7. Vector Splines

7.1. Characterization of Variational Vector Spline Functions

7.1.1. Direct Sum of Semi-Hilbert Spaces

Let $X_1, ..., X_n$ be arbitrary Hilbert spaces. The direct sum of the Hilbert spaces $X = \oplus_{i=1}^n X_i$ consists of vectors $u = (u_1, ..., u_n)$, $u_i \in X_i$, $i = 1, ..., n$ and is a Hilbert space with respect to the scalar product

$$(u, v)_X = \sum_{i=1}^{n} (u_i, v_i)_{X_i}. \tag{7.1}$$

Lemma 7.1. Any linear continuous functional $L \in X^*$ is of the form

$$L(u) = L_1(u_1) + ... + L_n(u_n) \tag{7.2}$$

where $L_i \in X_i^*$. The representation of the functional L in the form of sum (7.2) is uniquely defined.

Proof. Let us introduce the reproducing mappings $\pi : X^* \to X$, $\pi_i : X_i^* \to X_i$, $i = 1, ..., n$. Then (7.2) is implied by the equalities

$$L(u) = (\pi(L), u)_X = \sum_{i=1}^{n} (\pi(L)_i, u_i)_{X_i} = \sum_{i=1}^{n} \pi_i^{-1}(\pi(L)_i)(u_i)$$

where $\pi(L) = (\pi(L)_1, ..., \pi(L)_n)$ is an element of the direct sum. The uniqueness of representation (7.2) is also obvious. This completes the proof of the Lemma. □

Assume now that in $X_1, ..., X_n$ is defined the additional semi-Hilbert structure with scalar semi-products $(\cdot, \cdot)_{P_i}$, where P_i are closed subspaces in X_i, $i = 1, ..., n$. In the space X, define the additional scalar semi-product and semi-norm in the following way:

$$(u, v)_P = \sum_{i=1}^{n} (u_i, v_i)_{P_i}, \tag{7.3}$$

$$|u|_P = \left(\sum_{i=1}^{n} (u_i, u_i)_{P_i} \right)^{1/2} = \left(\sum_{i=1}^{n} |u_i|_{P_i}^2 \right)^{1/2}. \tag{7.4}$$

The space P is a direct sum $P_1 \oplus ... \oplus P_n$,, i.e. a closed subspace in $X = X_1 \oplus ... \oplus X_n$.

Theorem 7.1. The direct sum of the Hilbert spaces $X = X_1 \oplus ... \oplus X_n$ with additional scalar semi-product (7.3) and semi-norm (7.4) is a semi-Hilbert space.

Proof. Under the assumption X_i are semi-Hilbert spaces, i.e. there exist constants $c_i > 0$ such that

$$|u_i|_{P_i} \leq c_i \|u_i\|_{X_i}, \quad \forall u_i \in X_i \tag{7.5}$$

and the factor spaces X_i/P_i are Hilbert spaces with respect to scalar products

$$(u_i + P_i, v_i + P_i)_i = (u_i, v_i)_{P_i}. \tag{7.6}$$

Making use of inequalities (7.5), the definitions of the norms and scalar products in the space X, we have

$$|u|_P = \left(\sum_{i=1}^{n} |u_i|_{P_i}^2 \right)^{1/2} \leq \left(\sum_{i=1}^{n} c_i^2 \|u_i\|_{X_i}^2 \right)^{1/2} \leq \max_{i=1,\dots,n} c_i \|u\|_X.$$

The semi-norm introduced is thus majorized by the norm, and condition (2.8) is satisfied.

It is obvious that the factor space X/P is a direct sum of the spaces X_i/P_i, $i = 1, ..., n$, and by virtue of (7.3) the scalar product in the space X/P is defined similarly to (7.1):

$$(u + P, V + P)_* = \sum_{i=1}^{n} (u_i + P_i, V_i + P_i)_i.$$

It means that the space X/P is a Hilbert space as a direct sum of the Hilbert spaces. This completes the proof of the Theorem. □

Now as $X_1, ..., X_n$ choose semi-Hilbert spaces of functions with the reproducing kernels $G_1, ..., G_n$. Then the direct sum $X = \oplus_{i=1}^{n} X_i$ will be called a semi-Hilbert space of vector functions.

Theorem 7.2. Let $L_1, ..., L_n$ be representation (7.2) for the functional $L \in X^*$. Then the mapping $\pi_P : X^* \to X$:

$$\pi_P(L) = (L_1 G_1, ..., L_n G_n) \tag{7.7}$$

is reproducing for the semi-Hilbert space of vector functions of X.

Proof. It is necessary to verify equality (2.7). Let the functional $L \in X^*$ vanish on the space P. Then (7.2) implies that the functionals $L_1, ..., L_n$ vanish on the space $P_1, ..., P_n$, respectively. Then equality (2.7) is implied by the properties of the reproducing kernels $G_1, ..., G_n$:

$$L(u) = \sum_{i=1}^{n} L_i(u_i) = \sum_{i=1}^{n} (L_i G_i, u_i)_{P_i} = (\pi_P(L), u)_P.$$

□

7.1.2. Analytical Representations of Vector Spline-Functions

Let $L_1, ..., L_N$ be a linear independent set of functionals in X^*. In Chapter 2, we have introduced a variational interpolating spline as the solution to the constrained optimization problem

$$\begin{aligned} &L_i(u) = r_i, \quad i = 1, ..., N, \quad u \in X \\ &|\sigma|_P = \min |u|_P. \end{aligned} \tag{7.8}$$

The reader knows that if the space P is finite-dimensional, the functionals $L_1, ..., L_N$ form an L-set for the space P, the solution to problem (7.8) exists and is unique. Henceforth, we will assume these hypotheses to be satisfied.

In the case where X is a direct sum of semi-Hilbert subspaces of functions, the interpolating spline σ which is the solution to problem (7.8) will be called a vector spline function. Our aim is to formulate the theorem on the characterization of vector spline functions based on the fact that we know reproducing mapping (7.7).

