Bull. Nov. Comp. Center, Num. Anal., 5 (1994), 87–103

Numerical solution of elliptic problems with factorized operators*

S.B. Sorokin

The paper presents the method of constructing the difference analogs of elliptic operators based on the use of their factorized structure (for second order equations as an example). For the Poisson equations the structure of the difference operators obtained allows us to suggest a new efficient method for solving the difference problems in the domains of standard shape – the method of part-by-part inversion. The number of operations for obtaining a solution by this method coincides, in its order, with the number of operations necessary to realize conventional efficient direct methods: sweeping, fast Fourier transform, cyclic reduction technique.

Introduction

Along with certain practicaly merits, presenting equations of the elliptic kind in the operator form

$$Au = f, \tag{1}$$

does not reflect the interior structure of the operator of the problem having mainly the following form:

$$A = R^* B R. \tag{2}$$

The one

$$-\frac{d}{dx}\Big(k(x)\frac{du}{dx}\Big) = f \tag{3}$$

and two-dimensional

$$-\frac{d}{dx}\Big(k^1(x,y)\frac{du}{dx}\Big) - \frac{d}{dy}\Big(k^2(x,y)\frac{du}{dy}\Big) = f \tag{4}$$

heat conductivity equations can serve as simple examples.

*This work was supported in part by Russian Fund of Fundamental Research under Grant 93-012-494.

In the first case:

$$R = \frac{d}{dx}, \quad B = k(x), \quad R^* = -\frac{d}{dx}. \tag{5}$$

In the second case:

$$R = \begin{bmatrix} R_x \\ R_y \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}, \ B = \begin{bmatrix} k^1 & 0 \\ 0 & k^2 \end{bmatrix}, \ R^* = [R_x, R_y] = \left[-\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}\right]. \tag{6}$$

For more complicated problems of mathematical physics, the form of the operators R and B may be found, for example, in [1–3].

The factorized structure (2) of the operator A allows a realization of the following method of constructing a finite-difference approximation of equation (1):

- approximate the operator R by the operator R_h;
- select an operator adjoint to R_h, i.e., R_h^*, as an approximation of the operator R^*;
- approximate B by the operator $B_h = B_h^* > 0$.

Then the finite-dimensional analogue A_h of the operator A is of the form

$$A_h = R_h^* B_h R_h.$$

Positive features of the method are: the necessity to approximate a differential operator of less dimensionality than the dimensionality of the equation (actually, the first derivative operator), the fact that symmetry and positiveness of A_h directly follows from its form.

This paper is aimed at the illustration of this approach and description of a new efficient direct method of solving difference equations, corresponding to (3), (4).

Note that the method of solving difference equations for one-dimensional case, which is similar to the method proposed here, is described in [4]. However, in [4] only the case of constant coefficients is studied.

1. Construction of difference schemes

1.1. One-dimensional case

Consider equation (3) which in accordance with (5) is written in the form

$$R^* BRu = f, \quad x \in (a, b), \tag{1.1}$$

with the Dirichlet boundary conditions

$$u(a) = u(b) = 0. \tag{1.2}$$

In the interval (a,b) construct two grids with the mesh size $h = \frac{(b-a)}{N+1}$

$$W = \left\{x_i : \ i = \overline{0, N+1}, \quad x_i = a + ih\right\},$$
$$W_{\frac{1}{2}} = \left\{x_{i+\frac{1}{2}} : \ i = \overline{0, N}, \quad x_{i+\frac{1}{2}} = a + \left(i + \frac{1}{2}\right)h\right\}.$$

By $U = U_{N+2}$ and $V = V_{N+1}$ we mean the spaces of the grid functions given on the grids W and $W_{\frac{1}{2}}$, respectively. The subscript means the space dimension. Introduce in U and V the scalar products

$$(u^1, u^2)_U = \sum_{i=0}^{N+1} u_i^1 u_i^2 h, \quad u^1, u^2 \in U,$$
$$(v^1, v^2)_V = \sum_{i=0}^{N} v_{i+\frac{1}{2}}^1 v_{i+\frac{1}{2}}^2 h, \quad v^1, v^2 \in V.$$

Denote by R_h the operator acting from U to V according to the formula

$$(R_h u)_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{h}, \quad i = \overline{0, N}. \tag{1.3}$$

It is easy to see that the rectangular matrix with $(N+1)$ lines and $(N+2)$ columns correspond to this linear operator. We also denote this matrix as R_h:

$$R_h = \frac{1}{h} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} : \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N+1} \end{bmatrix} \longrightarrow \begin{bmatrix} v_{\frac{1}{2}} \\ v_{1+\frac{1}{2}} \\ \vdots \\ v_{N+\frac{1}{2}} \end{bmatrix}. \tag{1.4}$$

