11. $\Sigma\Pi$-Approximations and Data Compression

The problem of $\Sigma\Pi$-approximation in a simple form is the following: let $f(x,y)$ be a real function of two real variables x and y; we want to replace this function by the finite sum of products of one-variable functions

$$\sum_{k=1}^{s} \Phi_k(x)\Psi_k(y)$$

and to provide some given accuracy of approximation. This problem is important in various applications, like data compression in digital image processing, in decomposition of two-dimensional digital filters into the one-dimensional filters and so on. In the beginning of our century E. Schmidt (1907) considered this problem in the analytical form and found the connection between optimal $\Sigma\Pi$-approximation and singular values of the integral operator with the kernel $f(x,y)$. After that many mathematicians become interested in this problem, but usually in the analytical form without using numerical algorithms. In this chapter, we consider the so-called finite dimensional $\Sigma\Pi$-approximations in the general form and in the examples, and give the numerical algorithm for them.

11.1. General Consideration

Let $n_x \geq 1$, $n_y \geq 1$ be integers and $\Omega_x \subset R^{n_x}$, $\Omega_y \subset R^{n_y}$ be some domains. In these domains, we define the Hilbert spaces $X(\Omega_x)$ and $Y(\Omega_y)$ of the real-value functions of the vector arguments $x \in \Omega_x$ or $y \in \Omega_y$ correspondingly. Let $\Omega = \Omega_x \times \Omega_y$ and

$$Z(\Omega) = X(\Omega_x) \otimes Y(\Omega_y) \tag{11.1}$$

be a tensor product and, also, the Hilbert space which is complete with respect to any cross-norm $\|\cdot\|_{Z(\Omega)}$ (see Chapter 8),

$$\begin{aligned} &\forall\varphi(x) \in X(\Omega_x) \quad \forall\psi(y) \in Y(\Omega_y) \\ &\qquad \|\varphi(x)\cdot\psi(y)\|_{Z(\Omega)} = \|\varphi\|_{X(\Omega_x)} \cdot \|\Psi\|_{Y(\Omega_y)}. \end{aligned} \tag{11.2}$$

Let X_n be n-dimensional subspace in $X(\Omega_x)$ with the basis $\varphi_1(x), ..., \varphi_n(x)$, and Y_m be m-dimensional subspace in $Y(\Omega_y)$ with the basis $\psi_1(y), ..., \psi_m(y)$. The finite-dimensional $\Sigma\Pi$-approximation is expression of the form

$$\sum_{k=1}^{s} \Phi_{n,k}(x)\Psi_{m,k}(y), \quad \Phi_{n,k} \in X_n, \quad \Psi_{m,k} \in Y_m. \tag{11.3}$$

We say that this expression is reduced if the functions $\Phi_{n,1}, \Phi_{n,2}, ..., \Phi_{n,s}$ are linear independent and $\Psi_{m,1}, \Psi_{m,2}, ..., \Psi_{m,s}$ are also linear independent. Moreover, if these systems of functions are orthogonal (1-st system in X-scalar product and 2-nd in Y-scalar product), then we say that $\Sigma\Pi$-approximation is orthogonal.

Lemma. Every reduced $\Sigma\Pi$-approximation can be transformed to the orthogonal one.

Proof. Let us introduce functions

$$\begin{aligned} \bar{\Phi}_{n,k}(x) &= \sum_{l=1}^{s} a_{kl}\Phi_{n,l}(x), \quad k = 1,2,...,s, \\ \bar{\Psi}_{m,k}(y) &= \sum_{l=1}^{s} b_{kl}\Psi_{m,l}(y), \quad k = 1,2,...,s, \end{aligned} \tag{11.4}$$

with any coefficients a_{kl}, b_{kl}, and reguire

$$\sum_{k=1}^{s} \Phi_{n,k}(x)\Psi_{m,k}(y) = \sum_{k=1}^{s} \bar{\Phi}_{n,k}(x)\bar{\Psi}_{m,k}(y) \tag{11.5}$$

and, in addition,

$$(\bar{\Phi}_{n,k}, \bar{\Phi}_{n,l})_{X(\Omega_x)} = (\bar{\Psi}_{m,k}, \bar{\Psi}_{m,l})_{Y(\Omega_y)} = 0, \quad k \neq l. \tag{11.6}$$