According to (7.2) write down the expansions of functionals

$$\begin{aligned} &L_1(u) = L_{1,1}(u_1) + ... + L_{1,n}(u_n) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &L_N(u) = L_{N,1}(u_1) + ... + L_{N,n}(u_n) \end{aligned} \tag{7.9}$$

and by l_i denote the sets of functionals over X_i:

$$l_i = (L_{1,i}, ..., L_{N,i}), \quad i = 1, ..., n.$$

Despite the fact that the set of functionals $L_1, ..., L_N$ is linearly independent and forms L-set for the space P, the sets of functionals l_i may prove to be linearly dependent in the subspaces X_i and may not form L-set for the spaces P_i, $i = 1, ..., n$.

Theorem 7.3. The interpolating vector spline function σ in the semi-Hilbert space of vector functions X is of the following form:

$$\begin{aligned} &\sigma_1 = \sum_{j=1}^{N} \lambda_j L_{j,1}(G_1) + \sum_{j=1}^{M_1} c_{j,1} p_{j,1} \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\sigma_n = \sum_{j=1}^{N} \lambda_j L_{j,n}(G_n) + \sum_{j=1}^{M_n} c_{j,n} p_{j,n} \end{aligned} \tag{7.10}$$

where $p_{1,i}, ..., p_{M_i,i}$ is the basis of the space P_i, $i = 1, ..., n$. The vectors of coefficients $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_N)^T$, $\boldsymbol{c}_i = (c_{1,i}, ..., c_{M_i,i})^T$, $i = 1, ..., n$, are determined from the following system of linear algebraic equations:

$$\left[\begin{array}{c|c} \sum_{k=1}^{n}(G_k)_{l_k l_k} & (P_1)_{l_1} \dots (P_n)_{l_n} \\ \hline (P_1)_{l_1}^T & \\ \cdot & \\ \cdot & \mathbf{0} \\ \cdot & \\ (P_n)_{l_n}^T & \end{array}\right] \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{c}_1 \\ \cdot \\ \cdot \\ \boldsymbol{c}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \\ \mathbf{0} \\ \cdot \\ \cdot \\ \mathbf{0} \end{bmatrix}.$$

Here, the matrices $(G_k)_{l_k l_k}$ have the elements $g_{ij}^{(k)} = L_{i,k}(L_{j,k}G_k)$, and the matrices $(P_k)_{lk}$ have the elements $p_{ij}^{(k)} = L_{i,k}(p_{j,k})$.

Proof. Let us make use of Theorems 2.12 and 7.3. Substituting expression (7.7) into equality (2.33), we have

$$\sigma = \sum_{j=1}^{N} \lambda_j (L_{j,1}G_1, ..., L_{j,n}G_n) + \sum_{j=1}^{M} b_j p_j. \tag{7.11}$$

In the direct sum $P = P_1 \oplus ... \oplus P_n$, the basis can be chosen in the following way:

$$\begin{aligned} &p_1 = (p_{1,1}, 0, ..., 0), ..., p_{M_1} = (p_{M_1,1}, 0, ..., 0) \\ &p_{M_1+1} = (0, p_{1,2}, 0, ..., 0), ..., p_{M_1+M_2} = (0, p_{M_2,2}, 0, ...) \\ &p_{M-M_n+1} = (0, ..., 0, p_{1,n}, ..., p_M = (0, ..., 0, p_{M_n,n}). \end{aligned}$$

Then, changing the notation of the vector of coefficients $(b_1, ..., b_M)^T$ of representation (7.11) to $(c_1, ..., c_n)^T$, we obtain from (7.11) the sought form (7.10). The system of equations for determining the coefficients of vector spline function (7.10) coincides with system (2.48). This completes the proof of the Theorem. □

The following two corollaries are actually other formulations of Theorem 7.3 for the case of vector functions of two components.

Let X and Y be two semi-Hilbert functional spaces with the reproducing kernels G and H, respectively, and let P and Q be kernels of semi-norms in the spaces X and Y. Introduce sets of the functionals $\boldsymbol{v} = (v_1, ..., v_N)$ and $\boldsymbol{w} = (w_1, ..., w_N)$ in the spaces X and Y, respectively, such that the set of functionals $\boldsymbol{v}(u_1) + \boldsymbol{w}(u_2)$, $u_1 \in X$, $u_2 \in Y$ forms a linearly independent system in $X \oplus Y$ and is an L-set for $P \oplus Q$.

Corollary 7.1. Let $\boldsymbol{\delta}$ belong to $\mathbb{R}^N$, then the solution to problem

$$\begin{aligned} &\boldsymbol{v}(u_1) + \boldsymbol{w}(u_2) = \boldsymbol{\delta}, \quad u_1 \in X,\ u_2 \in Y \\ &|\sigma_1|_P^2 + |\sigma_2|_Q^2 = \min |u_1|_P^2 + |u_2|_Q^2 \end{aligned} \tag{7.12}$$

exists and is unique, and can be presented in the form

$$\begin{aligned}\sigma_1(s_1) &= \sum_{i=1}^{N} \lambda_i v_i(G(s_1,\cdot)) + \sum_{i=1}^{M_1} c_{1,i} p_i(s_1)\\ \sigma_2(s_2) &= \sum_{i=1}^{N} \lambda_i w_i(H(s_2,\cdot)) + \sum_{i=1}^{M_2} c_{2,i} q_i(s_2).\end{aligned} \tag{7.13}$$

Here, $p_1(s_1), ..., p_{M_1}(s_1)$ is the basis of the space P, and $q_1(s_2), ..., q_{M_2}(s_2)$ is the basis of the space Q. The coefficients of representation (7.13) are determined from the system of equations with the non-singular matrix:

$$\begin{bmatrix} G_{vv} + H_{ww} & P_v & Q_w \\ P_v^T & 0 & 0 \\ Q_w^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{7.14}$$

where the matrix G_{vv} consists of the elements $g_{ij} = v_i v_j(G(s_1, t_1))$, the matrix H_{ww} consists of the elements $h_{ij} = w_i w_j(H(s_2, t_2))$, the matrix P_v consists of the elements $p_{ij} = v_i(p_j(s_1))$ and the matrix Q_w consists of the elements $q_{ij} = w_i(q_j(s_2))$.