For any $u \in U$ and $v \in V$ the formula of summation by parts is valid

$$\sum_{i=0}^{N} \frac{u_{i+1} - u_i}{h} v_{i+\frac{1}{2}} h = -u_0 v_{\frac{1}{2}} h - \sum_{i=1}^{N} \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h} u_i h + u_N v_{N+\frac{1}{2}} h, \tag{1.5}$$

which is written, in terms of scalar products, as follows:

$$(R_h u, v)_V = (u, R_h^* v)_U.$$

Here the operator R_h^* acts from V to U and is of the form

$$(R_h^* v)_i = \begin{cases} -\dfrac{v_{\frac{1}{2}}}{h}, & i = 0, \\ -\dfrac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{h}, & i = \overline{1,N}, \\ \dfrac{v_{N+\frac{1}{2}}}{h}, & i = N+1. \end{cases} \tag{1.6}$$

Thus, we have constructed the operator R_h^* adjoint to R_h, the rectangular matrix with $(N+2)$ lines and $(N+1)$ columns (transposed to R_h) corresponding to it

$$R_h^* = \frac{1}{h}\begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & 1 & \ddots & \\ & & \ddots & -1 \\ & & & 1 \end{bmatrix} : \begin{bmatrix} v_{\frac{1}{2}} \\ v_{1+\frac{1}{2}} \\ \vdots \\ v_{N+\frac{1}{2}} \end{bmatrix} \longrightarrow \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N+1} \end{bmatrix}. \tag{1.7}$$

So far we have not made use of the form of the boundary conditions (1.2). Now let $\overset{\circ}{U} = \overset{\circ}{U}_N$ denote the subspace of N dimensionality (the number of internal nodes of the grid W) of the space U_{N+2} and be determined as follows:

$$\overset{\circ}{U} = \{u \in U: \ \ u_0 = u_{N+1} = 0\}.$$

Denote the narrowing of the operator R_h on $\overset{\circ}{U}$ by $\overset{\circ}{R}_h$. It is evident that $\overset{\circ}{R}_h : \overset{\circ}{U} \to V$ and the matrix with $(N+1)$ lines and N columns correspond to it

$$\overset{\circ}{R}_h = \frac{1}{h}\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix} : \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \longrightarrow \begin{bmatrix} v_{\frac{1}{2}} \\ v_{1+\frac{1}{2}} \\ \vdots \\ v_{N+\frac{1}{2}} \end{bmatrix}. \tag{1.8}$$

It directly follows from (1.5) that $\overset{\circ}{R}{}_h^* : V \to \overset{\circ}{U}$ and the matrix transposed to $\overset{\circ}{R}_h$, corresponds to this operator i.e.,

$$\overset{\circ}{R}{}_h^* = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} : \begin{bmatrix} v_{\frac{1}{2}} \\ v_{1+\frac{1}{2}} \\ \vdots \\ v_{N+\frac{1}{2}} \end{bmatrix} \longrightarrow \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}. \tag{1.9}$$

Determine the difference analogue of the multiplication operator by the function $Bu = k(x)u$, in the following manner:

$$(B_h v)_{i+\frac{1}{2}} = k_{i+\frac{1}{2}} v_{i+\frac{1}{2}}, \quad i = \overline{0, N},$$

where $k_{i+\frac{1}{2}} = k(x_{i+\frac{1}{2}})$. The diagonal matrix

$$B_h = \begin{bmatrix} k_{\frac{1}{2}} & & & \\ & k_{1+\frac{1}{2}} & & \\ & & \ddots & \\ & & & k_{N+\frac{1}{2}} \end{bmatrix} : \begin{bmatrix} v_{\frac{1}{2}} \\ v_{1+\frac{1}{2}} \\ \vdots \\ v_{N+\frac{1}{2}} \end{bmatrix} \longrightarrow \begin{bmatrix} v_{\frac{1}{2}} \\ v_{1+\frac{1}{2}} \\ \vdots \\ v_{N+\frac{1}{2}} \end{bmatrix}.$$

corresponds to operator $B_h : V \to V$.

Finally, as an approximation of the operator of problem (1.1)–(1.2), we take the expression

$$A_h = \mathring{R}_h^* B_h \mathring{R}_h. \tag{1.10}$$

Direct verification shows that

$$A_h u = -(k_{i-\frac{1}{2}} u_{\bar{x}})_x, \quad \forall u \in \mathring{U},$$

i.e., the proposed technique brings about the well-known approximation of problem (1.1)–(1.2) (see [5]).

We have considered the case of the Dirichlet boundary conditions. It is evident that this scheme can be also implemented for the Neumann boundary conditions and the case when a Dirichlet condition is given on one end of the segment (interval) while a Neumann condition is given on the other one.

1.2. Two-dimensional case

Consider equation (4), which with allowance for (6), is written as

$$R^* B R u = f, \quad (x, y) \in D = (a, b) \times (c, d). \tag{1.11}$$

Consider the Dirichlet conditions

$$U\big|_\Gamma = 0 \tag{1.12}$$

as conditions on the boundary Γ.

