \frac{1}{\mu}\bar{g}$.

To calculate a local right part vector of finite element Ω_m (i.e., contributions to FEM SLAE right part vector components from finite element Ω_m), it is necessary to replace integration area at right-hand side of equation (18) with finite element Ω_m, sampling function Ψ – with one of local basic functions $\psi_l, l = 1, \ldots, 4$, and vector-function $\vec{W}$ and first factors of integrand functions of equation (18) right-hand side latter items – with their basic functions ψ_j decomposition:

$$\vec{W}^k\Big|_{\Omega_m} = \sum_{j=1}^{4} \vec{w}_j \psi_j, \tag{23}$$

$$\left(\vec{J} - \beta \vec{U}^{k-1} - \gamma \vec{U}^{k-2} + \tilde{\alpha} \vec{W}^k + \tilde{\beta} \vec{W}^{k-1} + \tilde{\gamma} \vec{W}^{k-2}\right)\Big|_{\Omega_m} = \sum_{j=1}^{4} \vec{\xi}_j \psi_j. \tag{24}$$

It is not difficult to be convinced that, taking into account decomposition (23), contributions of a finite element Ω_m to local right part vector $\vec{f}_l^{\,1}$ from the first of two equation (18), the right-hand side items can be calculated through local rigidity matrices $\bar{g}$:

$$\vec{f}_l^1 = \left(\frac{1}{\mu^0} - \frac{1}{\mu}\right) \sum_{j=1}^{4} \bar{g}_{lj} \vec{w}_j, \tag{25}$$

The contributions to the local right part vector from equation (18) right-hand side latter items using decomposition (24) are easily calculated through local weight matrices $\bar{d}$:

$$\vec{f}_l^2 = \sum_{j=1}^{4} \bar{d}_{lj} \vec{\xi}_j. \tag{26}$$

Thus, the local matrix $\bar{p}$ and the right part vector $\bar{f}$ of finite element Ω_m are defined as

$$\bar{p} = \bar{c} + \alpha \bar{d}, \qquad \vec{f} = \vec{f}^1 + \vec{f}^2. \tag{27}$$

As follows from determining $\bar{p}$ relations (20) and (27), the local matrix is symmetric. Really, all diagonal block-elements $\bar{p}_{ll}$ are symmetric matrices, and for off-diagonal block-elements $\bar{p}_{lj} = \bar{p}_{lj}^T$ equality is valid. Therefore the FEM SLAE global matrix $\bar{P}$, assembled from such local matrices, is symmetric too. In addition, memory location quantity, necessary for store only non-zero matrix $\bar{P}$ components, is defined as:

$$k = 9k_g + 6k_d, \tag{28}$$

where k_d – number of matrix $\bar{P}$ diagonal elements (equal to number of nodes in finite element mesh), k_g – number of off-diagonal elements in FEM SLAE matrix bottom (or top) triangle's portrait for any scalar boundary value problem solved on considered finite element mesh, if this matrix is represented in sparse row format [4].

4. Computing scheme for the case of negligible displacement of current's influence and permeance homogeneous medium

4.1. Modeling technique

As original mathematical model, describing an electromagnetic field with negligible displacement of current's influence in permeance homogeneous medium, we consider the system of equations (8)–(9). As well as in previous section, we look for solution $(\vec{A}, V)$ of the equations system (8)–(9) as a sum of two solutions: solution $(\vec{A}^0, V^0)$ of two-dimensional (axially symmetric) boundary value problem for the equation system

$$-\frac{1}{\mu}\Delta\vec{A}^0 + \sigma^0\Big(\frac{\partial\vec{A}^0}{\partial t} + \operatorname{grad} V^0\Big) = \vec{J}^0, \tag{29}$$

$$-\operatorname{div}(\sigma^0 \operatorname{grad} V^0) - \operatorname{div}\Big(\sigma^0\frac{\partial\vec{A}^0}{\partial t}\Big) = F^0, \tag{30}$$