Note that the components of the spline $\sigma = (\sigma_1, \sigma_2)$ can formally be functions of different number of variables, i.e. the domains Ω_1 and Ω_2 of the functional spaces $X = X(\Omega_1)$ and $Y = Y(\Omega_2)$ can have different dimensions.

There frequently arises a situation where the derivation of the vector spline $\sigma = (\sigma_1, \sigma_2)$ brings about separate linear constraints imposed on the components σ_1 and σ_2, and there are also joint linear constraints. Let $\boldsymbol{x} = (x_1, ..., x_{N_1})$ and $\boldsymbol{z} = (z_1, ..., z_{N_3})$ be sets of functionals over the space X, while $\boldsymbol{y} = (y_1, ..., y_{N_2})$ and $\boldsymbol{t} = (t_1, ..., t_{N_3})$ be sets of functionals over the space Y.

Corollary 7.2. Let $\boldsymbol{\alpha} \in \mathbb{R}^{N_1}$, $\boldsymbol{\beta} \in \mathbb{R}^{N_2}$, $\gamma \in \mathbb{R}^{N_3}$ be vectors. Then the solution to problem

$$\begin{aligned}&\boldsymbol{x}(u_1) = \boldsymbol{\alpha}, \quad \boldsymbol{y}(u_2) = \boldsymbol{\beta}\\ &\boldsymbol{z}(u_1) + \boldsymbol{t}(u_2) = \gamma, \quad u_1 \in X, \quad u_2 \in Y\\ &|\sigma_1|_P^2 + |\sigma_2|_Q^2 = \min |u_1|_P^2 + |u_2|_Q^2\end{aligned} \tag{7.15}$$

is presentable in the form

$$\begin{aligned}\sigma_1 &= (\boldsymbol{\rho}_1, G_x) + (\boldsymbol{\kappa}, G_z) + (\boldsymbol{c}_1, P)\\ &= \sum_{i=1}^{N_1} \rho_{i,1} x_i(G) + \sum_{i=1}^{N_3} \kappa_i z_i(G) + \sum_{i=1}^{M_1} c_{i,1} p_i\\ \sigma_2 &= (\boldsymbol{\rho}_2, H_y) + (\boldsymbol{\kappa}, H_t) + (\boldsymbol{c}_2, Q)\\ &= \sum_{i=1}^{N_2} \rho_{i,2} y_i(H) + \sum_{i=1}^{N_3} \kappa_i t_i(H) + \sum_{i=1}^{M_2} c_{i,2} q_i.\end{aligned} \tag{7.16}$$

The coefficients of the expansions are determined from the following system of equations:

$$\begin{bmatrix} G_{xx} & 0 & G_{xz} & P_x & 0 \\ 0 & H_{yy} & H_{yt} & 0 & Q_y \\ G_{zx} & H_{ty} & G_{zz}+H_{tt} & P_z & Q_t \\ P_x^T & 0 & P_z^T & 0 & 0 \\ 0 & Q_y^T & Q_t^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\rho}_1 \\ \boldsymbol{\rho}_2 \\ \boldsymbol{\kappa} \\ \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{7.17}$$

Proof. Interpolation conditions (7.15) can be rewritten as follows:

$$\begin{aligned} &\boldsymbol{x}(u_1) + \mathbf{0}_{N_1}(u_2) = \alpha \\ &\mathbf{0}_{N_2}(u_1) + \boldsymbol{y}(u_2) = \beta \\ &\boldsymbol{z}(u_1) + \boldsymbol{t}(u_2) = \gamma \end{aligned} \tag{7.18}$$

Introducing the sets of functionals $\boldsymbol{v} = (\boldsymbol{x}, \mathbf{0}_{N_2}, \boldsymbol{z})$, $\boldsymbol{w} = (\mathbf{0}_{N_1}, \boldsymbol{y}, \boldsymbol{t})$ and, also, the vector $\boldsymbol{r} = (\alpha, \beta, \gamma)$, problem (7.15) can be reduced to problem (7.12). In this case, we have

$$G_{vv} = \begin{bmatrix} G_{xx} & 0 & G_{xz} \\ 0 & 0 & 0 \\ G_{zx} & 0 & G_{zz} \end{bmatrix}, \quad H_{ww} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & H_{yy} & H_{yt} \\ 0 & H_{ty} & H_{tt} \end{bmatrix}$$

$$P_v = \begin{bmatrix} P_x \\ 0 \\ P_z \end{bmatrix}, \quad Q_w = \begin{bmatrix} 0 \\ Q_y \\ Q_t \end{bmatrix}.$$

Note that the superfluous terms are removed in expansion (7.16) as compared to (7.13). The complete expansion is of the form

$$\sigma_1 = (\boldsymbol{\rho}_1, G_x) + (\boldsymbol{\rho}_2, G_{0_{N_2}}) + (\kappa, G_z) + (c_1, P),$$
$$\sigma_2 = (\boldsymbol{\rho}_1, H_{0_{N_1}}) + (\boldsymbol{\rho}_2, H_y) + (\kappa, H_t) + (c_2, Q),$$

and the coefficients $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_N)$ are determined as follows; $\boldsymbol{\lambda} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\kappa})$. This completes the proof of the corollary. □

7.1.3. Vector Splines on Subspaces

Let $E_1, ..., E_n$ be finite-dimensional subspaces in $X_1, ..., X_n$ containing the spaces $P_1, ..., P_n$, respectively.

Definition 7.1. *The function $\sigma \in E = E_1 \oplus ... \oplus E_n$ is said to be an interpolating vector spline-function on the subspace if it is the solution to constrained optimization problem as follows:*

$$\begin{aligned} &L_i u = r_i, \quad i = 1, ..., N, \quad u \in E_1 \oplus ... \oplus E_n \\ &|\sigma|_P = \min |u|_P. \end{aligned} \tag{7.19}$$

The spline in the subspace is thus the solution to minimization problem (7.8) not in the entire space X, but only in its finite-dimensional subspace. It is obvious that if the space E is equipped with the topology induced by that of the space X, spline (4.1) is a spline in the semi-Hilbert space E.