Denote by A and B two $s \times s$-matrices of the elements $a_{k,l}$ and $b_{k,l}$ respectively, and by F and G - two Gram matrices of the elements $f_{kl} = (\Phi_{n,k}, \Phi_{n,l})_{X(\Omega_x)}$ and $g_{kl} = (\Psi_{m,k}, \Psi_{m,l})_{Y(\Omega_y)}$. Since the initial $\Sigma\Pi$-approximation is reduced, we have $F = F^* > 0$, $G = G^* > 0$. Equations (11.5), (11.6) are equivalent to matrix equations

$$A^*B = E, \quad AFA^* = D_1, \quad BGB^* = D_2, \tag{11.7}$$

where D_1, D_2 are any diagonal matrices. Taking into account $B = (A^*)^{-1}$, we obtain

$$AFGA^{-1} = D_1D_2 = (A^*)^{-1}GFA^*. \tag{11.8}$$

Is it possible to find the matrix A with this property? Let us consider the generalized eigenvalue problem

$$FGu = \lambda u, \quad Gu = \lambda F^{-1}u. \tag{11.9}$$

Since $F^{-1} = (F^{-1})^* > 0$, there are s eigenvector $u_1, u_2, ..., u_s$, which are orthogonal in the scalar product $(F^{-1}u, v)$ and the corresponding eigenvalues

$\lambda_1, \lambda_2, ..., \lambda_s$ are positive, because $G = G^* > 0$. Let the matrix U be assembled with the vectors $u_1, ..., u_s$ as columns and $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_s)$. Let us normalize the eigenvectors by $U^* F^{-1} U = \Lambda^{-1/2}$. We have $GU = F^{-1} U \Lambda$, and let $A = U^{-1}$, $B = (A^*)^{-1} = U^*$. Then

$$AFA^* = U^{-1} F (U^*)^{-1} = \Lambda^{1/2},$$
$$BGB^* = U^* F^{-1} U \Lambda U^{-1} U = U^* F^{-1} U \Lambda = \Lambda^{-1/2} \cdot \Lambda = \Lambda^{1/2}.$$

Finally, $D_1 = D_2 = \Lambda^{1/2}$ and Lemma is proved; we can search for $\Sigma\Pi$-approximation only in the orthogonal form.

11.2. Optimal $\Sigma\Pi$-Approximations

Let us represent the functions $\Phi_{n,k}(x)$ and $\Psi_{m,k}(y)$ with the basic functions

$$\begin{aligned} \Phi_{n,k}(x) &= \sum_{i=1}^{n} \alpha_i^{(k)} \varphi_i(x), \\ \Psi_{m,k}(y) &= \sum_{j=1}^{m} \beta_j^{(k)} \psi_j(y), \quad k = 1, 2, ..., s. \end{aligned} \tag{11.10}$$

Let $f(x, y)$ belong to the tensor space $Z(\Omega)$. We need to find the optimal $\Sigma\Pi$-approximation from the minimization of norm

$$E_{n,m}^{(s)} = \left\| f(x,y) - \sum_{k=1}^{s} \left(\sum_{i=1}^{n} \alpha_i^{(k)} \varphi_i(x) \right) \cdot \left(\sum_{j=1}^{m} \beta_j^{(k)} \psi_j(y) \right) \right\|_{Z(\Omega)}^2 \tag{11.11}$$

with respect to the coefficients $\alpha_i^{(k)}$, $\beta_j^{(k)}$. Using lemma, we are able to require without loss of generality that:

$$\|\Phi_{n,k}\|_{X(\Omega_x)} = \|\Psi_{m,k}\|_{Y(\Omega_y)}, \quad k = 1, 2, ..., s \tag{11.12}$$

$$(\Phi_{n,k}, \Phi_{n,l})_{X(\Omega_x)} = (\Psi_{m,k}, \Psi_{n,k})_{Y(\Omega_y)} = 0, \quad k \neq l. \tag{11.13}$$