and solution of three-dimensional boundary value problem for the system of equations

$$-\frac{1}{\mu}\Delta\vec{A}^+ + \sigma^0\Big(\frac{\partial\vec{A}^+}{\partial t} + \operatorname{grad} V^+\Big) = -(\sigma - \sigma^0)\Big(\frac{\partial\vec{A}^0}{\partial t} + \operatorname{grad} V^0\Big) + \vec{J}^+, \tag{31}$$

$$-\operatorname{div}(\sigma \operatorname{grad} V^+) - \operatorname{div}\Big(\sigma\frac{\partial\vec{A}^+}{\partial t}\Big)$$
$$= \operatorname{div}\Big[(\sigma - \sigma^0)\frac{\partial\vec{A}^0}{\partial t}\Big] + \operatorname{div}[(\sigma - \sigma^0)\operatorname{grad} V^0] + F^+, \tag{32}$$

where

$$F^0 = -\operatorname{div}\vec{J}^0, \qquad F^+ = -\operatorname{div}\vec{J}^+. \tag{33}$$

Vector-potential A^0 and scalar potential V^0 in the equation system (31)–(32) are considered known functions, found as solution of the system of equations (29)–(30).

It is not difficult to be convinced that vector-potential $\vec{A} = \vec{A}^+ + \vec{A}^0$ and scalar potential V^0 satisfy the system of equations (8)–(9). For this purpose it is enough to combine equation (29) with (31), equation (30) – with (32) and to take into account designation (33).

The representation of the solution of the system of equations (8)–(9) as a sum of solutions of (29)–(30) and (31)–(32) gives the same advantages, as representation of the solution of (3) as a sum of solutions of equations (14) and (15). These advantages are described in detail in Subsection 3.1. The differences are only that coefficient μ is identical in equations (29) and (31) (and is equal to vacuum permeance μ_0), equations (29)–(32) do not contain medium dielectric constant ϵ (and, accordingly, $\vec{A}$ second time derivative), and the source member of (9) $F = -\operatorname{div}\vec{J}$ as well as and the vector of currents $\vec{J}$, is decomposed on two-dimensional (axially symmetric) part F^0 and on three-dimensional part F^+. Thus, in three-dimensional system of equations (31)–(32) the field $(\vec{A}^+, V^+)$ sources are located only in those places of calculation area Ω, where either σ does not coincide with two-dimensional approximation σ^0, or the sources $\vec{J}$ and $F = -\operatorname{div}\vec{J}$ cannot be precisely taken into account in two-dimensional problem for the system of equations (29)–(30).

4.2. Local and global matrix of finite element SLAE

Let us fulfill time and space approximation of equations (31)–(32) in the same way as it was made above for vector equation (11). As a result, finite element Ω_m local matrix $\tilde{p}$ will consist of block-elements $\tilde{p}_{lj}$ as

$$\tilde{p}_{lj} = \begin{pmatrix} c_{lj} & 0 & 0 & d^x_{lj} \\ 0 & c_{lj} & 0 & d^y_{lj} \\ 0 & 0 & c_{lj} & d^z_{lj} \\ d^x_{jl} & d^y_{jl} & d^z_{jl} & g_{lj}/\chi \end{pmatrix}, \tag{34}$$

where

$$c_{lj} = \frac{1}{\mu_0} \int_{\Omega_m} \operatorname{grad} \psi_j \cdot \operatorname{grad} \psi_l \, d\Omega + \int_{\Omega_m} \alpha \psi_j \psi_l \, d\Omega, \tag{35}$$

$$d^x_{lj} = \int_{\Omega_m} \sigma \frac{\partial \psi_j}{\partial x} \psi_l \, d\Omega, \quad d^y_{lj} = \int_{\Omega_m} \sigma \frac{\partial \psi_j}{\partial y} \psi_l \, d\Omega, \quad d^z_{lj} = \int_{\Omega_m} \sigma \frac{\partial \psi_j}{\partial z} \psi_l \, d\Omega, \tag{36}$$

$$g_{lj} = \int_{\Omega_m} \sigma \operatorname{grad} \psi_j \cdot \operatorname{grad} \psi_l \, d\Omega, \qquad \chi = \frac{d\eta_3(t_k)}{dt}. \tag{37}$$

