8. Tensor and Blending Splines

Variational formulations of interpolation and smoothing problems in tensor products of functional spaces were studied in (A.Imamov 1977; Yu.S. Zav'yalov, A.Imamov 1978) for the particular case of polynomial splines. This Chapter suggests variational formulations corresponding to the tensor product of spline interpolating and smoothing operators in the abstract real Hilbert spaces, gives convergence estimates for interpolating tensor splines and an algorithm for constructing tensor splines.

The variational formulations for tensor splines given in Section 8.3 for the tensor product of two spline operators are generalized in the last subsection to the case of $n > 2$ spline operators. The generalization of the algorithm for constructing tensor splines to the multi-dimensional case was given in (A.Yu.Bezhaev, A.J.Rozhenko 1989).

All the material is illustrated by using D^m-interpolation problems and their tensor products. Section 8.5 deals with the problem of processing the results of well measurements and gives the estimates of computer costs of the algorithm.

The first two sections being introductory are written using the monograph (W.A.Light, E.W.Cheney 1985) and Chapter 1 of this book.

8.1. Tensor Product of Spaces

8.1.1. Main Definitions

Let X and Y be the Banach spaces, X^* and Y^* be the spaces of linear continuous functionals over them, and $\mathcal{L}(X, Y)$ be the totality of linear bounded operators acting from X into Y.

Definition 8.1. *The algebraic tensor product $X \otimes Y$ is said to be a linear space of formal finite sums of the form $\sum_{i=1}^{n} x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$, $n \in N$, for which the operations of addition and multiplication by scalars are defined in a natural way, which is factorized in the subspace*

$$\left\{ \sum_{i=1}^{n} x_i \otimes y_i : \sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = 0 \quad \forall \varphi \in X^*, \quad \psi \in Y^* \right\}.$$

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Definition 8.2. *Let X, Y, U and V be the Banach spaces, $A \in \mathcal{L}(X, U)$ and $B \in \mathcal{L}(Y, V)$. Define the linear operator $A \otimes B : X \otimes Y \to U \otimes V$ by the formula*

$$A \otimes B \left(\sum_{i=1}^{n} x_i \otimes y_i \right) = \sum_{i=1}^{n} Ax_i \otimes By_i.$$

This definition is correct, i.e. a class of equivalence from $X \otimes Y$ is mapped into a class of equivalence from $U \otimes V$.

Example 8.1. Let S and T be compact Housdorff spaces, $C(S)$ and $C(T)$ be Banach spaces of continuous functions over them. The space $C(S) \otimes C(T)$ is isomorphic to a linear subspace in $C(S \times T)$. The isomorphism is of the form

$$\sum_{i=1}^{n} f_i \otimes g_i \to \sum_{i=1}^{n} f_i(s) g_i(t).$$

Under this mapping a class of equivalence from $C(S) \otimes C(T)$ is in correspondence with a function from $C(S \times T)$, and elements of this class (various equivalent formal sums) are in correspondence with different forms of this function. We associate $C(S) \otimes C(T)$ with the subspace in $C(S \times T)$, to which it is isomorphic.

Let $A \in \mathcal{L}(C(S), \mathbb{R}^N)$ be an operator of projection of the functions from $C(S)$ onto the mesh $\{s_i \in S,\ i = 1, ..., N\}$, and $B \in \mathcal{L}(C(T), \mathbb{R}^M)$ be a projector onto the mesh $\{t_j \in T,\ j = 1, ..., M\}$. Then $A \otimes B : C(S) \otimes C(T) \to \mathbb{R}^N \otimes \mathbb{R}^M$ can be identified with the operator of the projection of functions from $C(S) \otimes C(T)$ onto the mesh $\{(s_i, t_j),\ i = 1, ..., N,\ j = 1, ..., M\}$.

8.1.2. α_p- and β-norms

There are different techniques for introducing a norm on $X \otimes Y$. Consider α_p-norms, $1 \le p \le \infty$:

$$\alpha_p(z) = \inf \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \nu_p(y_1, ..., y_n),$$

where the lower bound is taken by using all representations of $z \in X \otimes Y$ in the form of the formal sums $\sum_{i=1}^{n} x_i \otimes y_i$, and

$$\nu_p(y_1, ..., y_n) = \sup \left\{ \left\| \sum_{i=1}^{n} a_i y_i \right\| : \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \le 1 \right\}.$$

For $p = \infty$, the corresponding expressions are replaced with max $\|x_i\|$ and max $|a_i|$.

Theorem 8.1. The following statements are valid:

(1) α_p-norms are cross-norms, i.e.

$\alpha_p(x \otimes y) = \|x\| \cdot \|y\| \quad \forall x \in X,\ y \in Y;$

(2) if $\quad 1 \le p_1 \le p_2 \le \infty$, then $\alpha_{p_1} \ge \alpha_{p_2}$;

(3) The operator $A \otimes B$ is continuous if the same α_p-norm is introduced on $X \otimes Y$ and $U \otimes V$.

The completion of the space $X \otimes Y$ in a certain norm α will be denoted by $X \otimes_\alpha Y$. If $\alpha = \alpha_p$, we give a simpler form $X \otimes_p Y$.

Theorem 8.2. The operator $A \otimes B$ is continuously prolonged up to the operator

$$A \otimes_p B \in \mathcal{L}(X \otimes_p Y,\ U \otimes_p V)$$

and $\|A \otimes_p B\| = \|A\| \cdot \|B\|$.

Theorem 8.3. Let X and Y be Hilbert spaces. The expression

$$\beta\left(\sum_{i=1}^{n} x_i \otimes y_i,\ \sum_{j=1}^{m} u_j \otimes \nu_j\right) = \sum_{i,j} (x_i, u_j)(y_i, \nu_j)$$

defines the scalar product on $X \otimes Y$, and the norm $\beta(z) = \sqrt{\beta(z, z)}$ induced by it coincides with α_2-cross-norm. The introduced scalar product is extended to $X \otimes_2 Y$.

Henceforth, we will write $X \otimes Y$ instead of $X \otimes_\beta Y$ and $X \otimes_2 Y$ keeping in mind that it is a Hilbert space with the scalar product β if X and Y are Hilbert spaces. The same concerns the tensor product of operators: we shall omit the subscript, which indicates the prolongation α_2- or β-norms. Let X, Y, U and V be the Hilbert spaces, $A \in \mathcal{L}(X, U), B \in \mathcal{L}(X, U), N(A)$ and $N(B)$ be the kernels of the operators A and B, while $R(A)$ and $R(B)$ be their images.

Theorem 8.4. $N(A \otimes B)$ coincides with the closure in $X \otimes Y$ of the linear space $N(A) \otimes Y + X \otimes N(B)$, and $R(A \otimes B)$ is closed in $U \otimes V$, if $R(A)$ and $R(B)$ are closed in U and V.

Let S and T be spaces of the finite measure, $L_p(S),\ L_q(T)$ be spaces of measurable p, q - summable, respectively, functions, $1 \le p, q \le \infty$. Consider a space $L_{p,q}(S \times T)$ of measurable p, q-summable functions with the norm

$$\|f\| = \left(\int_S \left(\int_T |f(s,t)|^q dt \right)^{p/q} ds \right)^{1/p}.$$

Theorem 8.5. The space $L_p(S) \otimes_p L_q(T)$ can be isomorphically imbedded to $L_{p,q}(S \times T)$. The norm of the imbedding operator is equal to unity.