Construct three grids in the domain D

$$W = \Big\{(x_i, y_j) : \ x_i = a + ih_x, \ i = \overline{0, N_x + 1},$$
$$y_j = c + jh_y, \ j = \overline{0, N_y + 1}\Big\},$$

$$W_x = \Big\{(x_{i-\frac{1}{2}}, y_j): \ x_{i-\frac{1}{2}} = a + \Big(i - \frac{1}{2}\Big)h_x, \ i = \overline{1, N_x+1},$$
$$y_j = c + jh_y, \ j = \overline{0, N_y+1}\Big\},$$
$$W_y = \Big\{(x_i, y_{j-\frac{1}{2}}): \ x_i = a + ih_x, \ i = \overline{0, N_x+1},$$
$$y_{j-\frac{1}{2}} = c + \Big(j - \frac{1}{2}\Big)h_y, \ j = \overline{1, N_y+1}\Big\}.$$

Here N_x, N_y are the number of internal nodes in W along the x and y directions, respectively. Thus, $h_x = (b-a)/(N_x+1)$, $h_y = (d-c)/(N_y+1)$.

In Section 1.1 we first constructed approximation of R irrespective of the type of boundary conditions. Then the operator approximation for the Dirichlet boundary conditions was constructed from the general form. The same approach can be used for problem (1.11) as well. However, in this Section, for simplicity, we will at once take into consideration the type of the boundary conditions (1.12). Thus, let

$$\overset{\circ}{U} = \{u_{i,j}: \ i = \overline{0, N_x+1}, \ j = \overline{0, N_y+1}, \ u = 0 \text{ on } \Gamma\},$$
$$\overset{\circ}{U}_x = \{v_{i+\frac{1}{2},j}: \ i = \overline{0, N_x}, \ j = \overline{0, N_y+1}, \ v = 0 \text{ on } \Gamma\},$$
$$\overset{\circ}{U}_y = \{w_{i,j+\frac{1}{2}}: \ i = \overline{0, N_x+1}, \ j = \overline{0, N_y}, \ w = 0 \text{ on } \Gamma\},$$

$$\dim \overset{\circ}{U} = N_x N_y, \quad \dim \overset{\circ}{U}_x = (N_x+1)N_y, \quad \dim \overset{\circ}{U}_y = N_x(N_y+1),$$

mean a subspace of spaces of the grid functions given on the grids W, W_x, W_y respectively.

The scalar product will be determined as follows:

$$(u^1, u^2)_{\overset{\circ}{U}} = \sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y+1} u^1_{ij} u^2_{ij} h_x h_y, \qquad \forall\ u^1, u^2 \in \overset{\circ}{U},$$
$$(v^1, v^2)_{\overset{\circ}{U}_x} = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y+1} v^1_{i+\frac{1}{2},j} v^2_{i+\frac{1}{2},j} h_x h_y, \quad \forall\ v^1, v^2 \in \overset{\circ}{U}_x,$$
$$(w^1, w^2)_{\overset{\circ}{U}_y} = \sum_{i=0}^{N_x+1} \sum_{j=0}^{N_y} w^1_{i,j+\frac{1}{2}} w^2_{i,j+\frac{1}{2}} h_x h_y, \quad \forall\ w^1, w^2 \in \overset{\circ}{U}_y.$$

Denote, by $R_x^h : \overset{\circ}{U} \to \overset{\circ}{U}_x$ and $R_y^h : \overset{\circ}{U} \to \overset{\circ}{U}_y$, the difference operators given by formulas

$$(R_x^h u)_{i+\frac{1}{2},j} = \begin{cases} \dfrac{u_{1j}}{h_x}, & i = 0, \\ \dfrac{u_{i+1,j} - u_{ij}}{h_x}, & i = \overline{1, N_x - 1},\ j = \overline{0, N_y + 1}, \\ \dfrac{-u_{N_x,j}}{h_x}, & i = N_x, \end{cases} \tag{1.13}$$

$$(R_y^h u)_{i,j+\frac{1}{2}} = \begin{cases} \dfrac{u_{i1}}{h_y}, & j = 0, \\ \dfrac{u_{i,j+1} - u_{ij}}{h_y}, & j = \overline{1, N_y - 1},\ i = \overline{0, N_x + 1}, \\ \dfrac{-u_{i,N_y}}{h_y}, & j = N_y. \end{cases} \tag{1.14}$$

As an R-approximation, we select the operator

$$R_h = \begin{bmatrix} R_x^h \\ R_y^h \end{bmatrix} : \overset{\circ}{U} \longrightarrow \begin{bmatrix} \overset{\circ}{U}_x \\ \overset{\circ}{U}_y \end{bmatrix} = \overset{\circ}{U}_{xy}$$

acting according to the rule $R_h u = \left[\begin{smallmatrix} R_x^h u \\ R_y^h u \end{smallmatrix}\right]$. Here $\overset{\circ}{U}_{xy} = \left[\begin{smallmatrix} \overset{\circ}{U}_x \\ \overset{\circ}{U}_y \end{smallmatrix}\right]$ the space of vectors $z = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$, such that $v \in \overset{\circ}{U}_x$, $w \in \overset{\circ}{U}_y$ with the scalar product

$$(z^1, z^2)_{\overset{\circ}{U}_{xy}} = (v^1, v^2)_{\overset{\circ}{U}_x} + (w^1, w^2)_{\overset{\circ}{U}_y}, \quad z^1 = \begin{bmatrix} v^1 \\ w^1 \end{bmatrix},\ z^2 = \begin{bmatrix} v^2 \\ w^2 \end{bmatrix}.$$