It is not difficult to be convinced that $\tilde{p}_{lj} = \tilde{p}^T_{jl}$, and it means that the local matrix $\tilde{p}$ is symmetric. As a result of local matrices of all finite elements addition, the symmetric global matrix for the system of equations (31)–(32), in which equation (32) is multiplied by coefficient χ, will be received.

Let us consider a situation, when the finite element Ω_m is located in medium with $\sigma = 0$. The quantity of non-zero elements in block-element $\tilde{p}_{lj}$ of this finite element's local matrix $\tilde{p}$ is significantly reduced. Equation (32) is not determined when $\sigma = 0$. Therefore, to keep homogeneity of weight vector $\vec{q}_j = (q^x_j, q^y_j, q^z_j, q^v_j)^T$ structure, for fourth components q^v_j of vector $\vec{q}_j$ the fictitious equation with diagonal coefficient, equal to 1, and zero off-diagonal coefficients and right part can be introduced. Then the block-element can be written as

$$\tilde{p}_{lj} = \begin{pmatrix} c_{lj} & 0 & 0 & 0 \\ 0 & c_{lj} & 0 & 0 \\ 0 & 0 & c_{lj} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{38}$$

Thus, to store the whole block $\tilde{p}_{lj}$ only one computer memory location, containing value c_{lj}, is necessary.

Let us calculate quantity of computer memory locations, necessary for storing of the global SLAE matrix, received as finite element approximation of the system of equations (31)–(32). Let k^σ_d be number of nodes in finite element mesh, laying inside and on boundary of area Ω subareas with $\sigma \neq 0$, and k_d^0 – number of nodes, laying inside subareas with $\sigma = 0$. Then to store only diagonal block-elements of finite element SLAE matrix $5k_d^\sigma + k_d^0$ memory locations is necessary. Similarly through k_g^σ let us designate number of off-diagonal elements in portrait [4] of one of matrix triangles (bottom or top) rows, corresponding to mesh nodes, which belong to subareas (including boundaries) with $\sigma \neq 0$, and through k_g^0 – number of off-diagonal elements in portrait of matrix bottom (or top) triangle rows, corresponding to nodes, laying inside subareas with $\sigma = 0$. Then to store all off-diagonal blocks-elements of FEM SLAE matrix $8k_g^\sigma + k_g^0$ memory locations is required. As a result, the number of memory locations, necessary for storing all non-zero elements of FEM SLAE matrix, is determined as

$$k = 5k_d^\sigma + k_d^0 + 8k_g^\sigma + k_g^0. \tag{39}$$

5. Peculiarities of the computing schemes. Computing expenses

Considered three-dimensional non-stationary electromagnetic fields calculation techniques, based on representation of vector equation (3) or the solutions of the system of equations (8)–(9) as sums of solutions of systems (10)–(11) or (29)–(32), permit to decrease computing expenses (required computer memory and calculation time) more than an order for numerical solving of important applied problems with error about several percents from solution and below. In addition, such high efficiency of offered techniques is equally typical by using both equations (3), and the system of equations (8)–(9) as original electromagnetic field mathematical model.

Let us compare models (3) and (8)–(9) on expenses of computer memory, required for solving appropriate three-dimensional boundary value problems. Let us carry out comparison for three-dimensional equations (11) and (31)–(32) solving procedures, received by using described technique of splitting of appropriate original equations (3) and (8)–(9).

