10. Classification of Spline Objects

This chapter is special in the sense that it represents a collection of the facts from the previous chapters, which underline the internal unity of these chapters. This is a selective observation which helps as to classify general methods and objects of variational spline theory. The Chapter was prepared on the basis of the paper (Bezhaev 1990).

General objects of investigation are the Hilbert spaces and linear continuous operators in the Hilbert spaces. Since other spaces except the Hilbert ones and other operators except the aforementioned are not used, then the words "Hilbert", "linear" and "continuous" are omitted almost everywhere.

Remember the abstract variational spline-interpolation problem, which we write down in the following form:

$$\begin{cases} A\sigma = z, & \sigma \in X, \\ \|T\sigma\|_Y = \min. \end{cases} \tag{10.1}$$

Notation (10.1) contains the spaces X, Y, Z and the operators $A : X \to Z$, $T : X \to Y$. As earlier, the interpolating spline $\sigma \in X$ is an element satisfying the operator equation $Au = z$, $z \in Z$ (line 1), and minimizing the energy functional $\|T\sigma\|_Y$ (line 2). The previous chapters contain various examples of concrete spline functions reduced to (10.1).

With the help of the suggested classification we will try to teach the reader how to get new problems of spline interpolation from the known classical one. The suggested scheme of classification concerns the spline-interpolation, but it may be easily spread out on the smoothing splines.

On the whole, the classification of spline objects is connected with the fundumental operations over the Hilbert spaces and operators (Kirillov, Gvishiani 1979), which allow one to organize new spaces and operators. In Section 10.1, we describe five general operations which in combinations help us to construct new spaces, operator and spline-interpolating methods.

In Section 10.2 we describe five composed spline objects, which are obtained by merging of the usual variational spline functions. More general composed spline objects can be by merging these principal five obtained spline objects. The respective examples are given.

10.1. Fundamental Operations Over Hilbert Spaces

10.1.1. Closed Subspaces and Restriction of Operators on Subspaces

Consider an operator $A : X \to Z$ and a closed subspace E in X. It is known that E is a Hilbert space with the Hilbert norm and scalar product induced from X. The restriction of the operator A in the space E, denoted by $A|_E : E \to Z$, is obviously a linear continuous operator.

10.1.2. Space of Traces on Manifolds and Trace of Operator

Let X/E be a set of the factor-classes $u_E = u + E,\ u \in X$. Then one can introduce in the space X/E the following norm

$$\|u_E\| = \min_{e \in E} \|u + e\|_X,$$

which generates the scalar product, changing X/E in the Hilbert space.

Assume that the operator $A : X \to Z$ is annihilated on the space E, i.e. $Ae = 0,\ \forall e \in E$. Let us put into correspondance to this operator the operator $A : X/E \to Z$ of the same name defined by the formula:

$$Au_E = Au, \quad \forall u_E \in X/E.$$

Readily, the operator is correctly defined, linear and continuous.

Let us present a particular realization of the factor-space as the space of traces. The general operator A in the factor-space will be changed on the trace of the operator. Let $X = X(\Omega)$ be a functional space on the domain Ω, Γ be a manifold in Ω. Introduce the space of traces

$$X(\Gamma) = \{u : \Gamma \to \mathbb{R} : \quad \exists w \in X(\Omega),\ w|_\Gamma = u\}$$

with the norm

$$\|u\|_{X(\Gamma)} = \inf_{w|_\Gamma = u} \|w\|_{X(\Omega)}.$$

Remark. If one defines the closed subspace

$$X_{0,\Gamma}(\Omega) = \{u \in X(\Omega) : u|_\Gamma = 0\},$$

then it is clear that the factor space $X(\Omega)/X_{0,\Gamma}(\Omega)$ is isomorphic to $X(\Gamma)$.

Let the operator A be annihilated in the space $X_{0,\Gamma}(\Omega)$. The trace of the operator A on the manifold Γ denoted by $A|_\Gamma : X(\Gamma) \to Z$ is defined as follows:

$$(A|_\Gamma)(u) = Aw, \quad \text{where} \quad w|_\Gamma = u.$$

It is easy to verify that the trace of the operator $A|_\Gamma$ is uniquely defined and continuous.