Let

$$\begin{aligned} \bar{u}_j &= [u_{1j}, u_{2j}, \dots, u_{N_x j}]^T, & j &= \overline{1, N_y}, \\ \bar{v}_j &= [v_{\frac{1}{2},j}, \dots, v_{N_x+\frac{1}{2},j}]^T, & j &= \overline{1, N_y}, \\ \bar{w}_{j+\frac{1}{2}} &= [w_{ij+\frac{1}{2}}, \dots, w_{N_x j+\frac{1}{2}}]^T, & j &= \overline{0, N_y}, \end{aligned}$$

E be a unit matrix of N_x-dimension, and P be the matrix from $(N_x + 1)$ lines and N_x columns of the form

$$P = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix},$$

then the matrices

$$R_x^h = \frac{1}{h_x}\begin{bmatrix} P & & & \\ & P & & \\ & & \ddots & \\ & & & P \end{bmatrix} : \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{N_y} \end{bmatrix} \longrightarrow \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N_y} \end{bmatrix}, \qquad (1.15)$$

$$R_y^h = \frac{1}{h_y}\begin{bmatrix} E & & & \\ -E & E & & \\ & -E & \ddots & \\ & & \ddots & E \\ & & & -E \end{bmatrix} : \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{N_y} \end{bmatrix} \longrightarrow \begin{bmatrix} \bar{w}_{\frac{1}{2}} \\ \bar{w}_{1+\frac{1}{2}} \\ \vdots \\ \bar{w}_{N_y+\frac{1}{2}} \end{bmatrix} \qquad (1.16)$$

correspond to linear operators (1.13) and (1.14). The matrix R_x^h is block-diagonal, with N_y blocks on the diagonal, and the matrix R_y^h consists of (N_y+1) block lines and N_y block columns.

Here and in what follows, as in the previous section, we denote both the operator and the appropriate matrix by the same letter.

Using formulas of summation by parts (1.5), $\forall u \in \overset{\circ}{U}$ and $\forall z = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \overset{\circ}{U}_{xy}$, we have

$$(R_h u, z)_{\overset{\circ}{U}_{xy}} = (R_x^h u, v)_{\overset{\circ}{U}_y} + (R_y^h u, w)_{\overset{\circ}{U}_y} = (u, R_h^* z)_{\overset{\circ}{U}}.$$

Here $R_h^* = [(R_x^h)^*, (R_y^h)^*] : \overset{\circ}{U}_{xy} \to \overset{\circ}{U}$ is the matrix-operator adjoint to R_h, acting on the element $z = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \overset{\circ}{U}_{xy}$ by the rule $R_h^* z = [(R_x^h)^* v + (R_y^h)^* w]$, while the operators $(R_x^h)^* : \overset{\circ}{U}_x \to \overset{\circ}{U}$, $(R_y^h)^* : \overset{\circ}{U}_y \to \overset{\circ}{U}$ are adjoint to R_x^h and R_y^h. The matrices

$$(R_x^h)^* = \frac{1}{h_x}\begin{bmatrix} P^* & & & \\ & P^* & & \\ & & \ddots & \\ & & & P^* \end{bmatrix} : \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N_y} \end{bmatrix} \longrightarrow \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{N_y} \end{bmatrix}, \qquad (1.17)$$

$$(R_y^h)^* = \frac{1}{h_y}\begin{bmatrix} E & -E & & & \\ & E & -E & & \\ & & \ddots & \ddots & \\ & & & E & -E \end{bmatrix} : \begin{bmatrix} \bar{w}_{\frac{1}{2}} \\ \bar{w}_{1+\frac{1}{2}} \\ \vdots \\ \bar{w}_{N_y+\frac{1}{2}} \end{bmatrix} \longrightarrow \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{N_y} \end{bmatrix} \qquad (1.18)$$

correspond to them.

As an approximation of the operator $B = \begin{bmatrix} k^1 & 0 \\ 0 & k^2 \end{bmatrix}$, we take the operator

$$B_h = \begin{bmatrix} B_h^1 & 0 \\ 0 & B_h^2 \end{bmatrix} : \overset{\circ}{U}_{x,y} \to \overset{\circ}{U}_{x,y}$$

acting on the element $z = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \in \overset{\circ}{U}_{xy}$ according to the rule

$$B_h z = \begin{bmatrix} B_h^1 & 0 \\ 0 & B_h^2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} B_h^1 v \\ B_h^2 w \end{bmatrix}.$$