8.2. Some Extracts from General Spline Theory

Let X, Y and Z be the Hilbert spaces, $A \in \mathcal{L}(X, Z)$ and $T \in \mathcal{L}(X, Y)$. Assume that $R(A)$ and $R(T)$ are closed in Z and Y, respectively. Remember that the operator $(T, A) : X \to Y \times Z$ is said to be a spline pair, if $N(A) + N(T)$ is closed and $N(A) \cap N(T) = \{0\}$.

8.2.1. Interpolation

Theorem 8.6. The following statements are equivalent:

(1) (T, A) is a spline pair;

(2) $\rho(u, \nu) = (Tu, T\nu) + (Au, A\nu)$ is a scalar product in X, which induces the norm $\rho(u) = \sqrt{\rho(u, u)}$ equivalent to the original one;

(3) for any $z \in R(A)$ there exists a unique solution to the constrained optimization problem

$$\sigma = \arg \min_{u \in A^{-1}(z)} \|Tu\|^2. \tag{8.1}$$

which is said to be an interpolating spline.

Theorem 8.7. For $\sigma \in A^{-1}(z)$ to be an interpolating spline, it is necessary and sufficient that we have

$$(T\sigma, Tu) = 0 \quad \forall u \in N(A). \tag{8.2}$$

The operator $\tilde{A} \in \mathcal{L}(X, \tilde{Z})$, where $\tilde{Z}$ is a Hilbert space, is said to be T-compatible with A if $R(\tilde{A})$ is closed in $\tilde{Z}$, $N(A) \subset N(\tilde{A})$ and $(T, \tilde{A})$ is a spline pair. Introduce in X the scalar product $\tilde{\rho}(u, \nu) = (Tu, T\nu) + (\tilde{A}u, \tilde{A}\nu)$.

Theorem 8.8. For $\sigma \in A^{-1}(z)$ to be an interpolating spline, it is necessary and sufficient that we have

$$\tilde{\rho}(\sigma, u) = 0 \quad \forall u \in N(A). \tag{8.3}$$

It is not difficult to verify the equivalence of conditions (8.2) and (8.3), which implies the statement of the theorem.

Example 8.2. (D^m-splines). Let Ω be a bounded simply connected domain in $\mathbb{R}^n$ with the Lipschitz boundary and $X = W_2^m(\Omega)$ be the Sobolev space. For a finite set of points $\omega \subset \Omega$, define the operator $A : W_2^m(\Omega) \to \mathbb{R}^{|\omega|}$ by the formula $Af = f|_\omega$. The operator A will be continuous if $m > n/2$. Let us introduce the operator $T : W_2^m(\Omega) \to [L_2(\Omega)]^K$

$$Tf = D^m f = \left\{ \sqrt{\frac{m!}{\alpha!}} D^\alpha f, \quad |\alpha| = m \right\}$$

where K is the number of different multi-indices $\alpha = (\alpha_1, ..., \alpha_n)$ such that $|\alpha| = \alpha_1 + ... + \alpha_n = m, \alpha_i \geq 0$.

Let us introduce an operator $\tilde{A}$, $\tilde{A}f = f|_{\tilde{\omega}}$, where $\tilde{\omega} \subset \omega$. It is obvious that $N(A) \subset N(\tilde{A})$. The operator $(T, \tilde{A})$ generates a spline pair if $\tilde{\omega}$ contains a set of $(n+m-1)!/n!/(m-1)!$ points, on which the problem of construction of the $(m-1)$-th degree Lagrange polynomial has a unique solution. It is obvious that if $(T, \tilde{A})$ is a spline pair, then (T, A) is a spline pair, and $\tilde{A}$ is T-compatible with A.

The norm $\tilde{\rho}$ can be defined as follows:

$$\tilde{\rho}(f) = (\|D^m f\|^2 + \|\tilde{A}f\|^2)^{1/2}$$
$$= \left(\int_{\Omega} \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha f)^2 d\Omega + \sum_{P \in \tilde{\omega}} f^2(P) \right)^{1/2}. \tag{8.4}$$

8.2.2. Smoothing

Theorem 8.9. The following statements are equivalent:

(1) (T, A) is a spline pair;

(2) for any $z \in R(A), \alpha > 0$, there exists a unique solution to problem

$$\sigma_\alpha = \arg \min_{u \in X} \alpha \|Tu\|^2 + \|Au - z\|^2, \tag{8.5}$$

which is said to be a smoothing spline.

If (T, A) is a spline pair, then $(\sqrt{\alpha}T, A)$ is a spline pair for any $\alpha > 0$. Therefore, according to Theorem 8.1 the scalar product

$$\rho_\alpha(u, \nu) = \alpha(Tu, T\nu) + (Au, A\nu)$$

induces the norm in X, which is equivalent to the original one.

Theorem 8.10. For $\sigma_\alpha \in X$ to be a smoothing spline, it is necessary and sufficient that we have

$$\rho_\alpha(\sigma_\alpha, u) = (z, Au) \quad \forall u \in X. \tag{8.6}$$

The spline interpolating operator associating the interpolating spline $\sigma \in X$ with the element $f \in X$ will be denoted by S_A. The spline smoothing operator associating the smoothing operator $\sigma_\alpha \in X$ for $\alpha > 0$ with the element $f \in X$ will be denoted by S_A^α.

Theorem 8.11. If (T, A) is a spline pair, then we have

(1) $S_A, S_A^\alpha \in \mathcal{L}(X, X)$;

(2) $N(S_A) = N(S_A^\alpha) = N(A)$;
(3) S_A is a projector, i.e. $S_A \cdot S_A = S_A$.

8.3. Variational Principle for Tensor Splines

8.3.1. Spline Pairs and Scalar Products

Let X_i, Y_i and Z_i be the Hilbert spaces, $A_i \in \mathcal{L}(X_i, Z_i)$ and $T_i \in \mathcal{L}(X_i, Y_i)$ be operators generating the spline pairs (T_i, A_i), $i = 1, 2$. Introduce spline operators S_{A_i} and $S_{A_i}^{\alpha_i}$ and consider their tensor products $S_{A_1} \otimes S_{A_2}$ and $S_{A_1}^{\alpha_1} \otimes S_{A_2}^{\alpha_2}$ acting in the Hilbert space $X_1 \otimes X_2$. By Theorem 8.2, these operators are bounded. In this section we give variational formulations of the problems of spline interpolation and smoothing in $X_1 \otimes X_2$, whose solutions are obtained by using the operators $S_A \otimes S_A$ and $S_A^\alpha \otimes S_A^\alpha$.

Let Y_i^1 and Y_i^2 be the Hilbert spaces, $C_i^1 \in \mathcal{L}(X_i, Y_i^1)$ and $C_i^2 \in \mathcal{L}(X_i, Y_i^2)$ be operators generating the spline pairs (C_i^1, C_i^2), $i = 1, 2$. By Theorem 8.1, the scalar products

$$a_i(u, \nu) = \sum_{j=1}^{2} (C_i^j u, C_i^j \nu)$$

induce norms in the spaces X_i, which are equivalent to the original ones. By Theorem 8.2, we have $C_1^k \otimes C_2^l \in \mathcal{L}(X_1 \otimes X_2, Y_1^k \otimes Y_2^l), k, l \in \{1, 2\}$, hence, we can define the bilinear form

$$a(u, \nu) = \sum_{k,l=1}^{2} (C_1^k \otimes C_2^l u, C_1^k \otimes C_2^l \nu)$$

for any $u, \nu \in X_1 \otimes X_2$. We can readily verify the following statement.