10.1.3. Direct Sum of Spaces and Operators

The direct sum of the spaces X_1 and X_2, denoted by $X_1 \oplus X_2$, is defined as a Hilbert space of pairs (u_1, u_2), $u_1 \in X_1$, $u_2 \in X_2$, whose scalar product and norm are determined as follows:

$$[(u_1, u_2), (v_1, v_2)]_X = (u_1, v_1)_{X_1} + (u_2, v_2)_{X_2},$$

$$\|(u_1, u_2)\|_X = \sqrt{\|u_1\|^2_{X_1} + \|u_2\|^2_{X_2}}.$$

Define the direct sum of the operators $A_1 : X_1 \to Z$, $A_2 : X_2 \to Z$ in the following form:

$$(A_1 \oplus A_2)(u) = A_1 u_1 + A_2 u_2.$$

The linearity and the boundedness of the operator $A_1 \oplus A_2 : X_1 \oplus X_2 \to Z$ are obvious.

10.1.4. Tensor Product of Spaces and Operators

Consider a bilinear mapping $B : X \times Y \to Z$, where $X \times Y$ is the Decart product of spaces. Clearly, the finite sums

$$\sum_{i=1}^{k} B(u_i, v_i), \quad u_i \in X,\ v_i \in Y,\ k \in \mathbb{N},$$

form in Z a linear subset. In the conventional manner, we change the notation $B(u, v)$ to $u \otimes v$, and denote the aforesaid linear subset in Z by $X \otimes Y$. In $X \otimes Y$, introduce the inner product in the following form:

$$\left(\sum_{i=1}^{k_1} u_i^1 \otimes v_i^1, \sum_{j=1}^{k_2} u_j^2 \otimes v_j^2 \right)_{X \otimes Y} = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} (u_i^1, u_j^2)_X (v_i^1, v_j^2)_Y.$$

It is known (Light, Cheney 1985), that the completion of the set $X \otimes Y$ according to the norm, induced by the latter inner product, is a Hilbert space. It is called a tensor product of the spaces X and Y. Further, it will be denoted in the same way, $X \otimes Y$.

Remark. One can see that this space depends on the bilinear mapping B and on the linear space Z. In reality, the space Z defines the concrete realization of the Hilbert space $X \otimes Y$, and the mapping B defines one of the equivalent parametrizations.

If $A_1 : X_1 \to X_1$ and $A_2 : X_2 \to Z_2$ are operators, then their tensor product $A_1 \otimes A_2 : X_1 \otimes X_2 \to Z_1 \otimes Z_2$ is defined as follows

$$(A_1 \otimes A_2) \left(\sum_{i=1}^{k} u_i \otimes v_i \right) = \sum_{i=1}^{k} (Au_i) \otimes (Av_i).$$

The extension of this operator to the whole $X \otimes Y$ is the linear continuous operator of Hilbert spaces.

10.1.5. Conjugate Space and Operator

The space of linear continuous functionals $k : X \to \mathbb{R}$ is called a conjugate space to X and is denoted by X^*. To introduce into it the Hilbert structure, remember the following definition.

Definition. *A mapping $\pi : X^* \to X$ is called a reproducing mapping of the space X, if*

$$l(u) = (\pi(l), u)_X, \quad \forall l \in X^*, \quad \forall u \in X.$$

By the Riesz theorem this mapping exists and is unique. Now it is easy to verify that the scalar product

$$(l_1, l_2)_{X^*} = (\pi(l_1), \pi(l_2))_X$$

introduces the Hilbert structure into the space X^*.

We associate with the operator $A : X \to Z$ the conjugate operator $A^* : Z^* \to X^*$ which is defined by the formula

$$A^*(\lambda)(u) = \lambda(Au), \quad \forall u \in X, \quad \forall \lambda \in Z^*.$$

The conjugate operator is linear and continuous as well as A.

10.2. Classification of Spline Objects and Methods of Their Merging

The subsections of this section are logical continuations of the respective subsections of Sect. 10.1. For example, we refer the reader to Sect. 10.1.1 to hear of the notations to Sect. 10.2.1.