The operators $B_h^1 : \overset{\circ}{U}_x \to \overset{\circ}{U}_x$ and $B_h^2 : \overset{\circ}{U}_y \to \overset{\circ}{U}_y$ are defined as follows:

$$\begin{aligned} (B_h^1 v)_{i+\frac{1}{2},j} &= k^1_{i+\frac{1}{2},j} v_{i+\frac{1}{2},j}, \qquad i = \overline{0, N_x},\ j = \overline{1, N_y}, \\ (B_h^2 w)_{i,j+\frac{1}{2}} &= k^2_{i,j+\frac{1}{2}} w_{i,j+\frac{1}{2}}, \qquad i = \overline{1, N_x},\ j = \overline{0, N_y}, \end{aligned} \tag{1.19}$$

where $k^1_{i+\frac{1}{2},j}$, $k^2_{i,j+\frac{1}{2}}$ can be given in different ways, for example, $k^1_{i+\frac{1}{2},j} = k^1(x_{i+\frac{1}{2}}, y_i)$ and $k^2_{i,j+\frac{1}{2}} = k^2(x_i, y_{j+\frac{1}{2}})$.

If one introduces the notations

$$K_j^1 = \begin{bmatrix} k^1_{\frac{1}{2},j} & & & \\ & k^1_{\frac{3}{2},j} & & \\ & & \ddots & \\ & & & k^1_{N_x+\frac{1}{2},j} \end{bmatrix}, \qquad j = \overline{1, N_y},$$

$$K^2_{j+\frac{1}{2}} = \begin{bmatrix} k^2_{1,j+\frac{1}{2}} & & & \\ & k^2_{2,j+\frac{1}{2}} & & \\ & & \ddots & \\ & & & k^2_{N_y,j+\frac{1}{2}} \end{bmatrix}, \quad j = \overline{0, N_y},$$

then the operators (1.19) (and hence B_h) can be written in the form of block-diagonal matrices

$$B_h^1 = \begin{bmatrix} k_1^1 & & & \\ & k_2^1 & & \\ & & \ddots & \\ & & & k^1_{N_y} \end{bmatrix} : \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N_y} \end{bmatrix} \longrightarrow \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N_y} \end{bmatrix},$$

$$B_h^2 = \begin{bmatrix} k^2_{\frac{1}{2}} & & & \\ & k^2_{1+\frac{1}{2}} & & \\ & & \ddots & \\ & & & k^1_{N_y+\frac{1}{2}} \end{bmatrix} : \begin{bmatrix} \bar{w}_{\frac{1}{2}} \\ \bar{w}_{1+\frac{1}{2}} \\ \vdots \\ \bar{w}_{N_y+\frac{1}{2}} \end{bmatrix} \longrightarrow \begin{bmatrix} \bar{w}_{\frac{1}{2}} \\ \bar{w}_{1+\frac{1}{2}} \\ \vdots \\ \bar{w}_{N_y+\frac{1}{2}} \end{bmatrix}.$$

Finally, as the difference analogue of the operator of problem (1.11), (1.12), we select the operator

$$A_h = R_h^* B_h R_h.$$

Direct verification shows that

$$A_h u = -(k^1(x_{i-\frac{1}{2}}, y_j)u_{\bar{x}})_x - (k^2(x_i, y_{j-\frac{1}{2}})u_{\bar{y}})_y, \quad \forall u \in \overset{\circ}{U},$$

i.e., it coincides with one of the standard approximations [5].

2. The new direct method of solution

In this section we present a new direct method for solving difference equations, which is based on the use of a factorized structure of constructed operators and is in the part-by-part inversion of each operator being a part of the representation A_h.

2.1. One-dimensional case

In the one-dimensional case, it is necessary to find a solution of the equation

$$\overset{\circ}{R}{}_h^* B_h \overset{\circ}{R}_h u = f, \tag{2.1}$$

where $\overset{\circ}{R}_h$ and $\overset{\circ}{R}{}_h^*$ are determined by the equalities (1.8), (1.9).

Before the presentation of the method, let us formulate two statements, whose proofs are trivial.

Statement 1. *The kernel of the matrix $\overset{\circ}{R}_h$, i.e., the set of vectors satisfying the equation $\overset{\circ}{R}_h \psi = 0$, consists of a zero element.*

Statement 2. *The kernel $\overset{\circ}{R}{}_h^*$, consists of vectors of the form*

$$\text{const}(1, 1, \ldots, 1)^T.$$

The factorized form of operator allows us to solve problem (2.1) in three stages.

Stage 1. Find the solution v of the equation

$$\overset{\circ}{R}{}_h^* v = f. \tag{2.2}$$

The solution (2.2) exists if and only if the right-hand side is orthogonal to the kernel of the adjoint operator $(\overset{\circ}{R}{}_h^*)^* = \overset{\circ}{R}_h$. Since the kernel $\overset{\circ}{R}_h$ consists only of a zero element, (2.2) is solvable for any f.

The solution of equation (2.2) is non-unique and is found within an element from the kernel $\overset{\circ}{R}{}_h^*$. Let v_0 be a solution to (2.2), then $v_\alpha = v_0 + \alpha\varphi$, $\varphi = (1, 1, \ldots, 1)^T$ for any α also solution (2.2). From all possible v_α we select such that

$$(B_h^{-1} v_\alpha, \varphi) = 0 \tag{2.3}$$

be fulfilled, i.e., we select α from the relation

$$\alpha = -\frac{(B_h^{-1} v_0, \varphi)}{(B_h^{-1} \varphi, \varphi)}.$$

This selection will be clear when describing the third stage.