Denote by $\bar{\alpha}^{(k)} = (\alpha_1^{(k)}, ..., \alpha_n^{(k)})^T$, $\bar{\beta}^{(k)} = (\beta_1^{(k)}, ..., b_m^{(k)})^T$. If A and B are the Gram matrices for basic elements,

$$A = \{(\varphi_i, \varphi_j)_{X(\Omega_x)}\}_{i,j=1}^{n},$$
$$B = \{(\psi_i, \psi_j)_{Y(\Omega_y)}\}_{i,j=1}^{m},$$

then we have instead of (11.12), (11.13) the following conditions:

$$\begin{aligned} (A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}) &= (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)}), \quad k = 1, 2, ..., s, \\ (A\bar{\alpha}^{(k)}, \bar{\alpha}^{(l)}) &= (B\bar{\beta}^{(k)}, \bar{\beta}^{(l)}) = 0, \quad k \neq l, \end{aligned} \tag{11.14}$$

where (,) means the usual scalar product in n-or-m-dimensional Euclidean space.

Let us introduce the rectangular $n \times m$-matrix F of elements

$$f_{ij} = (f(x,y), \quad \varphi_i(x)\psi_j(y))_{Z(\Omega)}.$$

Then the function $E_{n,m}^{(s)}(\bar{\alpha}, \bar{\beta})$ can be expressed by the formula

$$E_{n,m}^{(s)}(\bar{\alpha}, \bar{\beta}) = \|f\|^2_{Z(\Omega)} - 2\sum_{k=1}^{s}(F\bar{\beta}^{(k)}, \bar{\alpha}^{(k)}) + \sum_{k,l=1}^{s}(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(l)}) \cdot (B\bar{\beta}^{(k)}, \bar{\beta}^{(l)}).$$

Differentiation of the Lagrangian function

$$L(\bar{\alpha}, \bar{\beta}, \bar{\lambda}, \bar{\nu}, \bar{\mu}) = E_{n,m}^{(s)}(\bar{\alpha}, \bar{\beta}) + \sum_{k=1}^{s}\lambda_k\left[(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}) - (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)})\right] + \sum_{k\neq l}\left[\nu_{kl}(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(l)})^2 + \mu_{kl}(B\bar{\beta}^{(k)}, \bar{\beta}^{(l)})^2\right]$$

leads us to the following conditions at the point of minimum for every $k = 1, 2, ..., s$

$$-F\bar{\beta}^{(k)} + \sum_{l=1}^{s}(B\bar{\beta}^{(k)}, \bar{\beta}^{(l)})A\bar{\alpha}^{(k)} + \lambda_k A\bar{\alpha}^{(k)} + \sum_{l\neq k}\nu_{lk}(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(l)})A\bar{\alpha}^{(l)} = 0,$$

$$-F^*\bar{\alpha}^{(k)} + \sum_{l=1}^{s}(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(l)})B\bar{\beta}^{(k)} - \lambda_k B\bar{\beta}^{(k)} + \sum_{l\neq k}\mu_{lk}(B\bar{\beta}^{(k)}, \bar{\beta}^{(l)})B\bar{\beta}^{(l)} = 0.$$

Thus, the following non-linear system appears

$$\begin{aligned} &F\bar{\beta}^{(k)} = (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)})A\bar{\alpha}^{(k)} + \lambda_k A\bar{\alpha}^{(k)}, \\ &F^*\bar{\alpha}^{(k)} = (A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)})B\bar{\beta}^{(k)} - \lambda_k B\bar{\beta}^{(k)}, \\ &(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}) = (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)}), \quad k = 1, 2, ..., s, \\ &(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(l)}) = (B\bar{\beta}^{(k)}, \bar{\beta}^{(l)}) = 0, \quad k \neq l, \quad k, l = 1, 2, ..., s. \end{aligned} \tag{11.15}$$

It is clear from this system that

$$(F\bar{\beta}^{(k)}, \bar{\alpha}^{(k)}) = ((B\bar{\beta}^{(k)}, \bar{\beta}^{(k)}) + \lambda_k) \cdot (A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}),$$
$$(F^*\bar{\alpha}^{(k)}, \bar{\beta}^{(k)}) = ((A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}) - \lambda_k) \cdot (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)}).$$