10.2.1. Splines on Subspaces

Consider the operator equation $A|_E\sigma = z$ as an approximation of the operator equation $A\sigma = z$. Then, interpolating problem (10.1) can be approximated with the help spline on subspace:

$$\begin{cases} A|_E\sigma = z \\ \|T\sigma\|_Y = \min \end{cases} \tag{10.2}$$

Since E is a Hilbert space, and operator A is continuous, then this problem is a particular case of more general problem (10.1).

The general approach to this problem was stated in Chapters 4,5,9. Now we formulate only one result concerning the representation of spline on the finite-dimensional subspace E. Assume A is an operator with the finite-dimensional range $\mathbb{R}^n$. If $\omega_1, ..., \omega_n$ stands for the basis of E, then the spline σ being the solution of (10.2) is of the following form

$$\sigma = \sum_{i=1}^{n} \sigma_i \omega_i.$$

The vector of coefficients $\bar{\sigma} = (\sigma_1, ..., \sigma_n)^T$ is determined from the SLAE

$$\begin{bmatrix} \bar{T} & \bar{A}^T \\ \bar{A} & 0 \end{bmatrix} \begin{bmatrix} \bar{\sigma} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix},$$

where the matrix $\bar{T}$ has the common element $t_{ij} = (T\omega_i, T\omega_j)$, the matrix A has the common element $a_{ij} = (A\omega_j)_i$, the matrix $\bar{A}^T$ is transposed to the matrix $\bar{A}$.

Spline method of subspaces was proposed in (Vasilenko 1973). Realization of this method is described in the instructions to the package FINEL of the library (LIDA-3 1987).

10.2.2. Splines on Manifolds

In order to simplify the description of splines on manifolds, consider a particular case of the spherical manifold

$$\Gamma = \{(\varphi_i, \eta_i) : \varphi_i \in [0, 2\pi),\ \eta_i \in [0, \pi]\}.$$

Take the points $(\varphi_1, \eta_1), ..., (\varphi_n, \eta_n)$ on the unit sphere Γ in $\mathbb{R}^3$, and the real numbers $r_1, ..., r_n$. Assume that we need to solve interpolating problem

$$\sigma(\varphi_i, \eta_i) = r_i, \quad i = 1, ..., n. \tag{10.3}$$

One approach to the solution consists in the definition of classes of the Sobolev functions on the sphere Γ having the Hilbert and semi-Hilbert structures, in finding the energy functionals and reproducing kernels, and in further solution of the problem as we have done in Chapter 2 for the general case. This approach was implemented by (W.Freeden 1981) and (G.Wahba 1981) independently. The authors based on the operators Laplas-Beltrami on sphere.

Another approach, suggested by (Bezhaev 1984) and called the method of traces of spline on manifolds, consists in utilization of the available method of interpolation in $\mathbb{R}^n$ (Duchon 1977). The set of points on the sphere is considered as a chaotic set of points $P_1, ..., P_n$ in $\mathbb{R}^3$, more exactly

$$P_i = (x_i, y_i, z_i), \quad i = 1, ..., n,$$

where

$$x_i = \sin\varphi_i \sin\eta_i,$$
$$y_i = \cos\varphi_i \sin\eta_i,$$
$$z_i = \cos\eta_i.$$

Consider D^m-spline σ which is the solution to the following problem

$$\begin{cases} \sigma(P_i) = r_i, \quad i = 1, ..., n \\ \int\limits_{\mathbb{R}^n} \sum\limits_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha \sigma)^2 dX = \min . \end{cases}$$

Its trace $\sigma|_\Gamma$ is declared as the solution of interpolating problem (10.3) and is called a spline on a manifold. In detail this is described in Chapter 6.

Now we are interested in formal notation of interpolating problem on manifold:

$$\begin{cases} A\sigma|_\Gamma = z, \quad \sigma \in X, \\ \|T\sigma\|_Y = \min . \end{cases} \tag{10.4}$$

Here $X = X(\Omega)$ is the Hilbert space of the functions, defined in the domain Ω including the manifold Γ.