We consider that boundary value problems for equations (11) and (31)–(32) are solved on the same tetrahedral mesh. The main expenses of computer memory are for storing of finite element SLAE matrices. The number of memory locations, necessary for storing of SLAE matrices, received as a result of finite element approximation of equation (11) and the system of equations (31)–(32) on tetrahedral mesh, is determined accordingly by formulas (28) and (39). When equations (11) and (31)–(32) are solved on the same tetrahedral mesh, the parameters k_g and k_d from (28) are connected to parameters k_g^σ and k_g^0 from (39) as $k_g = k_g^\sigma + k_g^0$, $k_d = k_d^\sigma + k_d^0$.

Thus, to store non-zero elements of SLAE matrix, received as a result of the system of equations (31)–(32) finite element approximation, computer memory locations are required $\delta k = k_g^\sigma + 8k_g^0 + k_d^\sigma + 5k_d^0$ less than memory locations, necessary for storing of SLAE matrix, received as a result of finite element approximation of equation (11). It is obvious, that δk can reach very essential values, if the subareas with $\sigma \neq 0$ occupy relatively not large calculation area Ω part and contain (along with their boundaries) relatively small number of finite element mesh nodes. Note, that by required computer memory volume the fact that for finite element approximation of system (31)–(32) it is necessary four unknowns (and four right part vector components) per node, while for finite element approximation of equation (11) – only three, play against this system. The volume of additional memory, necessary for storing of auxiliary arrays for solving finite element SLAE by one of the most effective iterative methods – by conjugate gradient method, is increased in the same proportions. However this loss is completely exceeded by economies of memory, required for storing of finite element SLAE matrices, if any significant number of mesh nodes (for example, even if the third or fourth their part) is inside subareas with $\sigma = 0$.

Note another very important peculiarity of equations (3) and (8)–(9) (and, accordingly, (11) and (31)–(32)). Received as a result of these equations finite element approximation, the numerical solutions differently approximate fields $\vec{B}$ and $\vec{E}$, even if the same finite element mesh is used. And at worse approximation it is necessary to subdivide calculation area Ω on more small finite elements with all following consequences – increasing of required memory and calculation time. As it will be shown below, the system (8)–(9) appears also more preferable for the problems with discontinuous σ, than equation (3) by this criterion.

The use of the system of equations (8)–(9) (and received from it (29)–(32) for splitted field) can give additional advantages in comparison with equation (3) (and received from it (10)–(11)), when it is necessary to receive potential part of electrical field. This situation can occurs, for example, when an electrical field is studied by created potential difference in two points of medium. The solution $\vec{A}$ of equation (3) permits to calculate directly only total electrical field $\vec{E} = -\partial\vec{A}/\partial t$ in any space point. The solution $(\vec{A}, V)$ of the system of equations (8)–(9) permits to calculate separately rotational $\vec{E}_r = -\partial\vec{A}/\partial t$ and potential $\vec{E}_p = -\operatorname{grad} V$ parts of total electrical field $\vec{E} = \vec{E}_r + \vec{E}_p$ at once. It is easily to be convinced that the value $-\partial\vec{A}/\partial t$ for solution of the system of equations (8)–(9) is really pure rotational part of total field $\vec{E}$, by applying operation div: since, as was shown in Section 2, equality $\operatorname{div}\vec{A} = 0$ for solution $(\vec{A}, V)$ of the system of equations (8)–(9) is valid in whole area Ω, then $\operatorname{div}(-\partial\vec{A}/\partial t) = 0$ everywhere in Ω, that indicates pure rotational nature of the field $-\partial\vec{A}/\partial t$ when the system of equations (8)–(9) is used for electromagnetic field description. Therefore the remaining part $-\operatorname{grad} V$ of total field $\vec{E}$ is in this case the complete potential part of electrical field. Using equation (3) for description of electromagnetic field, it is also possible to receive complete potential part of an electrical field, but only by solving additional boundary value problem for equation $-\operatorname{div}\operatorname{grad} V = \operatorname{div}\vec{E}$, where $\vec{E} = -\partial\vec{A}/\partial t$ and $\vec{A}$ is the solution of (3).