Taking the difference of these expressions we obtain $\lambda_k = 0$ when $(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}) \neq 0$, i.e. $\bar{\alpha}^{(k)} \neq 0$. Trivial solutions $\bar{\alpha}^{(k)} = \bar{\beta}^{(k)} = 0$ are not interesting for $\Sigma\Pi$-approximation and finally we have non-linear system

$$\begin{aligned} F\bar{\beta}^{(k)} &= (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)})A\bar{\alpha}^{(k)}, \\ F^*\bar{\alpha}^{(k)} &= (A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)})B\bar{\beta}^{(k)}, \quad k = 1, 2, ..., s, \end{aligned} \tag{11.16}$$

with the natural normalization condition $(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}) = (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)})$, $k = 1, 2, ..., s$ and orthogonal property $(A\bar{\alpha}^{(k)}, \bar{\alpha}^{(l)}) = (B\bar{\beta}^{(k)}, \bar{\beta}^{(l)}) = 0$ for $k, l = 1, 2, ..., s$, $k \neq l$. It is clear from our consideration that the vector $[\bar{\alpha}^{(k)}, \bar{\beta}^{(k)}]^T$ is always some eigenvector for the generalized eigenvalue problem

$$\begin{pmatrix} 0 & F \\ F^* & 0 \end{pmatrix} \begin{pmatrix} u \\ \nu \end{pmatrix} = \lambda \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} u \\ \nu \end{pmatrix}. \tag{11.17}$$

Let us discuss in details this eigenvalue problem. Since the block matrix in the right-hand side is symmetric and positive, there exist $(n+m)$ eigenvectors $[u_i, \nu_i]^T$ corresponding to the real eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n+m}$. These vectors can be made orthonormal in the sense of special scalar product

$$(Au_i, u_j) + (B\nu_i, \nu_j) = \delta_{ij}. \tag{11.18}$$

If $\lambda_i \neq 0$, we have $(Au_i, u_i) = (B\nu_i, \nu_i)$. Actually,

$$\begin{aligned} (F^*u_i, \nu_i) &= \lambda(B\nu_i, \nu_i), \\ (F\nu_i, u_i) &= \lambda(Au_i, u_i), \end{aligned}$$

and $\lambda[(Au_i, u_i) - (B\nu_i, \nu_i)] = 0$. Moreover, if eigenvectors $[u_i, \nu_i]^T$ and $[u_j, \nu_j]^T$ correspond to the various eigenvalues $\lambda_i > 0$ and $\lambda_j > 0$, then we have the block orthogonal property

$$(Au_i, u_j) = (B\nu_i, \nu_j) = 0.$$

Really we have

$$\begin{aligned} &F\nu_i = \lambda_i Au_i, \quad F\nu_j = \lambda_j Au_j, \\ &F^*u_i = \lambda_i B\nu_i, \quad F^*u_j = \lambda_j B\nu_j, \\ &(Au_i, u_j) + (B\nu_i, \nu_j) = 0. \end{aligned}$$

Hence,

$$\begin{aligned} &(F\nu_i, u_j) = \lambda_i(Au_i, u_j) = (F^*u_j, \nu_i) = \lambda_j(B\nu_i, \nu_j) \\ &\lambda_i(Au_i, u_j) = -\lambda_j(Au_i, u_j), \quad -\lambda_i(B\nu_i, \nu_j) = \lambda_j(B\nu_i, \nu_j) \\ &(\lambda_i + \lambda_j)(Au_i, u_j) = (\lambda_i + \lambda_j)(B\nu_i, \nu_j) = 0. \end{aligned}$$

If $\lambda_i > 0$ is a multiple eigenvalue, then the corresponding eigenvectors can be made with the block orthogonal property. And the last evident property: if $\lambda > 0$ is the eigenvalue corresponding to the vector $[u, \nu]$, then $(-\lambda)$ is also an eigenvalue, because the suitable eigenvector is $[u, -\nu]$.