Remark. Our classification contains two types of spline-interpolating problems (10.2) and (10.4), therefore one can consider their mergings. Denote this in the following schematic form:

$$\boxed{\begin{array}{l} A|_E\sigma = z, \ \sigma \in E, \\ \|T\sigma\|_Y = \min \end{array}} \quad \longleftrightarrow \quad \boxed{\begin{array}{l} A\sigma|_\Gamma = z, \ \sigma \in X, \\ \|T\sigma\|_Y = \min \end{array}} \, .$$

This combination may be considered as 1) utilization of the method of splines on subspace, when solving the interpolation problem on manifold, or 2) utilization of the method of traces when interpolating on a subspace.

The first can be interpreted as an introduction of the finite-element space E in the space of functions, defined on the manifold Γ and subsequent minimization of some energy functional on E under the interpolating conditions. The latter functional may be the Laplace-Beltrame form like in the aforesaid papers by Freeden and Wahba.

The second kind of merged splines is connected with consideration of a finite-element subspace in the domain Ω, included the manifold Γ, with the subsequent solution of the interpolating problem on a subspace and with the final consideration of the trace of the obtained solution on manifold. This kind of splines is attentively investigated in Sect. 6.2.

10.2.3. Vector Splines

Consider a pair of abstract spline-interpolating problems with $z_1 \in Z$, $z_2 \in Z$:

$$\begin{cases} A_1\sigma_1 = z_1, \ \sigma_1 \in X_1, \\ \|T_1\sigma_1\|_{Y_1} = \min, \end{cases} \qquad \begin{cases} A_2\sigma_2 = z_2, \ \sigma_2 \in X_2, \\ \|T_2\sigma_2\|_{Y_2} = \min . \end{cases} \tag{10.5}$$

Definition. *Take $z \in Z$. An element $\sigma = (\sigma_1, \sigma_2)$ is called a vector spline, iff it is the solution of the following problem*

$$\begin{cases} A_1\sigma_1 + A_2\sigma_2 = z, \quad \sigma_1 \in X_1,\ \sigma_2 \in X_2, \\ \|T_1\sigma_1\|^2_{Y_1} + \|T_2\sigma_2\|^2_{Y_2} = \min . \end{cases} \tag{10.6}$$

In conformity with Sect. 10.1.3 construct the direct sums of the spaces $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ and the direct sum of the operators $T = T_1 \oplus T_2$. Then one can readily see that problem (10.5) is equivalent to general problem (10.1). Write down the problem for the vector spline of several components $\sigma = (\sigma_1, ..., \sigma_k)$:

$$\begin{cases} A_1\sigma_1 + A_2\sigma_2 + ... + A_k\sigma_k = z \\ \|T_1 \oplus T_2 \oplus ... \oplus T_k\sigma\|_Y = \min, \end{cases} \tag{10.7}$$

which we will use further only for $k = 2$. Chapter 7 contains the solution to this problem making use of the reproducing mappings or reproducing kernels of the spaces $X_1, ..., X_k$. One can say that if there is a method of the solution to problems (10.5), then one can easily construct the method of the solution to problem (10.6).

Example. The solution to the problem about the location of a thin elastic plate attached at some points of $\mathbb{R}^3$ is modelled (Smolyak 1971; Harder, Desmarais 1972) with the help of the following spline-interpolating problem

$$\begin{cases} \sigma(P_i) = z_i,\ i = 1, ..., n,\ \sigma \in W_2^2(\Omega), \\ \int\limits_\Omega (\sigma^2_{xx} + 2\sigma^2_{xy} + \sigma^2_{yy}) dxdy = \min . \end{cases} \tag{10.8}$$

The latter problem may be solved with the aid of analytical or finite element methods. Here, the interpolating conditions simulate the stiff fixation of the plate, described by the two-dimensional function $\sigma(x, y)$, and the energy functional approximates the potential energy of elasticity.