Denote by $\varphi_1, ..., \varphi_K$ the basis of the space E. Any function of the space E can be presented in form

$$u = \sum_{i=1}^{K} u_i \varphi_i = (\boldsymbol{u}, \boldsymbol{\varphi}) \tag{7.20}$$

where $\boldsymbol{u} = (u_1, ..., u_K)$ and $\boldsymbol{\varphi} = (\varphi_1, ..., \varphi_K)$. Introduce interpolating and energy matrices A and T in the following way:

$$A = \begin{bmatrix} L_1\varphi_1 & \dots & L_1\varphi_K \\ \vdots & & \vdots \\ L_N\varphi_1 & \dots & L_N\varphi_K \end{bmatrix}, \quad T = \begin{bmatrix} (\varphi_1, \varphi_1)_P & \dots & (\varphi_1\varphi_K)_P \\ \vdots & & \vdots \\ (\varphi_K\varphi_1)_P & \dots & (\varphi_K\varphi_K)_P \end{bmatrix}.$$

Then, we arrive at

Theorem 7.4. If the functionals $L_1, ..., L_N$ are linearly independent, form L-set for the space P and are non-contradictory in the subspace E, i.e. there exists at least one function from E, which satisfies the interpolation conditions, then the spline σ in the subspace exists and is unique. If its expansion in the basis is written in form

$$\sigma = \sum_{i=1}^{K} d_i \varphi_i = (\boldsymbol{d}, \boldsymbol{\varphi}), \tag{7.21}$$

the expansion coefficients are determined from the following system of linear algebraic equations:

$$\begin{bmatrix} T & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{r} \end{bmatrix}. \tag{7.22}$$

Proof. This theorem with a modification in system of equations (7.22) was proved in Chapter 4 by the Lagrange multiplier method. Here, we will give another proof based on the verification of orthogonal property.

The second group of equations $A\boldsymbol{d} = \boldsymbol{r}$ of system (7.22) represents interpolation conditions. Indeed,

$$(A\boldsymbol{d})_i = \sum_{j=1}^{K} L_i \varphi_j d_j = L_i \sum_{j=1}^{K} d_j \varphi_j = L_i \sigma.$$

The orthogonal property for the spline in the semi-Hilbert space E is as follows:

$$(\sigma, u)_P = 0 \quad \forall u \in E, \quad A\boldsymbol{u} = 0. \tag{7.23}$$

Let us show that this orthogonality is implied by the first group of equations in (7.22). To this end, rewrite condition (7.23) using the coefficients of the expansions. We have

$$(\sigma, u)_P = \left(\sum_{i=1}^{K} d_i \varphi_i, \sum_{i=1}^{K} u_i \varphi_i \right)_P = (T\boldsymbol{\sigma}, \boldsymbol{u}) = 0.$$

Whence,

$$(T\boldsymbol{\sigma}, \boldsymbol{u}) = -(\boldsymbol{\lambda}, A\boldsymbol{u})$$

and since $A\boldsymbol{u} = 0$, from the first group of equations in (7.22) we obtain the orthogonal property.

The spline σ thus satisfies the orthogonal property and interpolation conditions. Hence, it is the solution to constrained optimization problem (7.19). Note that the vector $\boldsymbol{\lambda}$ plays the role of the Lagrange multipliers.

Let us show now that the solution to problem (7.22) always exists. Indeed, the solvability in the vector $\boldsymbol{\sigma}$ is obvious, because the spline exists and is unique. The vector $\boldsymbol{\lambda}$ is determined from equations

$$A^* \boldsymbol{\lambda} = f = -T\boldsymbol{\sigma}. \tag{7.24}$$

As is known from the theory of the Fredgolm operator, the solution to (7.24) exists if the right-hand side f is orthogonal to the kernel of the matrix A. But it is just orthogonality condition (7.23). This completes the proof of the theorem. □

Let $\omega_{i,1}, ..., \omega_{i,K_i}$ be bases of spaces E_i, $i = 1, ..., n$. Then the dimension of the space $E = E_1 \oplus ... \oplus E_n$ is equal to $M = K_1 + ... + K_n$.

Theorem 7.5. Under the hypotheses of Theorem 7.4, the vector spline function $\sigma = (\sigma_1, ..., \sigma_n)$ in the subspace $E = E_1 \oplus ... \oplus E_n$ exists and is unique. If we write down its expansion in the bases of the spaces E_i in the form

$$\begin{aligned} \sigma_1 &= d_{1,1}\omega_{1,1} + ... + d_{1,K_1}\omega_{1,K_1} \\ &\cdots\cdots\cdots\cdots\cdots \\ \sigma_n &= d_{n,1}\omega_{n,1} + ... + d_{n,K_n}\omega_{n,K_n} \end{aligned} \tag{7.25}$$

the coefficients of the expansion $\boldsymbol{d}_i = (d_{i,1}, ..., d_{i,K_i})^T$, $i = 1, ..., n$, are determined from the following system of linear algebraic equations:

$$\begin{bmatrix} T_1 & & & & A_1^T \\ & \cdot & \mathbf{0} & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & \mathbf{0} & & T_n & A_n^T \\ A_1 & \cdot & \cdot & A_n & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_1 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{d}_n \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{0} \\ \boldsymbol{r} \end{bmatrix}, \tag{7.26}$$

where the matrices A_i and T_i, $i = 1, ..., n$, are as follows:

$$A_i = \begin{bmatrix} L_{1,i}\omega_{i,1} & \dots & L_{1,i}\omega_{i,K_i} \\ \vdots & & \vdots \\ L_{N,i}\omega_{i,1} & \dots & L_{N,i}\omega_{i,K_i} \end{bmatrix},$$

$$T_i = \begin{bmatrix} (\omega_{i,1},\omega_{i,1})_{P_i} & \dots & (\omega_{i,1},\omega_{i,K_i})_{P_i} \\ \vdots & & \vdots \\ (\omega_{i,K_i},\omega_{i,1})_{P_i} & \dots & (\omega_{i,K_i},\omega_{i,K_i})_{P_i} \end{bmatrix}.$$