Thus, eigenvectors of generalized eigenvalue problem (11.17) satisfy the natural conditions of normalization and orthogonality for system (11.16). We find now the solution of this system with the help of the normalized eigen vectors $[u_k, \nu_k]^T$, $k = 1, 2, ..., s$. Let us find the solutions $\bar{\alpha}^{(k)}$, $\bar{\beta}^{(k)}$ in the form

$$\bar{\alpha}^{(k)} = C_k u_k, \quad \bar{\beta}^{(k)} = C_k \nu_k, \quad C_k = \text{const}. \tag{11.19}$$

After substitution to system (11.16) we obtain

$$F\bar{\beta}^{(k)} = C_k F \nu_k = \lambda_k C_k A u_k = C_k^3 (B\nu_k, \nu_k) A u_k,$$
$$F^* \bar{\alpha}^{(k)} = C_k F^* u_k = \lambda_k C_k A \nu_k = C_k^3 (B u_k, u_k) A \nu_k.$$

Since $(Au_k, u_k) + (B\nu_k, \nu_k) = 1$ and $(Au_k, u_k) = (B\nu_k, \nu_k)$, we have $(Au_k, u_k) = (B\nu_k, \nu_k) = 1/2$. Thus,

$$\lambda_k = C_k^2/2, \quad C_k = \sqrt{2\lambda_k}, \quad k = 1, 2, ..., s. \tag{11.20}$$

If we take s positive eigenvalues $\lambda_1 > 0, \lambda_2 > 0, ..., \lambda_s > 0$, we obtain for this choice

$$E_{n,m}^{(s)} = \|f\|^2_{Z(\Omega)} - 2\sum_{k=1}^{s} (F\bar{\beta}^{(k)}, \bar{\alpha}^{(k)}) + \sum_{k=1}^{s} (A\bar{\alpha}^{(k)}, \bar{\alpha}^{(k)}) \cdot (B\bar{\beta}^{(k)}, \bar{\beta}^{(k)})$$
$$= \|f\|^2_{Z(\Omega)} - \frac{1}{4}\sum_{k=1}^{s} (\sqrt{2\lambda_k})^4 = \|f\|^2_{Z(\Omega)} - \sum_{k=1}^{s} \lambda_k^2.$$

So, the following theorem is already proved:

Theorem 11.1. Let generalized eigenvalue problem (11.17) have Q positive eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_Q > 0$. Then for every $s \leq Q \leq (n+m)/2$ the optimal coefficients $\bar{\alpha}^{(k)}$, $\bar{\beta}^{(k)}$ for the best $\Sigma\Pi$-approximation are connected with the normalized eigenvectors $[u_k, \nu_k]^T$ by the formula

$$[\bar{\alpha}^{(k)}, \bar{\beta}^{(k)}]^T = \sqrt{2\lambda_k}[u_k, \nu_k]^T \tag{11.21}$$

and for the optimal $\Sigma\Pi$-approximation we have the following error estimate

$$E_{n,m}^{(s)} = \|f\|^2_{Z(\Omega)} - \sum_{k=1}^{s} \lambda_k^2. \tag{11.22}$$

It is important in practice that the eigenvalues and eigenvectors can be determined consequently in the order of decay for λ_k, and we obtain at every step the optimal $\Sigma\Pi$ -approximation with one term, with two terms, and so on. Formula (11.22) provides the effective accuracy control.

The generalized eigenvalue problem (11.17) with $(n+m) \times (n+m)$-matrix can be reduced to the usual eigenvalue problem with $n \times n$ or $m \times m$-matrix. At first we ought to Cholessky decomposition of the Gram matrix A and B to the triangular factors,

$$A = LL^*, \quad B = MM^*.$$

Then the initial relations

$$Fu = \lambda Au, \quad F^*u = \lambda B\nu$$

can be rewritten in the form

$$L^{-1}F(M^*)^{-1}z = \lambda\omega, \quad M^{-1}F^*(L^*)^{-1}\omega = \lambda z,$$

where $\omega = L^*u, \quad z = M^*\nu$. After a simple transformation we have a usual eigenvalue problem

$$[L^{-1}F(M^*)^{-1}]^*[L^{-1}F(M^*)^{-1}]z = \lambda^2 z \tag{11.23}$$

with $m\times m$-symmetric, non-negative matrix. If $\lambda > 0$ and z is the corresponding eigenvector, then

$$\nu = (M^*)^{-1}z, \quad u = \frac{1}{\lambda}(L^*)^{-1}L^{-1}F(M^*)^{-1}z = \frac{1}{\lambda}A^{-1}F\nu$$

give us the eigenvector of initial problem (11.17) after suitable normalization.