Consider the following problem of the vector spline interpolation

$$\begin{cases} \sigma_1(P_i) = z_i, \quad i = 1, ..., n_1,\ \sigma_1 \in W_2^2(\Omega), \\ \sigma_2(Q_i) = t_i, \quad i = 1, ..., n_2,\ \sigma_2 \in W_2^2(\Omega), \\ \sigma_1(z_i) - \sigma_2(z_i) = h_i, \quad i = 1, ..., n_3, \\ \int\limits_\Omega (\sigma^2_{1xx} + 2\sigma^2_{1xy} + \sigma^2_{1yy} + 2\sigma^2_{2xx} + 2\sigma^2_{2xy} + \sigma^2_{2yy}) dxdy = \min . \end{cases}$$

We assert that this problem simulates a physical problem of mutual location of two thin elastic plates attached at some points and additionally connected by stiff rod. This is a new problem, which illustrates fruitfullness of the vector approach in spline functions. Other examples are described in Chapter 7.

Remark 1. Remember again about the opportunity of the merging of various types of spline interpolation. We have considered the three ones: splines on subspaces, on manifolds, vector splines. Consider the following combination:

$$\boxed{\begin{array}{l} A|_E\sigma = z,\ \sigma \in E, \\ \|T\sigma\|_Y = \min \end{array}} \quad \longleftrightarrow \quad \boxed{\begin{array}{l} A_1\sigma_1 + A_2\sigma_2 = z,\ \sigma_1 \in X_1,\ \sigma_2 \in X_2, \\ \|T_1\sigma_1\|^2_{Y_1} + \|T_2\sigma_2\|^2_{Y_2} = \min \end{array}}$$

This corresponds to the merging of splines on subspaces with the vector functions, which bring about three new problems:

$$\begin{cases} A_1\sigma_1 + A_2\sigma_2 = z, \quad \sigma_1 \in E_1,\ \sigma_2 \in X_2, \\ \|T_1\sigma_1\|^2_{Y_1} + \|T_2\sigma_2\|^2_{Y_2} = \min . \end{cases} \tag{10.9}$$

$$\begin{cases} A_1\sigma_1 + A_2\sigma_2 = z, \quad \sigma_1 \in X_1,\ \sigma_2 \in E_2, \\ \|T_1\sigma_1\|^2_{Y_1} + \|T_2\sigma_2\|^2_{Y_2} = \min . \end{cases} \tag{10.10}$$

$$\begin{cases} A_1\sigma_1 + A_2\sigma_2 = z, \quad \sigma_1 \in E_1,\ \sigma_2 \in E_2, \\ \|T_1\sigma_1\|^2_{Y_1} + \|T_2\sigma_2\|^2_{Y_2} = \min . \end{cases} \tag{10.11}$$

Chapter 7 contains the algorithms for the solution to these problems. Problems (10.9) and (10.10) are very unusual. There, the analytical and finite element approaches are combined.

Remark 2. The following chart illustrates the merging of the method of splines on manifolds and vector-splines:

$$\boxed{\begin{array}{l} A\sigma|_\Gamma = z,\ \sigma \in E, \\ \|T\sigma\|_Y = \min \end{array}} \quad \longleftrightarrow \quad \boxed{\begin{array}{l} A_1\sigma_1 + A_2\sigma_2 = z,\ \sigma_1 \in X_1,\ \sigma_2 \in X_2, \\ \|T_1\sigma_1\|^2_{Y_1} + \|T_2\sigma_2\|^2_{Y_2} = \min \end{array}}$$

with the help of this combination one can solve the interpolating problem for spherical functions, having the pole singularities. To prove this, consider the following interpolating conditions:

$$\begin{cases} \sigma_1(\varphi_i, \eta_i) - r_i\sigma_2(\varphi_i, \eta_i) = 0 \\ \sigma_2(\varphi_i, \eta_i) = 0, \quad \sigma_1(\varphi_i, \eta_i) = \pm 1, \quad \text{if} \quad r_i = \pm\infty . \end{cases}$$

Evidently, these conditions are sufficient for the function $\sigma = \sigma_1/\sigma_2$ to satisfy the interpolating conditions $\sigma(\varphi_i, \eta_i) = r_i$ including the pole singularities $r_i = \pm\infty$. Additionally, considering energy functional

$$\int_\Omega (\sigma^2_{1xx} + 2\sigma^2_{1xy} + \sigma^2_{1yy} + \sigma^2_{2xx} + 2\sigma^2_{2xy} + \sigma^2_{2yy})dxdy \tag{10.12}$$

for D^m-splines, one can reduce the initial problem with pole singularities to the traces on vector D^m-splines on the manifold.