Note another important peculiarity of equation (3), creating additional difficulties for its numerical solving (and, accordingly, for solving of the system of equations (10)–(11), received from it), when coefficient σ is discontinuous on some internal boundary S^σ of calculation area Ω. For simplicity let us consider that the displacement currents are insignificant and they can be ignored, and for outside currents $\vec{J}$ relation $\operatorname{div}\vec{J} = 0$ is valid. Applying operation div to both parts of equation (3) and taking into account that $\vec{E} = -\partial\vec{A}/\partial t$, we receive

$$\operatorname{div}\sigma\vec{E} = 0. \tag{40}$$

From (40) directly follows that on boundary S^σ, where σ discontinuous, equality $\sigma_1 E_n^1 = \sigma_2 E_n^2$ should be satisfied, where σ_1 and E_n^1 – coefficient σ value and vector $\vec{E}$ projection E_n on normal $\vec{n}$ on the one hand of S^σ, and σ_2 and E_n^2 – values of σ and E_n in the same point of surface S^σ, but on the other hand. Thus, normal $\vec{E}$ component discontinuity on S^σ is proportional to σ discontinuity. Obviously, that the solution of the system of equations (3) potential $\vec{A}$ normal to surfaces S^σ component is also discontinuous. It means that if non-zero current's components, directed along normal to surface S^σ, present then potential cannot be determined at i finite element mesh node, laying on S^σ, only by three basic function ψ_i weight's values, appropriate to given node. These problems do not arise, when the system of equations (8)–(9) is used as mathematical model of electromagnetic field.

6. Calculation of fields of separate objects' influence in significantly three-dimensional problems

If practical problems are solved, there is the situation, when in significantly three-dimensional field it is necessary to distinguish separate object influence, and when the variation level of three-dimensional field, received from influence of researched object, can coincide or even be lower than level of approximation errors of calculation with use of mesh method (i.e., FEM or FDM). Application of considered in the given work mathematical modeling techniques permits to receive good results in such situations too.

Let us write original differential boundary value problem in the operator form:

$$Lu = f, \tag{41}$$

where u – three-dimensional field, taking into account researched object influence. Let u^0 be three-dimensional field, determined by the same sources and medium, as in problem (41), but only not taking into account researched object influence. The differential boundary value problem, determining the field u^0, can also be written in the operator form:

$$L^0 u^0 = f^0. \tag{42}$$

Let us consider that the scopes of differential-boundary operators L and L^0 are the same. Then the researched object influence field $u^+ = u - u^0$ satisfies to the operator equation

$$Lu^+ = f - f^0 - (L - L^0)u^0. \tag{43}$$

To be convinced, it is enough to subtract equation (42) from (41) and to carry out elementary transformations.

Thus, in spite of the fact that approximating problem (42) as well as the original problem (41), is three-dimensional, researched object influence field u^+ can be found with sufficiently high accuracy from problem (43) solution, if the operator L^0 and problem (42) source member f^0 are sufficiently close to operator L and to problem (41) source member f. In this case the influence of the operators difference $L - L^0$ on field u^0 is the main source of field u^+. The account of this circumstance permits to build effective meshes both for numerical solving of approximating problem (42), and for problem (43), determining researched object influence field u^+. Obviously, that for each of three-dimensional problems (42) and (43) it is possible to build appropriate finite element (or finite difference) mesh, taking into account specific character of problem to be solved. It, certainly, permits to increase field u^+ calculation accuracy much more (or significantly decrease problems (42) and (43) solving computing expenses), but, on the other hand, creates some difficulties for u^+ calculation algorithm realization. Really, for problem (43) solving, the problem (42) solution u^0 should be correctly recalculated on other three-dimensional mesh, used for problem (43) finite element (or finite difference) approximation. Significant difficulties can arise for problem (42) member $(L - L^0)u^0$ finite element (finite difference) approximation. Therefore, it is meaningful to consider other, significantly more simple for numerical realization, way of researched object influence field u^+ calculation under the condition of field u^0 significant three-dimensionality (for example, in boundary value problem for equation (11) the member $(L - L^0)u^0$ is defined by item