Proof. The proof is implied by Theorem 7.4. So, (7.25) evidently follows from (7.21). To this end, choose a basis in the direct sum of subspaces $E_1 \oplus \dots \oplus E_n$ in a conventional way:

$$\begin{matrix} \begin{bmatrix} \omega_{1,1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} \omega_{1,K_1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \omega_{2,1} \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} 0 \\ \omega_{2,K_2} \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \omega_{n,1} \end{bmatrix} & \dots & \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \omega_{n,K_n} \end{bmatrix} \\ \uparrow & & \uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \varphi_1 & \dots & \varphi_{K_1} & \varphi_{K_1+1} & \dots & \varphi_{K_1+K_2} & \dots & \varphi_{K-K_n+1} & \dots & \varphi_K \end{matrix}$$

Then, taking into account the choice of the basis, the choice of scalar semi-product (7.3) and representation (7.9) of functionals, we obtain the following representations of the matrices and system (7.22):

$$T = \begin{bmatrix} T_1 & & & \\ & \cdot & \mathbf{0} & \\ & & \cdot & \\ & \mathbf{0} & & \cdot \\ & & & T_n \end{bmatrix}, \quad A = [A_1, \dots, A_n].$$

This completes the proof of the Theorem. □

7.1.4. Merging of the Analytic Splines and Splines on Subspaces

Let us have to interpolate the data by the vector function (u_1, u_2) with two components $u_1 \in X_1(\Omega_1)$, $u_2 \in X_2(\Omega_2)$. Assume that we know the reproducing kernel $G(s,t)$ of the semi-Hilbert space $(X_1(\Omega_1),\ |\cdot|_P)$, but the reproducing kernel $F(s,t)$ of the semi-Hilbert space $(X_2(\Omega_2),\ |\cdot|_Q)$ is not known. How could we solve the interpolating spline problem (7.8)? One way is to consider the vector spline on subspaces, i.e. problem (7.19). Another way is also possible, when the analytic and finite-dimensional approaches are merged.

Consider a finite-dimensional subspace E_2 in X_2 and the following problem for finding the two-component spline $\sigma = (\sigma_1, \sigma_2)$:

$$\begin{aligned} &L_i u = r_i, \quad i = 1, \dots, N, \quad u \in X_1(\Omega_1) \oplus E_2, \\ &|\sigma_1|_P^2 + |\sigma_2|_Q^2 = \min |u_1|_P^2 + |u_2|_Q^2. \end{aligned} \tag{7.27}$$

Theorem 7.6. The interpolating vector spline function σ in the semi-Hilbert space $X_1(\Omega_1)) \oplus E_2$ is represented in the following form

$$\begin{aligned}\sigma_1 &= \sum_{i=1}^{N} \lambda_i L_{i,1}(G(s,t)) + \sum_{i=1}^{M_1} c_i p_i(s),\\ \sigma_2 &= \sum_{i=1}^{K_2} d_i \omega_i,\end{aligned} \tag{7.28}$$

where M_1 is the dimension of the space P, K_2 is dimension of the space E_2. The vectors of coefficients are determined from the following system of linear algebraic equations

$$\begin{bmatrix} (G_1)_{l_1 l_1} & (P_1)_{l_1} & A_2 \\ (P_1)^T_{l_1} & 0 & 0 \\ A_2^T & 0 & -T_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{7.29}$$

Here the definition of matrices may be taken from Theorems 7.3 and 7.4.

Proof. Representation (7.28) follows from more general one (7.13). The first group of equations (7.29) represents, evidently, the interpolating conditions for functions (7.28). The second and third group present the orthogonal property. Thus, the Theorem is proved. □

7.1.5. Smoothing Vector Spline Functions

Let X be a semi-Hilbert vector space with n components. The solution to the following problem with the real positive α, $\alpha_1, ..., \alpha_n$

$$\sigma_{\alpha} = \arg\min_{u \in X} \alpha \sum_{i=1}^{n} \frac{|u_i|^2_{P_i}}{\alpha_i} + \sum_{i=1}^{N} (L_i u - r_i)^2 \tag{7.30}$$

we will call a smoothing vector spline.

Theorem 7.7. The smoothing vector spline function $\sigma_\alpha = (\sigma_1, ..., \sigma_n)$ is represented in the following form

$$\begin{aligned}\sigma_1 &= \alpha_1 \sum_{i=1}^{N} \lambda_i L_{i,1}(G_1) + \sum_{i=1}^{M_1} c_{i,1} p_{i,1}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ \sigma_n &= \alpha_n \sum_{i=1}^{N} \lambda_i L_{i,n}(G_n) + \sum_{i=1}^{M_n} c_{i,n} p_{i,n}.\end{aligned} \tag{7.31}$$

The vectors of coefficients are determined from the following system of linear algebraic equations

$$\left[\begin{array}{c|c} \sum\limits_{k=1}^{n} \alpha_k (G_k)_{l_k l_k} + \alpha I & (P_1)_{l_1} \dots (P_n)_{l_n} \\ \hline (P_1)^T_{l_1} & \\ \cdot & \\ \cdot & \mathbf{0} \\ \cdot & \\ (P_n)^T_{l_n} & \end{array}\right] \begin{bmatrix} \boldsymbol{\lambda} \\ \\ \boldsymbol{c}_1 \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{c}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \\ \\ \mathbf{0} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{0} \end{bmatrix}. \qquad (7.32)$$

Compare (7.31), (7.32) to (7.10), (7.11).