Stage 2. Solve a system with a diagonal non-singular matrix B_h

$$B_h w = v_\alpha. \tag{2.4}$$

Stage 3. Find a solution of the system of linear algebraic equations

$$\overset{\circ}{R}_h u = w. \tag{2.5}$$

The system (2.5) is solvable not for all right-hand sides w, but only for those which are orthogonal to the kernel $\overset{\circ}{R}{}_h^*$. Since from (2.4) $w = B_h^{-1} v_\alpha$ and α was selected from the relation (2.3), then $(w, \varphi) = 0$ and the system (2.5) is solvable. Besides, the solution (2.5) is unique, as the kernel $\overset{\circ}{R}_h$ is empty.

It is easy to check that the solution u obtained at the third stage is the solution of the initial system (2.1). To obtain it, it is necessary as in the sweeping method, to make about $8N$ arithmetic operations. However, if the sweeping method includes $3N$ miltiplication operations, $2N$ additions and $3N$ additions, the method proposed here requires N multiplication operations, $2N$ divisions and $5N$ additions.

It is interesting to note that in the case when for problem (1.1) on one end of the segment the Neumann condition is given and the Dirichlet condition is given on the other end, then for obtaining a solution, only N divisions and $2N$ additions are required. This is due to the fact that in this case all matrices in the factorized presentation of A_h are quadratic and non-singular.

2.2. Two-dimensional case

In this part of the paper we will describe a new efficient direct method of solving difference equations approximating the Poisson equation to the rectangular ones. In this case, B_h is a unit matrix and it is necessary to find a solution of the equation

$$R_h^* R_h u = f. \tag{2.6}$$

To make further considerations clearer, let us present the matrices R_h and R_h^* in the form

$$R_h = \overset{N_y}{\left[\begin{array}{cccc} rP & & & \\ & rP & & \\ & & \ddots & \\ & & & rP \\ \hline sE & & & \\ -sE & sE & & \\ & -sE & \ddots & \\ & & \ddots & sE \\ & & & -sE \end{array}\right]} \begin{array}{c} N_y \\ \\ N_y+1 \end{array} : \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{N_y} \end{bmatrix} \to \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_{N_y} \\ \bar{w}_{\frac{1}{2}} \\ \bar{w}_{1+\frac{1}{2}} \\ \vdots \\ \bar{w}_{N_y+\frac{1}{2}} \end{bmatrix}. \tag{2.7}$$

In (2.7), there are given dimensions of the marked blocks, $r = \frac{1}{h_x}$ and $s = \frac{1}{h_y}$, the matrix P and the vectors $\bar{u}_j$, $j = \overline{1, N_y}$, $\bar{v}_j$, $j = \overline{1, N_y}$, $\bar{w}_{j+\frac{1}{2}}$, $j = \overline{0, N_y}$ is desribed in Section 1.2. The matrix R_h contains $2N_y + 1$ block lines and N_y block columns, or $(N_y + 1)N_x + (N_x + 1)N_y$ lines and $N_x N_y$ columns.

The matrix R_h^* is obtained by the transposition of R_h and is of the form

$$R_h^* = \left[\begin{array}{ccc|ccccc} rP^* & & & sE & -sE & & & \\ & \ddots & & & \ddots & \ddots & & \\ & & rP^* & & & & sE & -sE \end{array}\right] : \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_{N_y} \\ \bar{w}_{\frac{1}{2}} \\ \vdots \\ \bar{w}_{N_y+\frac{1}{2}} \end{bmatrix} \to \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{N_y} \end{bmatrix}. \tag{2.8}$$

It is proposed to solve equation (2.6) in the following manner. First, by introdusing the notation $v = R_h u$, find a solution of the equation

$$R_h^* v = f \tag{2.9}$$

and then, with v given, reconstruct u from the equation

$$R_h u = v. \tag{2.10}$$

The aspects of existence and uniqueness of solutions of the systems (2.9), (2.10) are determined by features of the kernels R_h and R_h^*. Now we will turn to the elucidation of these features and then describe the proposed algorithm of the solution (2.6) in more detail.

It is easy to show that the kernel of the matrix R_h consists only of a zero element. Our main task will be to describe the kernel of the matrix R_h^*.

Consider the grid vector-function $u^{kn} = \begin{bmatrix} u_1^{kn} \\ u_2^{kn} \end{bmatrix} \in \overset{\circ}{U}_{xy}$. The appropriate vector of dimension $M = (N_y + 1)N_x + (N_x + 1)N_y$ will be also denoted by u^{kn}.