Lemma 8.1. For any $u_1, \nu_1 \in X_1$ and $u_2, \nu_2 \in X_2$ we have

$$a(u_1 \otimes u_2, \nu_1 \otimes \nu_2) = a_1(u_1, \nu_1) \cdot a_2(u_2, \nu_2).$$

The spaces X_i with the scalar products $a_i(u, \nu)$ will be denoted by X_i^a, $i = 1, 2$. Lemma 8.1 and Theorem 8.4 imply that the norm $a(u) = \sqrt{a(u, u)}$ is α_2-cross-norm on algebraic tensor product $X_1^a \otimes X_2^a$, and scalar product $a(u, \nu)$ may be extended to $X_1^a \otimes X_2^a$ making it a Hilbert space.

Theorem 8.12. The norm $a(u)$ is equivalent to the standard norm in $X_1 \otimes X_2$.

Proof. Let $J_i : X_i^a \to X_i$ be canonical imbedding operators, which are continuous as $a_i(u)$ are equivalent normalizations of the original ones. By Theorem 1.2, the operator $J_1 \otimes J_2$ belongs to $\mathcal{L}(X_1^a \otimes X_2^a, X_1 \otimes X_2)$. It is obvious that this operator is a canonical imbedding. This completes the proof of the Theorem. □

8.3.2. Variational Formulation of the Interpolation Problem

Let (T_i, A_i), $i = 1, 2$, be spline pairs and $\tilde{A}_i$ be operators T_i-compatible with A_i (see Section 8.2.1). Similarly to Section 8.3.1, construct the scalar products

$$\tilde{\rho}_i(u, \nu) = (T_i u, T_i \nu) + (\tilde{A}_i u, \tilde{A}_i \nu)$$

in the spaces X_i and the scalar product

$$\tilde{\rho}(u, \nu) = (T_1 \otimes T_2 u, T_1 \otimes T_2 \nu) + (T_1 \otimes \tilde{A}_2 u, T_2 \otimes \tilde{A}_2 \nu) \\ + (\tilde{A}_1 \otimes T_2 u, \tilde{A}_1 \otimes T_2 \nu) + (\tilde{A}_1 \otimes \tilde{A}_2 u, \tilde{A}_1 \otimes \tilde{A}_2 \nu)$$

generated by them in $X_1 \otimes X_2$.

Theorem 8.13. Let $f \in X_1 \otimes X_2$ and $z = A_1 \otimes A_2 f$. Then,

$$S_{A_1} \otimes S_{A_2} f = \arg \min_{u \in (A_1 \otimes A_2)^{-1}(z)} \|T_1 \otimes T_2 u\|^2 \\ + \|\tilde{A}_1 \otimes T_2 u\|^2 + \|T_1 \otimes \tilde{A}_2 u\|^2.$$

Proof. Let us introduce operators $\tilde{A} = \tilde{A}_1 \otimes \tilde{A}_2$ and $T = (T_1 \otimes T_2, \tilde{A}_1 \otimes T_2, T_1 \otimes \tilde{A}_2)$. By Theorem 8.4, they have closed images. Prove that:

(a) $\tilde{A}$ is an operator T-compatible with $A_1 \otimes A_2$;

(b) $\tilde{\rho}(S_{A_1} \otimes S_{A_2} f, u) = 0 \quad \forall u \in N(A_1 \otimes A_2)$;

(c) $S_{A_1} \otimes S_{A_2} f \in (A_1 \otimes A_2)^{-1}(z)$. Then making use of Theorem 8.8 we obtain the statement of this theorem.

By Theorem 8.12, the norm $\tilde{\rho}(u) = (\tilde{\rho}(u, u))^{1/2}$ is equivalent to the norm of the space $X_1 \otimes X_2$; hence, by Theorem 8.6, the operators T and $\tilde{A}$ generate a spline pair. Then the conditions $N(A_i) \subset N(\tilde{A}_i)$ and Theorem 8.4 imply that $N(A_1 \otimes A_2) \subset N(\tilde{A}_1 \otimes \tilde{A}_2)$. This completes the proof of Statement (a).

The expression $\tilde{\rho}(S_{A_1} \otimes S_{A_2} f, u)$ is continuous and linear in the components f and u. Hence, it is sufficient to show Condition (b) on the elements $f = f_1 \otimes f_2$ and $u = u_1 \otimes u_2$. By Theorem 8.4, either u_1 belongs to $N(A_1)$, or u_2 belongs to $N(A_2)$. We have

$$\tilde{\rho}((S_{A_1} \otimes S_{A_2})(f_1 \otimes f_2), u_1 \otimes u_2) = \tilde{\rho}(S_{A_1} f_1 \otimes S_{A_2} f_2, u_1 \otimes u_2) \\ = \tilde{\rho}_1(S_{A_1} f_1, u_1) \tilde{\rho}_2(S_{A_2} f_2, u_2).$$

By Theorem 8.8, one of the last co-factors vanishes. This completes the proof of Statement (b).

Let us rewrite Condition (c) in the form

$$A_1 \otimes A_2 (S_{A_1} \otimes S_{A_2} f) = A_1 \otimes A_2 f.$$

It is again sufficient to show this equality on elements of the form $f_1 \otimes f_2$, for which it is obvious by virtue of the conditions $A_1 S_{A_1} f = A_1 f$ and $A_2 S_{A_2} f = A_2 f$. This completes the proof of the Theorem. □

8.3.3. Bicubic splines

The solutions to problems of construction of natural splines

$$\begin{cases} \sigma_1(x_i) = r_i^1, \quad i = 1, ..., N1 \\ \int_a^b [\sigma_1''(x)]^2 dx = \min \end{cases}$$

and

$$\begin{cases} \sigma_2(x_j) = r_j^2, \quad j = 1, ..., N2 \\ \int_c^d [\sigma_2''(x)]^2 dy = \min \end{cases}$$

are defined by the operators $S_{A_1} \in \mathcal{L}(W_2^2[a,b], W_2^2[a,b])$ and $S_{A_2} \in \mathcal{L}(W_2^2[c,d], W_2^2[c,d])$ of spline interpolation by the operators

$$A_1 u = (u(x_1), ..., u(x_{N1})), \quad A_2 u = (u(y_1), ..., u(y_{N2})).$$

The operators T_1 and T_2 are the second differentiation operators in this case. Let $x_1 = a$, $x_{N1} = b$, $y_1 = c$ and $y_{N2} = d$. Then the operators

$$\tilde{A}_1 u = (u(a), u(b)), \quad \tilde{A}_2 u = (u(c), u(d))$$

will be T_1- and T_2-compatible with A_1 and A_2, respectively. Indeed, the expressions

$$\int_a^b u''\nu'' dx + u(a)\nu(a) + u(b)\nu(b)$$

and

$$\int_c^d u''\nu'' dy + u(c)\nu(c) + u(d)\nu(d)$$

define the scalar products in $W_2^2[a,b]$ and $W_2^2[c,d]$, which are equivalent to the original ones. The scalar productin $W_{2,2}^{2,2}(\Omega) = W_2^2[a,b] \otimes W_2^2[c,d]$, $\Omega = [a,b] \times [c,d]$ generated by them is defined as follows:

$$\int_\Omega u_{xxyy}\nu_{xxyy} d\Omega + \int_{\partial\Omega} \frac{\partial^2 u}{\partial\tau^2}\frac{\partial^2 \nu}{\partial\tau^2} d\Gamma + u(a,c)\nu(a,c)$$
$$+ u(a,d)\nu(a,d) + u(b,c)\nu(b,c) + u(b,d)\nu(b,d),$$

where τ is a vector tangible to $\partial\Omega$.