We leave the proof of this Theorem to the reader as the proof of the following

Theorem 7.8. The smoothing vector spline function $\sigma_\alpha = (\sigma_1, ..., \sigma_n)$ on the finite-dimensional subspace $E = \oplus_{i=1}^n E_i$ is represented in form (7.25), where the coefficients of the representation are determined as follows

$$\begin{bmatrix} \beta_1 T_1 + A_1^T A_1 & A_1^T A_2 & \dots & A_1^T A_n \\ A_2^T A_1 & \beta_2 T_2 + A_2^T A_2 & \dots & A_2^T A_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ A_n^T A_1 & \dots & \dots & \beta_n T_n + A_n^T A_n \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_n \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{d}_n \end{bmatrix} = \begin{bmatrix} A_1^T r \\ \cdot \\ \cdot \\ \cdot \\ A_n^T r \end{bmatrix}.$$

Here $\beta_1 = \alpha/\alpha_1, ..., \beta_n = \alpha/\alpha_n$. Other vectors and matrices are defined in Theorem 7.5.

7.2. Rational Splines

This Section follows the results of the thesis by (Rozhenko 1990).

7.2.1. Object of Interpolation

Let us assume the function f of the following form $f(P) = f_1(P)/f_2(P)$ to be interpolated, where $f_1 \in W_2^m(\Omega)$, $f_2 \in W_2^m(\Omega)$, $W_2^m(\Omega)$ is the Sobolev space. Such representation in the form of the ratio is not evidently unique. An example is the function

$$f(x) = \begin{cases} 1, & \text{if} \quad 0 < x \le 1, \\ -1, & \text{if} \quad -1 < x \le 0. \end{cases}$$

If $X_1(\Omega) = X_2(\Omega) = W_2^m[-1,1]$, then one can prove that any functions $f_1(x) = |x| \cdot x^{m-1} \cdot p(x)$, $f_2(x) = x^m \cdot p(x)$ satisfy the equality $f(x) = f_1(x)/f_2(x)$. The function $p(x) > 0$ can be arbitrary from $C^\infty[-1,1]$. Other example illustrates that the ratio function can reach the infinite values. Naturally, if $f_1(x) = 1$, $f_2(x) = x$, then $f(x) = 1/x$ has the "break" at the point 0.

Now, let $W_2^m(\Omega) \oplus W_2^m(\Omega)$ be the direct sum consisting of the pairs $[u_1, u_2]$, which is the Hilbert space with respect to the norm

$$\|u\| = \left(\|u_1\|^2_{W_2^m(\Omega)} + \|u_2\|^2_{W_2^m(\Omega)}\right)^{1/2}.$$

Definition 7.2. *The point $P \in \overline{\Omega}$ is called a regular point for the pair $[u_1, u_2]$, iff $u_1^2(P) + u_2^2(P) \neq 0$. The set of all regular points of the pair is denoted by $\Omega_{[u_1,u_2]}$.*

Definition 7.3. *We call the pair $[u_1, u_2]$ regular in the set $A(\subset \overline{\Omega})$, iff any point of the set A is a regular point for the pair $[u_1, u_2]$.*

For the previous two examples, the set of non regular points consists of only one point $\{0\}$.

7.2.2. Interpolating Rational Splines

Let $\Omega \subset R^n$ is a bounded domain with Lipshitz boundary, f_1/f_2 be a representation of the function $f \in W_2^m(\Omega)/W_2^m(\Omega)$, $m > n/2$, $\omega \subset \boldsymbol{\Omega}$ be an interpolating set. Then, one says that the ratio σ_1/σ_2 interpolates the function f on the set ω, iff

$$f_2(P)\sigma_1(P) - f_1(P)\sigma_2(P) = 0, \quad \forall P \in \omega. \tag{7.33}$$

Clearly, equalities (7.33) determine the set of linear functional restrictments

$$l_P(\sigma) = 0, \quad P \in A \tag{7.34}$$

on the Hilbert space $W_2^m(\Omega) \oplus W_2^m(\Omega)$. Thus, we can consider the vector spline function minimizing the composite energy functional

$$\begin{aligned}\Phi(u) &= (|u_1|^2_m + |u_2|^2_m)^{1/2} \\ &= \left(\int_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!}\left[(D^\alpha u_1)^2 + (D^\alpha u_2)^2\right] dP\right)^{1/2}\end{aligned} \tag{7.35}$$

under restrictions (7.34). However, the solution to such a variational problem is equal to zero. Thus, we have to add one or more functional restrictions, like

$$l(\sigma) = l(f) \tag{7.36}$$

where $l(f) \neq 0$. For example, one can take a regular point $\overline{P}$ and consider two restrictions:

$$\sigma_1(\overline{P}) = f_1(\overline{P}), \quad \sigma_2(\overline{P}) = f_2(\overline{P}).$$

Definition 7.4. *The function $\sigma_1(P)/\sigma_2(P)$ is called the interpolating rational D^m-spline for the ratio function $f = f_1/f_2$ on the mesh $A \subset \overline{\Omega}$, iff*

$$[\sigma_1, \sigma_2] = \arg \min_{[u_1,u_2]\in I_A} |u_1|_m^2 + |u_2|_m^2,$$

where

$$I_A = A^{-1}(f) \cap L^{-1}(f),$$

$$A^{-1}(f) = \{[u_1, u_2] \in W_2^m(\Omega) \oplus W_2^m(\Omega) : f_2(P)\sigma_1(P) - f_1(P)\sigma_2(P) = 0, \quad \forall P \in A\}$$

$$L^{-1}(f) = \{[u_1, u_2] \in W_2^m(\Omega) \oplus W_2^m(\Omega) : l_i(u) = l_i(f), \quad i = 1, ..., N\}$$

Theorem 7.9. If $A^{-1}(f) \cap L^{-1}(0) \cap (P_{m-1} \oplus P_{m-1}) = \{0\}$, then the interpolating rational D^m-spline exists and is unique.

Definition 7.5. *The function* $g_1(P)/g_2(P)$ *is called the limit rational* D^m-*spline for the ratio function* f_1/f_2, *iff*

$$[g_1, g_2] = \arg \min_{[u_1,u_2]\in I_\Omega} |u_1|_m^2 + |u_2|_m^2.$$

Theorem 7.10. If $f_1/f_2 \neq p_1/p_2$, where $p_1, p_2 \in P_{m-1}$, then the limit D^m-spline exist and is unique.