Determine u_1^{kn} and u_2^{kn} in the following way:

$$(u_1^{kn})_{i+\frac{1}{2},j} = c_1 \cos\frac{k\pi(i+\frac{1}{2})}{(N_x+1)} \sin\frac{n\pi j}{(N_y+1)}, \quad i = \overline{0, N_x}, \quad j = \overline{1, N_y},$$

$$(u_2^{kn})_{i,j+\frac{1}{2}} = c_2 \sin\frac{k\pi i}{(N_x+1)} \cos\frac{n\pi(j+\frac{1}{2})}{(N_y+1)}, \quad i = \overline{1, N_x}, \quad j = \overline{0, N_y}.$$

Then, using elementary formulas, obtain that $(R_h^* u^{kn})_{ij}$ is equal

$$\left[c_1 \frac{1}{h_x} \sin\frac{k\pi}{2(N_x+1)} + c_2 \frac{1}{h_y} \sin\frac{n\pi}{2(N_y+1)}\right] 2\sin\frac{k\pi i}{(N_x+1)} \sin\frac{n\pi j}{(N_y+1)}.$$

Having selected, for example, $c_1 = \frac{1}{h_y} \sin\frac{n\pi}{2(N_y+1)}$ and $c_2 = -\frac{1}{h_x} \sin\frac{k\pi}{2(N_x+1)}$, obtain $(R_n^* u^{kn})_{ij} = 0$, $\forall i, j$. Hence for all $k = \overline{0, N_x}$ and $n = \overline{0, N_x}$ $(k + n \neq 0)$ the vector u^{kn} belongs to the kernel R_h^*. On the whole, there will be $(N_x + 1)(N_y + 1) - 1 = N$ vectors U^{kn}. It is not difficult to note that their number is equal to the dimension of the kernel R_h^*.

Having made trivial calculations and using the relation from [5], it is possible to obtain that $(u^{kn}, u^{lm})_{\overset{\circ}{U}_{xy}} = \delta_{kl}\delta_{nm}\|u^{kn}\|^2_{\overset{\circ}{U}_{xy}}$,

$$\|u^{kn}\|^2_{\overset{\circ}{U}_{xy}} = \frac{(b-a)(d-c)}{4}\left[\frac{1}{h_y^2}\sin^2\frac{n\pi}{2(N_y+1)} + \frac{1}{h_x^2}\sin^2\frac{k\pi}{2(N_x+1)}\right].$$

Here δ_{kl} is the Kronecker symbol.

Thus, N pair wise orthogonal elements of the kernel R_h^* are found, whose dimension is also equal to N. Hence, the vectors u^{kn} form the basis of the kernel of the matrix R_h^*.

Now let us describe in detail the method of part-by-part inversion for the solution of problem (2.6)

$$R_h^* R_h u = f.$$

Stage 1. Find the solution of (2.9)

$$R_h^* v = f.$$

Since the kernel of the matrix R_h transposed to R_h^* consists only of zero element, the system (2.9) is solvable for any right-hand side of f. The solution of (2.9) is found within an element from the kernel R_h^*. Let v_0 be some particular solution (2.9), then

$$v_\alpha = v_0 + \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} \alpha_{ij} u^{ij} \tag{2.11}$$

for any numbers α_{ij} is also the solution of (2.9).

A particular solution of (2.9) will be obtained as follows. Let us give by zero N components of the sought for the vector

$$v_0 = \left[\bar{v}_1^T, \bar{v}_2^T, \ldots, \bar{v}_{N_y}^T, \bar{w}_{\frac{1}{2}}^T, \ldots, \bar{w}_{N_y+\frac{1}{2}}^T\right]^T,$$

that is the component $v_{\frac{1}{2}1}, v_{\frac{1}{2}2}, \ldots, v_{\frac{1}{2}N_y}, \bar{w}_{\frac{1}{2}}, \ldots, \bar{w}_{N_y+\frac{1}{2}}$. The remaining $N_x N_y$ components of the vector v_0 satisfying (2.9), are uniquely defined from the equation

$$\begin{bmatrix} r\tilde{P}^* & & & \\ & r\tilde{P}^* & & \\ & & \ddots & \\ & & & r\tilde{P}^* \end{bmatrix} \begin{bmatrix} v_{\frac{3}{2},1} \\ \vdots \\ v_{N_x+\frac{1}{2},1} \\ \vdots \\ v_{\frac{3}{2},N_y} \\ \vdots \\ v_{N_x+\frac{1}{2},N_y} \end{bmatrix} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ f_{N_y} \end{bmatrix}. \tag{2.12}$$

where $\tilde{P}^*$ is the quadratic matrix of the form

$$\tilde{P}^* = \begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix}.$$

Stage 2. From all possible (2.11) of the solutions v_α of problem (2.9) select the one satisfying the relations

$$(v_\alpha, u^{kn})_{\overset{\circ}{U}_{xy}} = 0, \quad k = \overline{0, N_x}, \quad n = \overline{0, N_y}. \tag{2.13}$$

Due to orthogonality of the basis u^{kn}, from (2.13) it follows that

$$\alpha_{kn} = -\frac{(v_0, u^{kn})_{\overset{\circ}{U}_{xy}}}{(u^{kn}, u^{kn})_{\overset{\circ}{U}_{xy}}}, \quad k = \overline{0, N_x}, \quad n = \overline{0, N_y}.$$

Stage 3. Find the solution of the system of linear equations (2.10)

$$R_h u = v_\alpha.$$

Since v_α is selected from the condition of orthogonality (2.13) to the kernel R_h^*, (2.10) is solvable. The solution (2.10) is unique, since the kernel R_h is empty.