By Theorem 8.13, the operator $S_{A_1} \otimes S_{A_2}$ gives the solution to the problem

$$\sigma = \arg \min_{u \in A^{-1}(z)} \int_\Omega u_{xxyy}^2 d\Omega + \int_{\partial\Omega} \left(\frac{\partial^2 u}{\partial\tau^2}\right)^2 d\Gamma,$$

where $A = A_1 \otimes A_2$ is the projection operator of the function from $W_{2,2}^{2,2}(\Omega)$ onto the rectangular mesh $\{(x_i, y_j),\ i = 1, ..., N1,\ j = 1, ..., N2\}$.

8.3.4. Variational Formulation of the Smoothing Problem

Let $\rho_{\alpha_i}(u,\nu) = \alpha_i(T_i u, T_i \nu) + (A_i u, A_i \nu)$ be scalar products in X_i constructed according to Section 8.2.2. Let us introduce the bilinear form

$$\rho_\alpha(u,\nu) = \alpha_1\alpha_2(T_1 \otimes T_2 u, T_1 \otimes T_2 \nu) + \alpha_2(A_1 \otimes T_2 u, A_1 \otimes T_2 \nu)$$
$$+ \alpha_1(T_1 \otimes A_2 u, T_1 \otimes A_2 \nu) + (A_1 \otimes A_2 u, A_1 \otimes A_2 \nu)$$

which is the scalar product on $X_1 \otimes X_2$.

Theorem 8.14. Let $f \in X_1 \otimes X_2$, $z = A_1 \otimes A_2 f$. Then,

$$S_{A_1}^{\alpha_1} \otimes S_{A_2}^{\alpha_2} f = \arg\min_{u \in X_1 \otimes X_2} \alpha_1\alpha_2 \|T_1 \otimes T_2 u\|^2 + \alpha_2 \|A_1 \otimes T_2 u\|^2$$
$$+ \alpha_1 \|T_1 \otimes A_2 u\|^2 + \|A_1 \otimes A_2 u - z\|^2.$$

Proof. Let $T = (\sqrt{\alpha_1}T_1 \otimes \sqrt{\alpha_2}T_2, A_1 \otimes \sqrt{\alpha_2}T_2, \sqrt{\alpha_1}T_1 \otimes A_2)$. Similarly to Theorem 8.13, we can readily prove that the operator $(T, A_1 \otimes A_2)$ is a spline pair. If we prove that

$$\rho_\alpha(S_{A_1}^{\alpha_1} \otimes S_{A_2}^{\alpha_2} f, u) = (z, A_1 \otimes A_2 u) \quad \forall u \in X_1 \otimes X_2,$$

then by Theorem 8.10 we obtain the statement of this Theorem.

It is sufficient, as usual, to show the equality on elements of the form $f = f_1 \otimes f_2$ and $u = u_1 \otimes u_2$. We have

$$\rho_\alpha(S_{A_1}^{\alpha_1} \otimes S_{A_2}^{\alpha_2}(f_1 \otimes f_2), u_1 \otimes u_2) = \rho_\alpha(S_{A_1}^{\alpha_1} f_1 \otimes S_{A_2}^{\alpha_2} f_2, u_1 \otimes u_2)$$
$$= \rho_{\alpha_1}(S_{A_1}^{\alpha_1} f_1, u_1) \cdot \rho_{\alpha_2}(S_{A_2}^{\alpha_2} f_2, u_2) = (A_1 f_1, A_1 u_1) \cdot (A_2 f_2, A_2 u_2)$$
$$= (A_1 f_1 \otimes A_2 f_2, A_1 u_1 \otimes A_2 u_2) = (z, A_1 \otimes A_2 u).$$

Here, we have made use of the orthogonality conditions for smoothing splines (8.6):

$$\rho_{\alpha_i}(S_{A_i}^{\alpha_i} f, u) = (A_i f, A_i u) \quad \forall f, u \in X_i, \quad i = 1, 2.$$

This completes the proof of the Theorem. □

8.3.5. Variational Principle for n-Component Tensor Spline

Let X_i, Y_i, Z_i be Hilbert spaces, $A_i \in L(X_i, Z_i)$, $T_i \in L(X_i, Y_i)$ be operators generating a spline pair (T_i, A_i), $i = 1, ..., n$. Introduce the tensor products of spline operators

$$\otimes_{i=1}^n S_{A_i} = S_{A_1} \otimes ... \otimes S_{A_n}, \quad \otimes_{i=1}^n S_{A_i}^\alpha = S_{A_1}^{\alpha_1} \otimes ... \otimes S_{A_n}^{\alpha_n}. \tag{8.7}$$

We can define n-component tensor products in the consequent manner making use of the transitivity law $U \otimes V \otimes W = (U \otimes V) \otimes W = U \otimes (V \otimes W)$, where U, V, W are spaces or operators.

In this Section, we give variational formulations of spline interpolation and smoothing in $\otimes_{i=1}^n X_i$, solution whose are obtained by using operators (8.7).

Let us recall the definition of the direct sum of spaces $Y \oplus Z$. It consists of the pairs (y,z), $y \in Y$, $z \in Z$ and possesses the following norm and scalar product

$$\|(y,z)\|^2_{Y\oplus Z} = \|y\|^2_Y + \|z\|^2_Z,$$

$$((y_1,z_1),(y_2,z_2))_{Y\oplus Z} = (y_1,y_2)_Y + (z_1,z_2)_Z.$$

Now we can realize a spline pair (T,A) as an operator $(T,A) : X \to Y \oplus Z$, which maps an element $u \in X$ to the element $(Tu, Au) \in Y \oplus Z$.

Proposition 8.1. Let $f \in \otimes_{i=1}^n X_i$, $z = \otimes_{i=1}^n A_i f$. Then,

$$\otimes_{i=1}^n S_{A_i} f = \arg \min_{u \in (\otimes_{i=1}^n A_i)^{-1}(z)} \| \otimes_{i=1}^n (T_i, \tilde{A}_i) u\|_{\otimes_{i=1}^n Y_i \oplus Z_i},$$

where $\tilde{A}_i$ are operators T_i-compatible with A_i.

Proposition 8.2. Let $f \in \otimes_{i=1}^n X_i$, $z = \otimes_{i=1}^n A_i f$. Then

$$\otimes_{i=1}^n S_{A_i}^{\alpha_i} f = \arg \min_{u \in \otimes_{i=1}^n X_i} \Big\{ \| \otimes_{i=1}^n (\sqrt{\alpha_i} T_i, A_i) u\|^2_{\otimes_{i=1}^n (Y_i \otimes Z_i)}$$
$$- \| \otimes_{i=1}^n A_i u\|^2_{\otimes_{i=1}^n Z_i} + \| \otimes_{i=1}^n A_i u - z\|^2_{\otimes_{i=1}^n Z_i} \Big\}.$$

The propositions are proved in the same manner as Theorems 8.13 and 8.14 and we keep them to the reader.