Combining further the latter object with the method of spline on subspaces, one can arrive at the finite-element method for the interpolation of the spherical function with the pole singularities.

Besides the papers of the author and Chapter 7, vector splines have been studied in (Rozhenko 1983; Wahba 1984).

10.2.4. Tensor Splines

Again, consider a pair of abstract spline-interpolating problems, but now with $z_1 \in Z_1$, $z_2 \in Z_2$:

$$\begin{cases} A_1\sigma_1 = z_1, \ \sigma_1 \in X_1, \\ \|T_1\sigma_1\|_{Y_1} = \min, \end{cases} \qquad \begin{cases} A_2\sigma_2 = z_2, \ \sigma_2 \in X_2, \\ \|T_2\sigma_2\|_{Y_2} = \min. \end{cases} \tag{10.13}$$

On can put into correspondence to these problems two linear operators of the spline interpolation $S_1 : Z_1 \to X_1$, $S_2 : Z_2 \to X_2$, which in turn puts into correspondence to any elements $z_1 \in Z_1$, $z_2 \in Z_2$ the splines-solutions of problems (10.13). Clearly, the tensor product $S_1 \otimes S_2$ gives a solution to problem

$$A_1 \otimes A_2 u = z \tag{10.14}$$

for any element $z \in Z = Z_1 \otimes Z_2$. In practice, such problems arise when one utilizes the data on the regular meshes or, more generally, the Decart product of arbitrary meshes.

In Chapter 8, we have shown which variational functional satisfies the interpolating method $S_1 \otimes S_2$. It appears that the respective variational spline interpolating problem looks like as follows:

$$\begin{cases} A_1 \otimes A_2\sigma = z, \ \sigma \in X_1 \otimes X_2, \\ \|T_1 \otimes T_2\sigma\|_{Y_1\otimes Y_2} = \min. \end{cases} \tag{10.15}$$

The spline interpolating method $S_1 \otimes S_2$ is the unique solution to problem (10.15) if $N(T_1) = \{0\}$ and $N(T_2) = \{0\}$. Else, problem (10.15) is not uniquely defined, but among its solutions one can find the aforementioned $S_1 \otimes S_2$. Chapter 8 contains the modification in the energy functional of (10.15), which leads to the uniquely defined solution.

A very interesting trend in the tensor spline theory is blended methods. There we give their variational formulations.

For any space X, denote by I_X the identity operator in this space. Introduce the following operators

$$A_1 \otimes I_{X_2} : X_1 \otimes X_2 \to Z_1 \otimes X_2, \quad I_{X_1} \otimes A_2 : X_1 \otimes X_2 \to X_1 \otimes Z_2$$

and some elements $\bar{z}_1 \in Z_1 \otimes X_2$, $\bar{z}_2 \in X_1 \otimes Z_2$.

Theorem. The system of operator equations

$$\begin{cases}(A_1 \otimes I_{X_2})\sigma = \bar{z}_1, & \sigma \in X_1 \otimes X_2,\\ (I_{X_1} \otimes A_2)\sigma = \bar{z}_2\end{cases} \tag{10.16}$$

is compatible, if

$$(I_{Z_1} \otimes A_2)\bar{z}_1 = (A_1 \otimes I_{Z_2})\bar{z}_2 \stackrel{\text{def}}{=} \bar{z}. \tag{10.17}$$

The solution is determined by the following formula

$$\sigma = (S_1 \otimes I_{X_2})\bar{z}_1 + (I_{X_1} \otimes S_2)\bar{z}_2 - (S_1 \otimes S_2)\bar{z}, \tag{10.18}$$

where the element $\bar{z}$ is determined from conditions (10.17).