$$\operatorname{rot}\Big(\Big(\frac{1}{\mu^0} - \frac{1}{\mu}\Big)\operatorname{rot}\vec{A}^0\Big) - (\sigma - \sigma^0)\frac{\partial \vec{A}^0}{\partial t} - (\epsilon - \epsilon^0)\frac{\partial^2 \vec{A}^0}{\partial t^2},$$

and in boundary value problem for the system of equations (32)–(33) – by items

$$(\sigma - \sigma^0)\Big(\frac{\partial \vec{A}^0}{\partial t} + \operatorname{grad} V^0\Big)$$

and

$$\operatorname{div}\Big[(\sigma - \sigma^0)\frac{\partial \vec{A}^0}{\partial t}\Big] + \operatorname{div}\big[(\sigma - \sigma^0)\operatorname{grad} V^0\big].$$

Many researchers noticed that to distinguish separate object influence field in three-dimensional field the following way is very effective. Two problems are solved on the same mesh (finite element or finite difference): total problem taking into account influence of researched object (problem (41)) and the same problem, but without accounting of researched object influence (problem (42)). Then difference of these problems' solutions $u^+ = u - u^0$, that is field of researched object's influence, is calculated. In spite of the fact that the approximation error of either of these problems can be comparable by level with field u^+ (and even exceeded it), the calculation accuracy of field u^+ as a difference of fields u and u^0 appear rather high and quite sufficient for correct estimation of researched object's influence. This fact can be explained by interannihilation of main parts of problems (41) and (42) approximation errors when their solutions' difference is calculated. However without theoretical substantiation of this fact and understanding of its inducing reasons the researcher has not confidence in correctness of received results. The considered approaches to complex fields mathematical modeling permit not only to give a theoretical substantiation to the fact of approximation errors interannihilation in difference of the differential boundary value problems (41) and (42) solutions, but also to predict a level of possible error in field of researched object's influence and even to determine ways of its decreasing.

We rely on the fact that at successful choice of approximating problem (42) problem (43) numerical solution error can be significantly (an order and more) reduced in comparison with error of main problem (41) numerical solution by means of significant decreasing of the source members' influence in problem (43). When problems (41) and (42) are solved on the same mesh, this fact is the main reason of significant decreasing error of researched object influence field u^+ approximation, calculated as a difference of the solutions u and u^0 of problems (41) and (42), in comparison with errors of original field u or field u^0 approximation. Let us show it.

Let us consider that problems (41) and (42) are solved by FEM. As result of finite element approximation of these problems we receive two SLAEs:

$$\hat{L}\hat{u} = \hat{f}, \tag{44}$$

$$\hat{L}^0\hat{u}^0 = \hat{f}^0, \tag{45}$$

where $\hat{L}$ and $\hat{L}^0$ – finite element matrices, that are discrete analogs of differential-boundary operators L and L^0, $\hat{f}$ and $\hat{f}^0$ – FEM SLAE right part vectors, and $\hat{u}$ and $\hat{u}^0$ – vectors, that are the numerical solutions to differential-boundary value problems (41) and (42). Since the SLAE (44) and (45) are received as a result of finite element approximation of problems (41) and (42) on the same mesh, we can carry out the same operations with them, which have carried out above for operator equations (41) and (42). First, let us subtract SLAE (45) from SLAE (44), and then from left- and right-hand sides of received system subtract vector $\hat{L}^0\hat{u}^0$. After elementary conversions we receive

$$\hat{L}\hat{u}^+ = \hat{f} - \hat{f}^0 - (\hat{L} - \hat{L}^0)\hat{u}^0, \tag{46}$$

where $\hat{u}^+ = \hat{u} - \hat{u}^0$. Comparing (46) with (43) we are convinced that the vector $\hat{u}^+$ is the solution of SLAE, received as a result of differential boundary value problem (43) finite element approximation. Thus, when the same mesh is used for calculation of the numerical solutions $\hat{u}$ and $\hat{u}^0$ of differential boundary value problems (41) and (42), their difference $\hat{u}^+$ is the numerical solution to a differential boundary value problem (43) on this finite element mesh.