8.4. Convergence Estimates for Tensor Splines

8.4.1. Limits of Tensor Products of Operators

Let X and Y be Banach spaces. The sequence of the operators $P_i \in \mathcal{L}(X,Y)$, $i \in N$, strongly converges to $P \in \mathcal{L}(X,Y)$, if for any $x \in X$ the sequence $P_i x$ converges to Px in the norm of the space Y. By the Banach-Steinhouse theorem, the sequence of the operators P_i in this case will be totally bounded, i.e. $\sup \|P_i\| < \infty$.

Lemma 8.2. Let X, Y, U and V be Banach spaces, $P_i \in \mathcal{L}(X,U)$, $Q_i \in \mathcal{L}(Y,V)$, $i \in N$ be sequences of operators. If one of these sequences is totally bounded and the other strongly converges to zero operator, the sequence $P_i \otimes_p Q_i \in \mathcal{L}(X \otimes_p Y, U \otimes_p V)$ strongly converges to zero operator, $1 \le p \le \infty$.

Proof. Without loss of generality assume that P_i strongly converges to zero and Q_i is bounded. For elements of the form $x \otimes y$ we have

$$\|P_i \otimes_p \mathcal{Q}_i(x \otimes y)\| = \|P_i x\| \cdot \|\mathcal{Q}_i y\| \le \|P_i x\| \cdot \|\mathcal{Q}_i\| \cdot \|y\|.$$

The convergence of the above-given expression to zero is obvious, implying that $P_i \otimes_p \mathcal{Q}_i(x \otimes y)$ converges to zero. By virtue of linearity of the operators $P_i \otimes_p \mathcal{Q}_i$, we can readily prove that $\|P_i \otimes_p \mathcal{Q}_i z\| \to 0$ for any z from the algebraic space $X \otimes Y$.

Now let $z \in X \otimes_p Y$. Then there exists a sequence $\{z_j\} \in X \otimes Y$ converging to z. It is evident that we have

$$\begin{aligned}\|P_i \otimes_p \mathcal{Q}_i z\| &\le \|P_i \otimes_p \mathcal{Q}_i(z - z_j)\| + \|P_i \otimes_p \mathcal{Q}_i z_j\| \\ &\le \sup_i \|P_i\| \cdot \|\mathcal{Q}_i\| \cdot \|z - z_j\| + \|P_i \otimes_p \mathcal{Q}_i z_j\|\end{aligned}$$

This expression directly implies that $P_i \otimes_p \mathcal{Q}_i z$ converges to zero and completes the proof. □

Corollary. If $P_i \in \mathcal{L}(X, U)$ strongly converges to P and $\mathcal{Q}_i \in \mathcal{L}(Y, V)$ strongly converges to $\mathcal{Q}$, then $P_i \otimes_p \mathcal{Q}_i$ strongly converges to $P \otimes_p \mathcal{Q}$, $1 \le p \le \infty$.

The proof is implied by the theorem and the equality

$$\begin{aligned}P_i \otimes_p \mathcal{Q}_i - P \otimes_p \mathcal{Q} = (P_i - P) \otimes_p (\mathcal{Q}_i - \mathcal{Q}) + P \otimes_p (\mathcal{Q}_i - \mathcal{Q}) \\ + (P_i - P) \otimes_p \mathcal{Q}.\end{aligned}$$

8.4.2. Main Convergence Theorem

Let (T_i, A_i) be spline pairs and S_{A_i} be interpolating spline projectors, $i = 1, 2$. On the subspaces $N(A_i) \subset X_i$ we can introduce the norms

$$\|x\|_i = \|T_i x\|$$

equivalent to the original ones. Indeed, these norms are induced by the norms $(\|T_i u\|^2 + \|A_i u\|^2)^{1/2}$ which are equivalent to the original ones by definition of the operators T_i and A_i.

Let B_i be Banach spaces and $D_i \in \mathcal{L}(X_i, B_i)$, $i = 1, 2$. Then the following inequalities are valid:

$$\|D_i(x - S_{A_i}x)\| \le g_i \|T_i(x - S_{A_i}x)\| \quad \forall x \in X_i, \quad i = 1, 2,$$

where

$$g_i = \left\|D_i|_{N(A_i)}\right\|_i \overset{\text{df}}{=} \sup_{x \in N(A_i)} \frac{\|D_i x\|}{\|x\|_i}.$$

The problem of construction of error estimates for spline interpolation consists in approximate calculation of the constants g_i. The following theorem presents error estimates for the tensor spline interpolation via the corresponding estimates for the components.

Theorem 8.15. For any $x \in X_1 \otimes X_2$ the following estimate is valid:

$$\begin{aligned}\|D_1 \otimes D_2(x - S_{A_1} \otimes S_{A_2}x)\| &\le g_1 g_2 \|T_1(I_1 - S_{A_1}) \otimes T_2(I_2 - S_{A_2})x\| \\ &+ g_1\|T_1(I_1 - S_{A_1}) \otimes D_2 x\| + g_2\|D_1 \otimes T_2(I_2 - S_{A_2})x\|,\end{aligned}$$

where I_1 and I_2 are identity operators in X_1 and X_2 respectively.

Proof. Making use of the identity

$$\begin{aligned}I - S_{A_1} \otimes S_{A_2} &= I_1 \otimes (I_2 - S_{A_2}) + (I_1 - S_{A_1}) \otimes I_2 \\ &- (I_1 - S_{A_1}) \otimes (I_2 - S_{A_2}),\end{aligned}$$

where $I = I_1 \otimes I_2$ is the identity operator in $X_1 \otimes X_2$, we obtain

$$\begin{aligned}\|D_1 \otimes D_2(x - S_{A_1} \otimes S_{A_2}x)\| &\le \|D_1 \otimes D_2(I_1 - S_{A_1}) \otimes (I_2 - S_{A_2})x\| \\ &+ \|D_1 \otimes D_2(I_1 - S_{A_1}) \otimes I_2 x\| + \|D_1 \otimes D_2 I_1 \otimes (I_2 - S_{A_2})x\|.\end{aligned}$$

Let us estimate, for example, the second component of the sum.

Rewrite it in the form $\|D_1 \otimes \tilde{I}_2 \cdot (I_1 - S_{A_1}) \otimes D_2 x\|$, where $\tilde{I}_2$ is an identity operator in B_2. Note that the operator $(I_1 - S_{A_1}) \otimes D_2$ acts from $X_1 \otimes X_2$ into $N(A_1) \otimes B_2$. Since on $N(A_1)$ the norm $\|x\|_1 = \|T_1 x\|$ is equivalent to the original one, α_2-cross-norm induced by the norm $\|\cdot\|_1$ and the norm of the space B_2 is equivalent to the standard one on $N(A_1) \otimes B_2$. Hence, the operator $D_1 \otimes \tilde{I}_2$ is bounded in this cross-norm. Therefore,

$$\begin{aligned}\|D_1 \otimes \tilde{I}_2 \cdot (I_1 - S_{A_1}) \otimes D_2 x\| &\le \|D_1|_{N(A_1)}\|_1 \cdot \|T_1 \otimes \tilde{I}_2 \cdot (I_1 - S_{A_1}) \otimes D_2 x\| \\ &= g_1\|T_1(I_1 - S_{A_1}) \otimes D_2 x\|.\end{aligned}$$