Proof. Conditions of the compatibility (10.17) are necessary. This follows from the following equalities

$$(I_{Z_1} \otimes A_2)\bar{z}_1 = (I_{Z_1} \otimes A_2)(A_1 \otimes I_{X_1})\sigma = (A_1 \otimes A_2)\sigma =$$
$$(A_1 \otimes I_{Z_2})(I_{X_1} \otimes A_2)\sigma = (A_1 \otimes I_{Z_2})\bar{z}_2.$$

Prove now that formula (10.18) really gives a solution to system of the operator equations (10.16). To verify this, produce the following true transformations:

$$(A_1 \otimes I_{X_2})\sigma = (A_1 \otimes I_{X_2})(S_1 \otimes I_{X_2})\bar{z}_1 + (A_1 \otimes I_{X_2})(I_{X_1} \otimes S_2)\bar{z}_2 -$$
$$(A_1 \otimes I_{X_2})(S_1 \otimes S_2)\bar{z} = (I_{Z_1} \otimes I_{X_2})\bar{z}_1 + (A_1 \otimes S_2)\bar{z}_2 -$$
$$(A_1 \otimes I_{X_2})(S_1 \otimes S_2)(A_1 \otimes I_{Z_2})\bar{z}_2 = \bar{z}_1.$$

The latter proves the correctness of the first equation of (10.16), the second equation is similarly verified. □

Thus, we found the operator which gives a solution to problem (10.16). It turns out that this operator posseses one interesting property, it gives the solution with the minimal semi-norm $\|T_1 \otimes T_2 u\|_{Y_1 \otimes Y_2}$.

With the help of operator equations (10.16) one usually simulates the practical task about surface construction, put on the frame, consisting of two orthogonal families of curves in the Euclidean space $\mathbb{R}^3$. The first operator equation of (10.16) simulates the frame, consisting of curves from one family, and the second operator equation simulates the frame, consisting of curves from another family, the conditions of the compatibility signify that the surface must lye at the points of intersections of two orthogonal families of curves.

The next problem of the blending tensor spline interpolation helps when one finds the surface, which lies on the frame of two families of discrete curves.

In addition to the introduced operators, consider the operators $\bar{A}_1 : X_1 \to \bar{Z}_1$ such, that $\bar{A}_1 S_1 \bar{A}_1 = \bar{A}_1$, $\bar{A}_2 S_2 \bar{A}_2 = \bar{A}_2$. Then the problems

$$\begin{cases}\bar{A}_1\sigma_1 = z_1,\ \sigma_1 \in X_1,\\ \|T_1\sigma_1\|_{Y_1} = \min,\end{cases} \qquad \begin{cases}\bar{A}_2\sigma_2 = z_2,\ \sigma_2 \in X_2,\\ \|T_2\sigma_2\|_{Y_2} = \min\end{cases} \tag{10.19}$$

are uniquely solved for any elements $z_1 \in \bar{Z}_1$, $z_2 \in \bar{Z}_2$. Let $\bar{S}_1, \bar{S}_2$ (like operators S_1, S_2 for (10.13)) be the linear operators of spline interpolation for problems (10.19). Consider the system of two operator equations

$$\begin{cases} (A_1 \otimes \bar{A}_2)\sigma = \bar{z}_1, & \sigma \in X_1 \otimes X_2, \\ (\bar{A}_1 \otimes A_2)\sigma = \bar{z}_2 \end{cases} \tag{10.20}$$

for the elements $\bar{z}_1 \in Z_1 \otimes \bar{Z}_2$, $\bar{z}_2 \in \bar{Z}_1 \otimes Z_2$. Like problem (10.16), one can write out the condition of compatibility

$$\bar{z} \overset{\text{def}}{=} (\bar{A}_1 \otimes A_2)\bar{z}_1 = (A_1 \otimes \bar{A}_2)\bar{z}_2,$$

which gives the solution to (10.20). We gave here two simple blended tensor methods, further one can generalize the problems, using the more complicated Boolean sums of operators instead of those used here. Tensor splines were investigated in (Bezhaev, Rozhenko 1989, 1990; LIDA-3 1987; Zavialov, Imamov 1978; Cui Ming-gen et al. 1986; Gordon 1971).