So, we were convinced that the approximation errors in researched object's influence field $\hat{u}^+$, received by subtraction of the numerical solutions $\hat{u}$ and $\hat{u}^0$ of boundary value problems (41) and (42), are actually approximation errors of differential boundary value problem (43) numerical solving on the same mesh, as was used for numerical solving of problems (41) and (42). This fact is of great practical importance. It permits to substantiate the following practical recommendation. To decrease approximation error in field $\hat{u}^+$, received as a difference of the numerical solutions $\hat{u}$ and $\hat{u}^0$ of problems (41) and (42), it is necessary to make local node concentration not only in places of solution u and u^0 gradients abrupt changes, but also in places of abrupt changes of gradients u^+ even in spite of that the fields u and u^0 in these places can be sufficiently smooth (field u^+ can be much lower by level than fields u and u^0, and consequently even the abrupt changes of derivative in field u^+ can weakly affect on u smoothness).

Thus, the considered two ways of calculations of separate objects influence fields, excited by significantly three-dimensional sources, have their merits and demerits. So, when field u^+ is calculated by solving of differential boundary value problem (43), two different meshes, taking into account each of these problems' singularities can be used for numerical solving of problems (42) and (43). As a result, each of problems (42) and (43) can be solved on optimum mesh with minimum number of nodes, and it frequently permits to significantly reduce required computing expenses (memory and computing time) without decreasing of field u^+ calculation accuracy. As a demerit of this way we can attribute the fact that for its application it is necessary to develop procedures of correct recalculation of the three-dimensional problem solution from one three-dimensional mesh to another, and also procedures of calculation of the contributions from operator equation (43) member $(L - L^0)u^0$ to finite element (or finite difference) SLAE right parts, which in many cases are not in the least trivial.

The second way of u^+ calculation as a difference of problems (41) and (42) solutions does not require development of any additional procedures. But the mesh, used for the numerical solution of problems (41) and (42), should simultaneously take into account both singularities of the solution u^0 of problem (42), and singularities of the problem (43) solution u^+ (though problem (43) in this case is not solved). It results in that for achievement of required accuracy in solution u^+ the number of nodes in mesh increases, that have as a consequence increasing of computing resource expenses (memory and calculation time). Therefore the choice of either way of field u^+ calculation (under the condition of field u^0 significant three-dimensionality) depends on specific situation: if it is enough computing resources to solve problems (41) and (42) on mesh, taking into account the singularity u^0 and u^+, then it is easier to calculate u^+ as a difference of the solutions u and u^0 of problems (41) and (42); but if optimum meshes for problem (42) and problems (43) are distinguished significantly, and it is not enough computing resources to achieve required accuracy, it is meaningful to find u^+ as the solution to problem (43) with all following consequences (development of procedures for solution recalculation from mesh to mesh and accounting of operators L and L^0 difference influence on function u^0 member when problem (43) solved).

7. An example of developed technique using to solve practical problem?

As an example let us consider the following typical enough problem of electromagnetic earth logging. The electromagnetic field in medium with layered structure is studied. The layers' boundaries are close to planes $z \doteq$ const, but do not coincide with them. In depth of medium a three-dimensional irregular form body is located. Figure 1 shows calculation area base plane triangulation with images of three-dimensional objects. Figure 2 displays the tetrahedral mesh section by plane, perpendicular to base section. An electromagnetic field is induced by a current in isolated circular wire (i.e., field excitation is cleanly inductive).