The remaining components of the sum can be estimated in a similar way. This completes the proof of the Theorem. □

8.4.3. Some Applications of Main Theorem

Let $\{S_{A_1}^{(j)}\}$ and $\{S_{A_2}^{(j)}\}$ be sequences of interpolating spline projectors strongly converging to the identity operators in X_1 and X_2, and constructed by the sequences $\{A_1^{(j)}\}$ and $\{A_2^{(j)}\}$. Let $g_i^{(j)} = \|D_i|_{N(A_i^{(j)})}\|_i$, $i = 1, 2$. Then,

$$\|D_i(x - S_{A_i}^{(j)}x)\| \le g_i^{(j)}\|T_i(x - S_{A_i}^{(j)}x)\| = o(g_i^{(j)}).$$

The corollary to Lemma 8.2 implies that the sequence $\{S_{A_1}^{(j)} \otimes S_{A_2}^{(j)}\}$ strongly converges to the identity operator in $X_1 \otimes X_2$. The estimate of the convergence rate for these operators is given by

Theorem 8.16. Asymptotic estimates for the error of the tensor spline interpolation are of the form

$$\|D_1 \otimes D_2(x - S_{A_1}^{(j)} \otimes S_{A_2}^{(j)}x)\| = o(g_1^{(j)} g_2^{(j)} + g_1^{(j)} + g_2^{(j)})$$

for $j \to \infty$, $x \in X_1 \otimes X_2$.

The proof of the Theorem is directly implied by the Corollary of Lemma 8.2 and Theorem 8.15.

Example 8.3. Let condensable h-nets of the sets $\omega_i^{(j)} \subset \Omega_i$ be given on the domains $\Omega_i \subset \mathbb{R}^{n_i}$, i.e. dist $(\Omega_i, \omega_i^{(j)}) = h_{i,j} \to 0$ for $j \to \infty$, where dist is the Housdorff spacing. Let the operators $S_{A_i}^{(j)}$ be used to obtain solutions to the problems of D^{m_i}-spline interpolation

$$\begin{cases} \sigma|_{\omega_i^{(j)}} = f|_{\omega_i^{(j)}} \\ \|D^{m_i}\sigma\|^2_{L_2(\Omega_i)} = \min \end{cases}$$

i.e. $\sigma = S_{A_i}^{(j)} f$. Then the following estimates are valid (see Chapter 5):

$$\|D^{k_i}(f - S_{A_i}^{(j)} f)\|_{L_{p_i}(\Omega_i)} = o(g_i^{(j)}),$$

where $2 \le p_i \le \infty$, $k_i - n_i/p_i \le m_i - n_i/2$ (excluding $k_i = m_i - n_i/2$ and $p_i = \infty$) and, also,

$$g_i^{(j)} = \mathcal{O}(h_{i,j}^{m_i - k_i - n_i/2 + n_i/p_i}).$$

Theorem 8.17. The following equality is valid:

$$\|D^{k_1,k_2}(f - S_{A_1}^{(j)} \otimes S_{A_2}^{(j)} f)\|_{L_{p_1,p_2}(\Omega_1 \times \Omega_2)} = o(g_1^{(j)} g_2^{(j)} + g_1^{(j)} + g_2^{(j)}).$$

Proof. By Theorem 4.3, we have

$$\|D^{k_1} \otimes D^{k_2}(f - S_{A_1}^{(j)} \otimes S_{A_2}^{(j)} f)\|_{L_{p_1}(\Omega_1) \otimes L_{p_2}(\Omega_2)} = o(g_1^{(j)} g_2^{(j)} + g_1^{(j)} + g_2^{(j)}).$$

Making use of Theorems 8.1 and 8.5, we obtain the inequalities

$$\|\cdot\|_{L_{p_1,p_2}(\Omega_1 \times \Omega_2)} \le \|\cdot\|_{L_{p_1}(\Omega_1) \otimes_{p_1} L_{p_2}(\Omega_2)} \le \|\cdot\|_{L_{p_1}(\Omega_1) \otimes L_{p_2}(\Omega_2)}$$

which directly imply the statement of the Theorem. □

8.5. An Algorithm for Constructing Tensor Splines

8.5.1. $\mathcal{L}_A(U, B)$-Method

Most schemes for implementing the spline approximation methods are described by $\mathcal{L}_A(U, B)$-method.

Let $Z = \mathbb{R}^N$, $A : X \to Z$ be an operator with the finite-dimensional image. Let us prescribe $B \subset X$, which is a finite-dimensional space with the basis $\omega_1, \dots, \omega_M$, and $U : \mathbb{R}^N \to \mathbb{R}^M$ is a linear operator.

Definition 8.3. *The linear operator $\mathcal{L}_A : X \to X$ is said to be realized by $\mathcal{L}_A(U, B)$-method if we have*

$$\sigma = \mathcal{L}_A f = \sum_{i=1}^{M} \lambda_i \omega_i$$

where the vector $\bar{\lambda} = (\lambda_1, ..., \lambda_M)^T$ is determined by the scheme

$$\bar{\lambda} = U\bar{r}, \quad \bar{r} = Af.$$

Denote by $\bar{\omega} = (\omega_1, ..., \omega_M)^T$ the vector of basis functions. Then the operator $\mathcal{L}_A$ can be written in the convenient form

$$\mathcal{L}_A =< UA\cdot, \bar{\omega} >,$$

where $<,>$ is an extension of the scalar product

$$\mathcal{L}_A f =< UAf, \bar{w} >=< \bar{\lambda}, \bar{\omega} >= \sum_{i=1}^{M} \lambda_i \omega_i.$$

Example 8.4. The problem of interpolation of the function $f : \mathbb{R}^n \to \mathbb{R}$ given on the scattered mesh $P_1, ..., P_N$ with the values $f(P_1), ..., f(P_N)$ can be solved by the reproducing function method (see Chapters 2,5). One must seek the interpolating spline

$$\sigma(x) = \sum_{i=1}^{N} \alpha_i G_\gamma(x - P_i) + \sum_{|\beta| \le m-1} \mu_\beta \nu_\beta(x).$$

Here $m > n/2$ is an integer, $0 < \gamma < m$,

$$G_\gamma(t) = \begin{cases} |t|^{2\gamma} \ln |t|, & \gamma \text{ is an integer} \\ |t|^{2\gamma}, & \text{otherwise} \end{cases} .$$

In this case, $\nu_\beta(t) = t^\beta = t_1^{\beta_1} \times ... \times t_n^{\beta_n}$ are monomials of the degree not exceeding $m - 1$.

The vector $\bar{\lambda} = (a_1, ..., a_N,\ \mu_\beta : |\beta| \le m-1)^T$ of coefficients of the spline σ is sought from the linear system

$$\begin{bmatrix} K & B \\ B^* & 0 \end{bmatrix} \bar{\lambda} = \begin{bmatrix} \bar{r} \\ 0 \end{bmatrix}$$

where $\bar{r} = Af = (f(P_1), ..., f(P_N))^T$, K is the matrix with the coefficients $k_{ij} = G_\gamma(P_i - P_j)$, B is the matrix with the coefficients

$$b_{i,\beta} = \nu_\beta(P_i), \quad i = 1, ..., N, \quad |\beta| \le m - 1.$$

The number of arithmetic operations required to calculate the coefficients of the spline can be estimated by the quantity $\mathcal{O}(N^3)$, and the next calculation of the spline at a point requires $\mathcal{O}(N)$ operations. This method is preferable for small N (approximately up to 100 points).