We specially do not discuss new opportunities connected with merging the spline objects and methods, where one of the terms is a tensor object. First, this could occupate much place, secondly, we think we had convinced the reader that the merging is a very fruitful way to organize new objects.

10.2.5. Optimal Approximation of Linear Functionals

Let $A : X \to Z$ be a surjective operator with nonzero kernel, $k \in X^*$ be a linear continuous functional. Assume, that we have to estime the value $k(u)$ for u, satisfying the operator equation $Au = z$. The spline method of such an estimation consists in choosing the spline σ which is the solution of (10.1), and in considering $k(\sigma)$ as an estimate of sought for value. If one denotes by $S : Z \to X$ the operator of spline-restoration, then $k \circ S$ defines the linear method of solution to the introduced estimation problem.

A very interesting question arises again if the operator $k \circ S \in Z^*$ satisfies any variational principle. In Chapter 9, we have shown that the latter operator gives the solution to the following minimization problem

$$\Phi(\lambda) = \max_{\|Tu\|_Y=1} |k(u) - \lambda(Au)|,$$

which is considered for all elements $\lambda \in Z^*$, satisfying the condition $\lambda(p) = 0$, $\forall p \in \ker T$. With the help of definition of a conjugate operator, the minimization functional is rewritten in the following form:

$$\Phi(\lambda) = \max_{\|Tu\|_Y=1} |(A^*\lambda - k)(u)|.$$

Clearly, $\lambda \in Z^*$ is being found from the condition of minimal distance of the element $A^*\lambda$ to the functional k on the class of elements satisfying $\|Tu\|_Y = 1$. In spite of unlikeness of this variational problem on the spline formulation we include it in the general scheme. So, we underline its variational essence and remember about the opportunity of the merging with the earlier discussed methods.

Finishing the Chapter, we give Figure 10.1, which illustrates the classification of spline methods and objects.

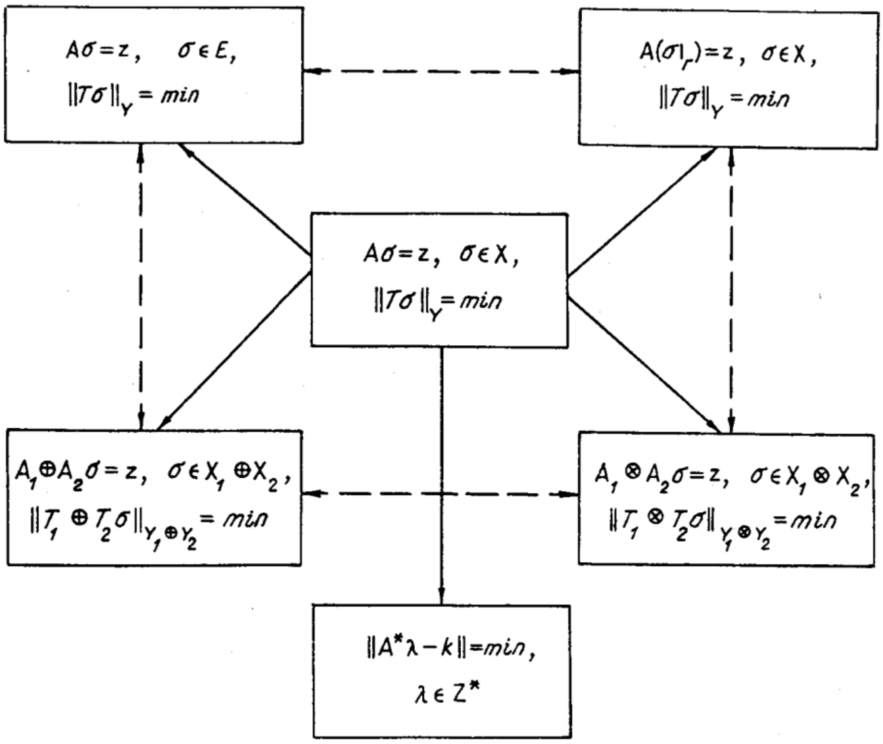

Fig. 10.1. Scheme of classification for variational spline methods and objects