Taking into account the form of R_h, the solution (2.10) can be derived from the equation

$$R_x^h u = (v_\alpha)_1, \tag{2.14}$$

where R_x^h are the first N block lines of the matrix $R_h = \begin{bmatrix} R_x^h \\ R_y^h \end{bmatrix}$ and $(v_\alpha)_1$ are the first of $(N_x + 1)N_y$ components of the vector $v_\alpha = \begin{bmatrix} (v_\alpha)_1 \\ (v_\alpha)_2 \end{bmatrix}$ (see (2.7)).

It is easy to check that the solution obtained at Stage 3 is the solution of the original problem (2.6). Evaluate the number of operations required for its obtaining.

Realization of Stages 1 and 3 requires the execution of $2N_xN_y$ addition operations, here a special form of the systems (2.12), (2.14) provides the possibility of parallelization.

At Stage 2 it is necessary to calculate

$$\alpha_{kn} = -\frac{(v_0, u^{kn})_{\overset{\circ}{U}_{xy}}}{(u^{kn}, u^{kn})_{\overset{\circ}{U}_{xy}}}, \quad k = \overline{0, N_x}, \quad n = \overline{0, N_y}.$$

The value $(u^{kn}, u^{kn})_{\overset{\circ}{U}_{xy}}$ is presented above. Due to the special choice of v_0, at the first stage:

$$(v_0, u^{kn})_{\overset{\circ}{U}_{xy}} = ((v_0)_1, u_1^{kn})_{\overset{\circ}{U}_{xy}} =$$
$$\frac{1}{h_y} \sin \frac{n\pi}{2(N_y+1)} \sum_{j=1}^{N_y} \left(\sum_{i=1}^{N_x} v_{i+\frac{1}{2},j} \cos \frac{n\pi(i+\frac{1}{2})}{(N_x+1)} h_x \right) \sin \frac{n\pi j}{(N_y+1)} h_y. \tag{2.15}$$

After α_{kn} are calculated, it is necessary to construct the vector v_α by formula (2.11). Since at the third stage it is sufficient to the solve system (2.14), only $(v_\alpha)_1$ is to be determined.

It follows from the said above that it is sufficient to calculate the components of the vector $\sum\limits_{k=0}^{N_x} \sum\limits_{n=0}^{N_y} \alpha_{kn} u_1^{kn}$, that is the values

$$\left[\sum_{n=0}^{N_y} \left(\sum_{k=0}^{N_x} \alpha_{kn} \cos \frac{k\pi(i+\frac{1}{2})}{(N_x+1)} \right) \sin \frac{n\pi j}{(N_x+1)} \right] \frac{1}{h_y} \sin \frac{n\pi}{2(N_y+1)}. \tag{2.16}$$

It is easy to see [5] that the cost of arithmetic operations to fulfil of formulas (2.15)–(2.16) are exactly equal to the cost required to implement the method of separation of variables (expansion in the double series) and are of the order:

$3N_xN_y \ln N_xN_y$ – addition and substraction operations,

$N_xN_y \ln N_xN_y$ – multiplication operations.

As a result, if one does not distinguish between arithmetic operations then the proposed method will take about $N_xN_y \ln(N_xN_y)$ arithmetic operations.

The following remark should be made about the algorithm stability to round-off errors. Numerical implementation of Stages 1 and 3 in fact reduces to calculations by formulas of the form

$$x_{i+1} = x_i + b_i,$$

where b_i are given, and x_i are calculated. It is evident that in calculating x_k, the error does not exceed $k\varepsilon$, where ε is determined by the stability of calculations at Stage 2, which is implementated with the help of the well examined algorithm.

To conclude, let us note that the proposed algorithm is generalized to the problems of greater dimension and to the other types of boundary conditions.

Acknowledgements. I wish to thank professor A.N. Konovalov for his interest, continuous support and contribution to this paper. He gives attention to principal capability of constructing the difference schemes for elliptic operators with help the method realized here.

References

[1] M.M. Bukenov and Yu.A. Kuznetsov, On a spectral problem of elasticity theory, Comp. Cent. Sib. Branch, USSR Acad. Sci., Preprint No 81, Novosibirsk, 1981 (in Russian).

[2] A.M. Konovalov and S.B. Sorokin, Structure of elasticity theory equations static problem, Comp. Cent. Sib. Branch, USSR Acad. Sci., Preprint No. 665, Novosibirsk, 1986 (in Russian).

[3] G. Strang, A framework for equilibrium equations, SIAM Review, Vol. 30, No 2, June 1988.

[4] D.J. Evans, An algorithm for the solution of certain threediagonal systems of linear equations, Com. J., No 15, 1972, 356–389.

[5] A.A. Samarsky and E.S. Nikolaev, Methods of solving grid equations, Nauka, Moscow, 1978 (in Russian).