7.2.3. Convergence of Rational D^m-splines

Here we only formulate the results, and do not give any proofs.

Theorem 7.11. If the meshes $A_1, A_2, ...$ form a embedded condensed h-net in the domain Ω, then the interpolating D^m-splines for the ratio function f on the meshes A_i:

$$[\sigma_1^i, \sigma_2^i] = \arg \min_{[u_1,u_2]\in I_{A_i}} |u_1|_m^2 + |u_2|_m^2$$

converge to the limit spline $[g_1, g_2]$, when $i \to \infty$.

Denote by $Q = \overline{\Omega}_{[f_1,f_2]} \cap \overline{\Omega}_{[g_1,g_2]}$ the intersection of the set of regular points of the pairs $[f_1, f_2]$ and $[g_1, g_2]$.

Theorem 7.12. There a point-wise convergence takes place of the ratios $\sigma_1^i(P)/\sigma_2^i(P)$ to the ratio $f_1(P)/f_2(P)$ on the set Q.

Let $\varepsilon > 0$. Introduce $Q_\varepsilon = \{P \in Q : B(P, \varepsilon) \subset Q\}$. The set Q_ε is called the ε-interiority of the set Q. Fix a constant $M > 0$ and separate Q_ε on two sets:

$$Q_{1,\varepsilon}^M = \{P \in Q_\varepsilon : |f(P)| \leq M\},$$

$$Q_{2,\varepsilon}^M = \{P \in Q_\varepsilon : |f(P)| \leq M\}.$$

Theorem 7.13.. The ratio functions $\sigma^i = \sigma_1^i/\sigma_2^i$ uniformely converge to the function $f = f_1/f_2$ on the set $Q_{1,\varepsilon}^M$, and the function σ_i^{-1} uniformely converge to the function f^{-1} on the set $Q_{2,\varepsilon}^M$. In addition, the following asymptotic estimates take place:

$$\|D^k(\sigma - f)\|_{L^p(Q_{1,\varepsilon}^M)} + \|D^k(\sigma^{-1} - f^{-1})\|_{L^p(Q_{2,\varepsilon}^M)} = o(h^{m-k-n/2+n/p}) \quad (7.37)$$

Proof. We want to give a plan of the demonstration.

1. If $f, g \in W_2^m(\Omega)$, then $fg \in W_2^m(\Omega)$, i.e. the space $W_2^m(\Omega)$ is C^* - (Diksmiex) algebra.

2. If $f, g \in W_2^m(\Omega)$ and $\min_{p\in\overline{\Omega}} |g(P)| \geq \kappa$, then $f/g \in W_2^m(\Omega)$ and

$$\|D^m(f/g)\|_{L^2(\Omega)} \leq C/\kappa^{m+1}\|f\|_{W_2^m(\Omega)}\|g\|^m_{W_2^m(\Omega)}. \quad (7.38)$$

3. If $\omega \subset \Omega$ is an arbitrary subset, then for all $\delta > 0$ there exists $\omega_\delta \subset \Omega$, consisting of finite number of domains with the Lipshitz boundary, such that

$$\omega \subset \omega_\delta \subset B(\omega, \delta) = \left\{ \bigcup_{t\in\omega} B(t,\delta) \right\}.$$

The proof of estimates (7.37) is based on Lemma 5.5 about the Sobolev functions with condensed zeros. First note that we can prove the estimates for the difference $(\sigma - g)$ instead of $(\sigma - f)$, because $f = g$ on the set $Q_{1,\varepsilon}^M$. Since ω_δ contains the condensed mesh of zeros for $(\sigma - g)$, then by Lemma 5.5 we can have

$$\|D^k(\sigma - g)\|_{L^p(Q_{1,\varepsilon}^M} \leq \|D^k(\sigma - g)\|_{L^p(\omega_\delta)}$$
$$\leq ch^{m-k-n/2+n/p}\|D^m(\sigma - g)\|_{L^2(\omega_\delta)}.$$

Further, on the basis of (7.38) we obtain

$$\|D^m(\sigma - g)\|_{L^2(\Omega)} = \|D^m \frac{\sigma_1 g_2 - \sigma_2 g_1}{\sigma_2 g_2}\|_{L^2(\omega_\delta)}$$
$$\leq \frac{c}{\kappa^{m+1}}\|\sigma_1 g_2 - \sigma_2 g_1\|_{W_2^m(\omega_\delta)}\|\sigma_2 g_2\|^m_{W_2^m(\omega_\delta)}.$$

The expression $\|\sigma_2 g_2\|_{W_2^m(\omega_\delta)}$ is bounded, because of the convergence $\sigma_2 \to g_2$ and the fact that $W_2^m(\omega_\delta)$ is C^*-algebra and, consequently, $\sigma_2 g_2 \in W_2^m(\omega_\delta)$. The expression $\|\sigma_1 g_2 - \sigma_2 g_1\|_{W_2^m(\omega_\delta)}$ converges to zero, because

$$\|\sigma_1 g_2 - \sigma_2 g_1\|_{W_2^m(\omega_\delta)} \leq \|(\sigma_1 - g_1)g_2\|_{W_2^m(\omega_\delta)} + \|g_1(\sigma_2 - g_2)\|_{W_2^m(\omega_\delta)}.$$

Estimates for $\|D^k(\sigma^{-1} - p^{-1})\|_{L^p(Q_{2,\varepsilon}^n)}$ may be analogously proved. □

7.3. Application of Vector Spline-Functions

7.3.1. Curve Approximation by Parametric Cubic Spline

Let the curve Γ on the plane be given in the parametric form

$$\Gamma = \begin{cases} x = x(s) \\ \\ y = y(s) \end{cases}, \quad s \in [0, S].$$

Assume the parametrization to be natural, i.e. the length of the curve beetween the points $(x(S_1), y(S_1))$ and $(x(S_2), y(S_2))$ be equal to $S_2 - S_1$.