8.5.2. Implementation of the Tensor $\mathcal{L}_A(U, B)$-Method

Let us prescribe two $\mathcal{L}_A(U, B)$-methods in the spaces X_1 and X_2:

$$\mathcal{L}_{A_1} =< U_1 A_1 \cdot, \bar{\omega}_1 >, \quad \mathcal{L}_{A_2} =< U_2 A_2 \cdot, \bar{\omega}_2 > .$$

Theorem 8.18. The following equality is valid:

$$\mathcal{L}_{A_1} \otimes \mathcal{L}_{A_2} =< (U_1 \otimes U_2)(A_1 \otimes A_2) \cdot, \bar{\omega}_1 \otimes \bar{\omega}_2 > .$$

Proof. Let us elucidate the sense of the Theorem. It is stated that the tensor product of $\mathcal{L}_A(U, B)$-methods is again $\mathcal{L}_A(U, B)$-method.

It is sufficient, generally, to verify the statement of the Theorem on elements of the form $f = f_1 \otimes f_2$. We have

$$\begin{aligned} &< U_1 \otimes U_2 A_1 \otimes A_2 f_1 \otimes f_2, \bar{\omega}_1 \otimes \bar{\omega}_2 > \\ &=< U_1 A_1 f_1 \otimes U_2 A_2 f_2, \bar{\omega}_1 \otimes \bar{\omega}_2 >=< \bar{\lambda}_1 \otimes \bar{\lambda}_2, \bar{\omega}_1 \otimes \bar{\omega}_2 > \\ &= \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \lambda_i^{(1)} \lambda_j^{(2)} \omega_i^{(1)} \otimes \omega_j^{(2)} = \sum_{i=1}^{M_1} \lambda_i^{(1)} \omega_i^{(1)} \otimes \sum_{j=1}^{M_2} \lambda_j^{(2)} \omega_j^{(2)} \\ &=< \bar{\lambda}_1, \bar{\omega}_1 > \otimes < \bar{\lambda}_2, \bar{\omega}_2 >=< U_1 A_1 f_1, \bar{\omega}_1 > \otimes < U_2 A_2 f_2 \bar{\omega}_2 > \\ &= \mathcal{L}_{A_1} f_1 \otimes \mathcal{L}_{A_2} f_2 = \mathcal{L}_{A_1} \otimes \mathcal{L}_{A_2} (f_1 \otimes f_2) = \mathcal{L}_{A_1} \otimes \mathcal{L}_{A_2} f. \end{aligned}$$

This completes the proof of the Theorem. □

The scheme for implementing the standard $\mathcal{L}_A(U, B)$-method consists of two stages:

(1) calculation of the vector $\bar{\lambda} = U\bar{r}$; it usually consists in solving a linear algebraic system;

(2) calculation of the characteristics required by using the explicit representation $\sum_{i=1}^{M} \lambda_i \omega_i$.

When implementing the tensor $\mathcal{L}_A(U, B)$-method we make use of the identity

$$U_1 \otimes U_2 = (U_1 \otimes I_{M_2})(I_{N_1} \otimes U_2)$$

where I_{M_2} and I_{N_1} are identity operators in $\mathbb{R}^{M_2}$ and $\mathbb{R}^{N_1}$ and obtain the following scheme:

(1) calculation of the vector $\bar{\mu} = I_{N_1} \otimes U_2 \bar{r}$; the calculation process falls into N_1 independent problems of the form $\bar{x} = U_2 \bar{y}$ with different right-hand sides;

(2) calculation of the vector $\bar{\lambda} = U_1 \otimes I_{M_2} \bar{\mu}$; the calculation process falls into M_2 independent problems of the form $\bar{x} = U_1 \bar{y}$;

(3) calculation of the characteristics required by using the explicit representation $\sum_{i,j} \lambda_{i,j} \omega_i^{(1)} \otimes \omega_j^{(2)}$.

Example 8.5. A simple problem of processing the well measurements is as follows. Let $P_1, ..., P_{N_1}$ be well coordinates on the plane, which form a scattered mesh. Assume that measurements of a function f are carried out at the depths $z_1, ..., z_{N2}$ in each well. In other words, the values of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ are taken at $N1 \times N2$ points:

$$r_{ij} = f(P_i, z_j), \quad i = 1, ..., N1, \quad j = 1, ..., N2.$$

It is necessary to reconstruct the function in the entire domain.

The direct calculation of the spline by using the reproducing kernels from Example 8.4 requires $\mathcal{O}(N1^3 N2^3)$ arithmetic operations for finding the coefficients and $\mathcal{O}(N1 \cdot N2)$ operations for calculating the spline at any point.

We suggest that the tensor spline be constructed, i.e. the solution be found in the form $\mathcal{L}_{A_1} \otimes \mathcal{L}_{A_2} f$, where $\mathcal{L}_{A_1}$ is the operator of spline approximation of data at a scattered mesh on the plane and $\mathcal{L}_{A_2}$ is the operator of one-dimensional spline approximation. When constructing the tensor spline we can additionally economize on the solution of a series of one-type problems. With this taken into account, the number of operations reduces up to $\mathcal{O}(N1^3 + N1^2 \cdot N2 + N1 \cdot N2)$ for finding the coefficients and to $\mathcal{O}(N1)$ for calculating the spline at any point.

A typical case of using the tensor methods is that of regular meshes on rectangles or n-dimensional parallelepipeds. It shows that the tensor methods can be more widely used by combining approximation methods of different types.

8.6. Blending Splines in Tensor Product of Spaces

Consider the linear continuous mappings of the Hilbert spaces $A_1 \in \mathcal{L}(X_1, Z_1)$, $A_2 \in \mathcal{L}(X_2, Z_2)$, $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$, which generate two spline pairs (T_1, A_1) and (T_2, A_2). Let S_{A_1}, S_{A_2} be interpolating spline projectors. In Sect. 8.3, we presented the variational principle for the spline projector $S_{A_1} \otimes S_{A_2}$, acting in the tensor product of the spaces $X_1 \otimes X_2$. Here we propose the variational problem for the interpolating spline projector $S_{A_1} \oplus S_{A_2}$, where the sign $\oplus$ stands for the Boolean sum[1].