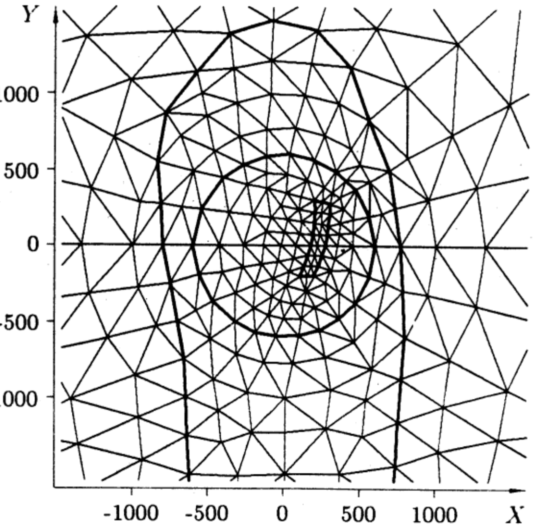

Figure 1. Base plane mesh fragment.

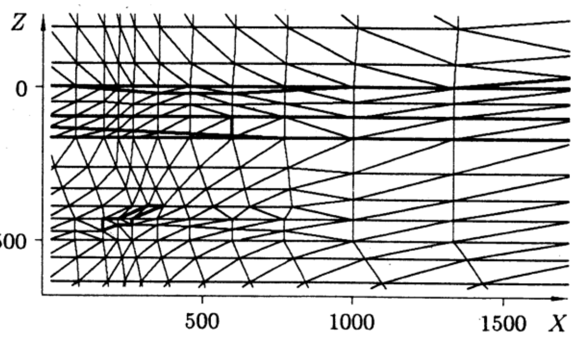

Figure 2. Fragment of tetrahedral mesh cross-section

The splitting of this problem was carried out as follows. As first an axially symmetric problem in horizontal-layered medium with outside current $\vec{J}^0 = \vec{J}$ in circular isolated wire was solved. In this case scalar potential V^0 is identically equal to zero, and vector-potential $\vec{A}^0$ in cylindrical system of coordinates has only one non-zero component A^φ, satisfying the equation

$$-\frac{1}{\mu_0}\Delta A^\varphi + \frac{A^\varphi}{\mu_0 r^2} + \sigma\frac{\partial A^\varphi}{\partial t} = J^\varphi. \tag{47}$$

The two-dimensional triangular finite element mesh, containing about 8000 nodes, was used for solving of this axially symmetric problem. The A^φ calculations were carried out on 350 time layers.

As a second the three-dimensional problem for equations (31)–(32) was solved, in which all deviations of medium layers' boundaries of original problem from axially symmetric problem layers' boundaries, and also three-dimensional object, located in medium depth were taken into account. To achieve required accuracy (error about 2–3% in $\partial B_z/\partial t$ values for required total field), three-dimensional tetrahedral mesh, containing about 15000 units, was used. The $(\vec{A}^+, V^+)$ calculations were carried out on 35 time layers.

To solve the system of equations (8)–(9) for considered problem by FEM directly (without splitting) with the same required accuracy (error about 2–3% in $\partial B_z/\partial t$ values of required field), even taking into account potentialities of TELMA software package, a tetrahedral mesh, containing more than 100000 nodes, would be required, and the calculations would be necessary to carry out on 350 time layers. It would result in increase of required memory volume about ten times in comparison with memory volume, necessary for solving equations (47) and (31)–(32) with the same total accuracy. The calculation time would increase much more significantly – not less than 100 times, since not only the FEM SLAE dimension would much increase, but also number of time layers for solving of three-dimensional boundary value problem for system (8)–(9) in comparison with solving of a three-dimensional boundary value problem for system (31)–(32) (the increasing of calculation time ten times gives ten times increasing of time layers' quantity, which should be the same as number of time layers for axially symmetric problem solving to achieve necessary time approximation accuracy for solving boundary value problems for system (8)–(9)). Such level of computing resources expenses decreasing with using considered technique is typical practically for all three-dimensional problems of electromagnetic earth logging, and also for other three-dimensional problems, having good two-dimensional (axially symmetric) approximation.

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