Definition. *The curvature of the curve Γ at the point $(x(s), y(s))$ is the expression $k(s) = \sqrt{x''^2(s) + y''^2(s)}$. The integral curvature of the curve Γ is the value $K(\Gamma) = \int_0^S k^2(s)ds$.*

Obviously,

$$K(x, y) = K(\Gamma) = \int_0^S x''^2(s) + y''^2(s)ds.$$

Let $(x_i, y_i) = (x(s_i), y(s_i)), \quad i = 1, ..., N$ be points of the unknown curve Γ. Introduce the problem of finding the curve with the minimal integral curvature, which lies at above-mentioned points:

$$\begin{cases} (u_1(s_1), u_2(s_i)) = (x_i, y_i), \quad i = 1, ..., N, \quad u_1, u_2 \in W_2^m[0, S], \\ K(\sigma_1, \sigma_2) = \min K(u_1, u_2). \end{cases}$$

It is easy to see that this problem is reduced to two independent spline problems:

$$\begin{cases} u_1(s_i) = x_i, \quad i = 1, ..., N \\ u_1 \in W_2^m[0, S], \\ \int_0^S \sigma_1''^2(s)ds = \min \int_0^S u_1''^2(s)ds, \end{cases} \qquad \begin{cases} u_2(s_i) = y_i, \quad i = 1, ..., N \\ u_2 \in W_2^m[0, S], \\ \int_0^S \sigma_2''^2(s)ds = \min \int_0^S u''^2(s)ds. \end{cases}$$

The latter problems are classical ones. These are the problems of cubic spline interpolation.

Now introduce the problem, where the vector spline approach is essential. Let in addition to the interpolating condition one hase to satisfy the "slope" conditions at given points, for example at the extremal points $(x(s_1), y(s_1))$ and $(x(s_N), y(s_N))$. Let the slope at the point $(x(s_1), y(s_1))$ be equal to (α_1, β_1), and the slope at the point $(x(s_N), y(s_N))$ be equal to (α_N, β_N). Then, at the extremal points we have to satisfy the equalities:

$$\alpha(x'(s_1), y'(s_1)) = (\alpha_1, \beta_1), \quad \beta(x'(s_N), y'(s_N)) = (\alpha_N, \beta_N)$$

with some constants α and β. To avoid the difficulties with the choice of the constants α, β introduce the following vector spline problem

$$\begin{cases} (u_1(s_i), u_2(s_i)) = (x_i, y_i), \quad i = 1, ..., N, \quad u_1, u_2 \in W_2^m[0, S], \\ \beta_1 u_1'(s_1) - \alpha_1 u_2'(s_1) = 0, \\ \beta_N u_1'(s_N) - \alpha_N u_2'(s_N) = 0, \\ K(\sigma_1, \sigma_2) = \min K(u_1, u_2). \end{cases}$$

Obviously the spline (σ_1, σ_2) satisfies the interpolating conditions and the "slope" conditions at the final points. This problem is not reduced to two independent problems like in the previous case, and it is possible to apply the methods of spline construction described in Section 7.1. Really, we can choose an easier way. We know that the solution to the latter problem is the pair of cubic splines. Thus, we can take the Hermite spline as basis. The interpolating conditions will give us the coefficients of expansion in a part of basic functions immediately. The remaining coefficients will be determined from the "slope" conditions and minimization principle. This way was considered in detail in (Rozhenko 1983) for a similar problem.

7.3.2. Rational Splines with the Given Derivatives

In Section 7.2, we considered the interpolation with prescribed finite or infinite values. Here we show that the described approach allows us to find the rational splines with prescribed finite derivatives.

Since $\sigma = \sigma_1/\sigma_2$, then the condition $\sigma'(x_i) = s_i$ is reduced to

$$\sigma'(x_i) = \left(\frac{\sigma_1(x_i)}{\sigma_2(x_i)}\right)' = \frac{\sigma_1'(x_i)\sigma_2(x_i) - \sigma_1(x_i)\sigma_2'(x_i)}{\sigma_2^2(x_i)} = s_i.$$

Applying the interpolating condition $\sigma_1(x_i)/\sigma_2(x_i) = r_i$ we come to $\sigma_1'(x_i) - (r_i\sigma_2'(x_i) + s_i\sigma_2(x_i)) = 0$. Thus, we can introduce the following problem

$$\begin{cases} \sigma_1(x_i) - r_i\sigma_2(x_i) = 0, \\ \sigma_1'(x_i) - (r_i\sigma_2'(x_i) + s_i\sigma_2(x_i)) = 0, \quad i = 1, ..., N, \\ l(\sigma_1, \sigma_2) = 1, \quad \sigma_1, \sigma_2 \in W_2^m[a, b], \\ \int\limits_s^b \sigma_1''^2(x) + \sigma_2''^2(x)dx = \min, \end{cases}$$

whose solution interpolates the values r_i and derivatives s_i, $i = 1, ..., N$. Here the functional l is introduced to avoid zero solution. The infinite interpolating values r_i may be satisfied in the conventional manner, but at these points we cannot give the derivatives.

7.3.3. Collocation Method for Differential Equations

The differential equation of the first order with n components

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0 \\ \boldsymbol{u}(t) = A(t)\boldsymbol{u}(t) + \boldsymbol{b}(t), \quad t \geq 0 \end{cases}$$

we propose discretization as follows:

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0 \\ \boldsymbol{u}(t_i) = A(t_i)\boldsymbol{u}(t_i) + \boldsymbol{b}(t_i), \quad 0 < t_1 < t_2 < \ldots < t_N. \end{cases}$$

and to add the minimization functional

$$|\boldsymbol{u}|_P = \left(\sum_{i=1}^{n} |u_i|_{P_i}^2 \right)^{1/2},$$

which has the reproducing mapping in an explicit form. Then, according to Section 7.1 the discretized problem may be solved with the help of a system of linear algebraic equations.

This approach is more important for the more difficult problems, when there do not exist the explicit methods like the Runge-Kutta method. Our example was only illustrative.