Theorem 8.19. Let $\tilde{A}_1, \tilde{A}_2$ be T_1-, T_2-compatible operators with A_1, A_2, respectively, f be an element from $X_1 \otimes X_2$. The interpolating spline problem with the following interpolating conditions

$$\begin{cases} A_1 \otimes I_2 u = A_1 \otimes I_2 f, \\ I_1 \otimes A_2 u = I_1 \otimes A_2 f, \end{cases} \tag{8.8}$$

and with the following minimizing functional

[1] The Boolean sum of operators S and T is defined as $S \oplus T = S \otimes I_2 + I_1 \otimes T - S \otimes T$. We shall its completion $S \oplus_2 T = S \otimes_2 I_2 + I_1 \otimes_2 T - S \otimes_2 T$, as previously, $S \oplus T$ for the sake of minimal notations.

$$\|T_1 \otimes T_2 u\|^2_{Y_1 \otimes Y_2} + \|\tilde{A}_1 \otimes T_2 u\|^2_{\tilde{Z}_1 \otimes Y_2} + \|T_1 \otimes \tilde{A}_2 u\|^2_{Y_1 \otimes \tilde{Z}_2} \tag{8.9}$$

has the unique solution $\sigma \in X_1 \otimes X_2$, which is called a blending spline. It may be defined with the help of the Boolean sum

$$\sigma = S_{A_1} \oplus S_{A_2} f. \tag{8.10}$$

Proof. Replace interpolating conditions (8.8) by one operator equation $Au = Af$. First demonstrate that

$$N(A) = N(A_1) \otimes N(A_2).$$

We have $N(A) = N(A_1 \otimes I_2) \cap N(I_1 \otimes A_2) = N(A_1) \otimes X_2 \cap X_1 \otimes N(A_2) = N(A_1) \otimes N(A_2)$. Secondly, verify the interpolating conditions

$$A_1 \otimes I_2 (S_{A_1} \oplus S_{A_2}) f = A_1 \otimes I_2 f,$$
$$I_1 \otimes A_2 (S_{A_1} \oplus S_{A_2}) f = I_1 \otimes A_2 f.$$

We check only the first condition, since the second is proved analogously. So, $A_1 \otimes I_2 (S_{A_1} \oplus S_{A_2}) = (A_1 \otimes I_2)(S_{A_1} \otimes I_2 + I_1 \otimes S_{A_2} - S_{A_1} \otimes S_{A_2}) = A_1 \otimes I_2 - A_1 \otimes S_{A_2} - A_1 \otimes S_{A_2} = A_1 \otimes I_2$. Here we use the interpolating properties of the spline projectors $A_1 S_{A_1} = A_1, \quad A_2 S_{A_2} = A_2$.

Define the scalar products in the spaces X_1, X_2 -

$$\tilde{\rho}_1(u,v) = (\tilde{A}_1 u, \tilde{A}_1 v) + (T_1 u, T_1 v),$$
$$\tilde{\rho}_2(u,v) = (\tilde{A}_2 u, \tilde{A}_2 v) + (T_2 u, T_2 v).$$

The necessary and sufficient conditions for the operators S_{A_1} and S_{A_2} to be interpolating ((T_1, A_1) and (T_2, A_2) to be a spline pair) are of the following form

$$\begin{aligned} &\tilde{\rho}_1(S_{A_1} f_1, u_1) = 0, \quad \tilde{\rho}_2(S_{A_2} f_2, u_2) = 0, \\ &\forall\, f_1 \in X_1,\ f_2 \in X_2,\ u_1 \in N(A_1),\ u_2 \in N(A_2). \end{aligned} \tag{8.11}$$

Introduce the cross scalar product in $X_1 \otimes X_2$

$$\tilde{\rho}(u_1 \otimes u_2, v_1 \otimes v_2) = \tilde{\rho}_1(u_1, u_2) \cdot \tilde{\rho}_2(v_1, v_2) \tag{8.12}$$

extending it with the help of the linearity and completion. Now we can verify the orthogonality property

$$\tilde{\rho}(S_{A_1} \oplus S_{A_2} f, u) = 0, \quad \forall u \in N(A). \tag{8.13}$$

Since $N(A) = N(A_1) \otimes N(A_2)$, then it is sufficient to prove (8.13) for the elements $f = f_1 \otimes f_2,\ f_1 \in X_1,\ f_2 \in X_2$ and $u = u_1 \otimes u_2,\ u_1 \in N(A_1),\ u_2 \in N(A_2)$. From (8.12) we have

$$\begin{aligned} \tilde{\rho}(S_{A_1} \oplus S_{A_2} f_1 \otimes f_2, u_1 \otimes u_2) &= \tilde{\rho}(S_{A_1} f_1 \otimes f_2 + f_1 \otimes S_{A_2} f_2 \\ &\quad - S_{A_1} f_1 \otimes S_{A_2} f_2, u_1 \otimes u_2) = \tilde{\rho}_1(S_{A_1} f_1, u_1) \cdot \tilde{\rho}_2(f_2, u_2) \\ &\quad + \tilde{\rho}_1(f_1, u_1) \tilde{\rho}_2(S_{A_2} f_2, u_2) - \tilde{\rho}_1(S_{A_1} f_1, u_1) \cdot \tilde{\rho}_2(S_{A_2} f_2, u_2). \end{aligned}$$

And from (8.11) it follows that all three last items are equal to zero. □

Remark. Let us explain the sense of interpolating conditions (8.8). Assume A_1 to be the trace operator on the mesh $\{x_1 < ... < x_N\}$ and A_2 be the trace operator on the mesh $\{y_1 < ... < y_N\}$. Then one can see (Fig. 8.1, 8.2), which meshes are obtained then using tensor splines and blending splines.

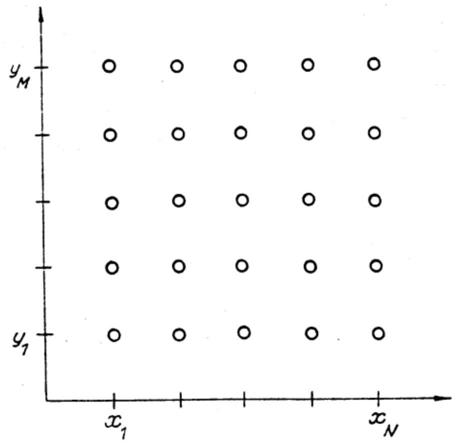

Fig. 8.1. Tensor interpolating mesh

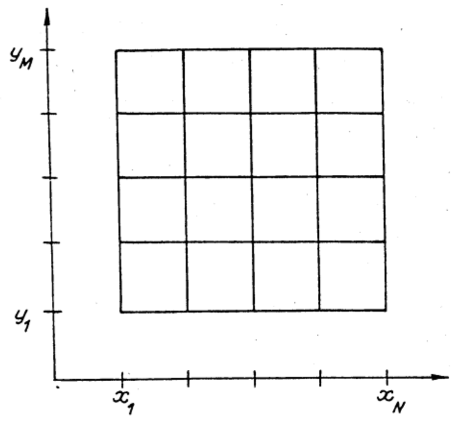

Fig. 8.2. Blending interpolating mesh

In the first case we obtain a discrete set of points organizing a grid and in the second case - the totality of crossing lines.

Theorem 8.20. If the following estimates for interpolation processes with the operators S_{A_1} and S_{A_2}

$$\|D_1(I - S_{A_1})f_1\| \le g_1, \quad \|D_2(I - S_{A_2})f_2\| \le g_2$$

are valid, then for every $f \in X_1 \otimes X_2$ the following estimate for the blending spline operator $S_{A_1} \oplus S_{A_2}$

$$\|D_1 \otimes D_2(I - S_{A_1} \oplus S_{A_2})f\| \le g_1 g_2 \tag{8.14}$$

is valid, too.

Proof. This follows from the equalities

$$I - S_{A_1} \oplus S_{A_2} = (I - S_{A_1}) \otimes (I - S_{A_2}),$$
$$D_1 \otimes D_2(I_1 - S_{A_1} \oplus S_{A_2}) = D_1 \otimes D_2(I - S_{A_1}) \otimes (I - S_{A_2})$$

and on the basis of arguments like those used in the proofs of Theorems 8.15, 8.16. □

One can see that estimate (8.14) is better than one from Theorem 8.16, i.e. the blending method is better than the tensor one. But, unfortunately, the first method demands more information.