Theorem 11.2. If $f(x,y) \in X_n \otimes Y_m$, in the other words, $f(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\varphi_i(x)\psi_j(y)$, then its optimal $\Sigma\Pi$-approximation is equal to $f(x,y)$ for some $s \le (n+m)/2$.

Proof. Denote by α the rectangular $n \times m$-matrix of the elements $\alpha_{ij}, \quad i = 1,2,..,n, \quad j = 1,2,...,m$. The element f_{ij} of the matrix F can be written by the formula

$$f_{ij} = (f(x,y),\varphi_i(x)\psi_j(y))_{Z(\Omega)} = \sum_{k=1}^{n}\sum_{l=1}^{m}\alpha_{kl}(\varphi_k,\varphi_i)_X\cdot(\psi_l,\psi_j)_Y.$$

It means that $F = A\alpha B$, where A and B are the Gram matrices of the elements $\{\varphi_k\}_{k=1}^n$ and $\{\psi_l\}_{l=1}^m$. Eigenvalue problem (11.17) is reduced to

$$\begin{aligned} A\alpha B\nu &= \lambda Au, \\ B\alpha^* Au &= \lambda B\nu. \end{aligned} \tag{11.24}$$

By substituation of the Cholessky decompositions $A = LL^*, \quad B = MM^*$ we obtain

$$\begin{aligned} L^*\alpha Mz &= \lambda\omega, \quad \omega = L^*u, \\ M^*\alpha^*L\omega &= \lambda z, \quad z = M^*\nu. \end{aligned}$$

For this eigenvalue problem with a block symmetric matrix, the sum of squared eigenvalues is equal to the square of the spherical norm for the matrix (the spherical norm is invariant with respect to orthogonal transformation!). But λ and $(-\lambda)$ are eigenvalues simultaneously. Therefore,

$$\sum_{\lambda_k>0} \lambda_k^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\sum_{k=1}^{n} \sum_{l=1}^{m} l_{ki} \alpha_{kl} m_{lj} \right]^2, \tag{11.25}$$

where l_{ki}, m_{lj} are elements of the matrices L and M. On the other hand

$$\begin{aligned} \|f\|_{Z(\Omega)}^2 &= \left(\sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_{kl} \varphi_k(x) \psi_l(y), \sum_{r=1}^{n} \sum_{s=1}^{m} \alpha_{rs} \varphi_r(x) \psi_s(y) \right)_{Z(\Omega)} \\ &= \sum_{k,r=1}^{n} \sum_{l,s=1}^{m} \alpha_{kl} \alpha_{rs} a_{kr} b_{ls}. \end{aligned} \tag{11.26}$$

But

$$a_{kr} = \sum_{i=1}^{n} l_{ki} l_{ri}, \quad b_{ls} = \sum_{j=1}^{m} m_{lj} m_{sj}.$$

After substitution we have for the right-hands sides in (11.25), (11.26)

$$\sum_{i,k,r=1}^{n} \sum_{j,l,s=1}^{m} l_{ki} \alpha_{kl} m_{lj} l_{ri} \alpha_{rs} m_{sj} = \sum_{k,r,i=1}^{n} \sum_{l,s,j=1}^{m} \alpha_{kl} \alpha_{rs} l_{ki} l_{ri} m_{lj} m_{sj}.$$

Finally, $\|f\|_{Z(\Omega)}^2 - \sum_{\lambda_k>0} \lambda_k^2 = 0$ and the number of positive eigenvalues is not more than $(n+m)/2$. Theorem is proved. □

11.3. Examples of ΣΠ-Approximations

11.3.1. Two-Dimensional Polynomial Splines and ΣΠ-Approximation

Let $\Omega_x = [a, b]$, $\Omega_y = [c, d]$, $\Omega = \Omega_x \times \Omega_y$, and $X(\Omega_x) = W_2^{m_x}(a, b)$, $Y(\Omega_y) = W_2^{m_y}(c, d)$. The Hilbert tensor product of these spaces is the space $W_{2,2}^{m_x, m_y}(\Omega)$ with the cross norm

$$\begin{aligned} \|u\|_{W_{2,2}^{m_x,m_y}(\Omega)} = \Big(&\|u\|_{L_2(\Omega)}^2 + \|D^{m_x,0} u\|_{L_2(\Omega)}^2 + \|D^{0,m_y} u\|_{L_2(\Omega)}^2 \\ &+ \|D^{m_x,m_y} u\|_{L_2(\Omega)}^2 \Big)^{1/2} \end{aligned} \tag{11.27}$$

because the norms in the spaces X and Y can be introduced by formulae

$$\begin{aligned} \|u\|_{W_2^{m_x}(a,b)} &= \left(\int_a^b [u^2 + (u^{(m_x)})^2] dx \right)^{1/2}, \\ \|u\|_{W_2^{m_y}(c,d)} &= \left(\int_c^d [u^2 + (u^{(m_y)})^2] dy \right)^{1/2}. \end{aligned} \tag{11.28}$$

Let us introduce two meshes

$$\Delta_x = \{a = x_1 < x_2 < ... < x_{N_x} = b\},$$
$$\Delta_y = \{c = y_1 < y_2 < ... < y_{N_y} = b\},$$

and connect with them two spaces $S^{K_x}(\Delta_x)$, $S^{K_y}(\Delta_y)$ of polynomial splines of the degrees K_x and K_y and of defect 1, i.e. $S^{K_x}(\Delta_x) \subset C^{K_x-1}[a,b]$, $S^{K_y}(\Delta_y) \subset C^{K_y-1}[a,b]$. It means that $S^{K_x}(\Delta_x)$ is a subspace in $W_2^{m_x}(a,b)$ if $K_x \geq m_x$, and $S^{K_y}(\Delta_y) \subset W_2^{m_y}(c,d)$ for $K_y \geq m_y$. It is a well-known fact that these subspaces have the basises of local B-splines (see Chapter 5), and the Gram matrices A and B become band ones.

By Theorem 11.2 every bivariate spline obtained, for example, by the interpolation of the function on the rectangular mesh, can be represented exactly as $\Sigma\Pi$-approximation. If we have some available level of accuracy of approximation, then we are able to compress the data (coefficients of the interpolating spline on the huge mesh) by $\Sigma\Pi$-approximation.

Remark. It is possible to repeat this construction (especially, for compression of a digital image) for the case of discrete splines. The mesh analogues of norms (11.28) are natural

$$\|u\|^2_{\tilde{W}_2^{m_x}} = \sum_{i=1}^{N_x} u^2(i) + \sum_{i=1}^{N_x-m_x} (\Delta_{m_x} u)^2(i),$$
$$\|\nu\|^2_{\tilde{W}_2^{m_y}} = \sum_{j=1}^{N_y} \nu^2(j) + \sum_{j=1}^{N_y-m_y} (\Delta_{m_y} \nu)^2(j),$$

where Δ_{m_x} and Δ_{m_y} mean the divided difference of the orders m_x, m_y. The corresponding cross-norm is also natural

$$\|f\|^2_{\tilde{W}_{2,2}^{m_x,m_y}} = \sum_{i,j=1}^{N_x,N_y} f^2(i,j) + \sum_{i,j=1}^{N_x-m_x,N_y} (\Delta_{m_x} f)^2(i,j)$$
$$+ \sum_{i,j=1}^{N_x,N_y-m_y} (\Delta_{m_y} f)^2(i,j) + \sum_{i,j=1}^{N_x-m_x,N_y-m_y} (\Delta_{m_x}\Delta_{m_y} f)^2(i,j).$$

The discrete analogues of B-splines can be constructed with the help of a few convolutions of simple discrete "step-functions" of the type

$$B(i) = \begin{cases} 1, & -\omega \leq i \leq \omega \\ 0 & \text{otherwise}, \end{cases}$$

where $\omega > 0$ is integer parameter.

11.3.2. Fourier Expansions and $\Sigma\Pi$-Approximations

Example 1. Let $\Omega = [0,1] \times [0,1]$ be the unit square and $W_{2,2}^{m_x,m_y}(\Omega) = W_2^{m_x}(0,1) \otimes W_2^{m_y}(0,1)$ be the tensor Hilbert space with cross-norm (11.27). Let us introduce in $W_2^{m_x}(0,1)$ and in $W_2^{m_y}(0,1)$ the finite-dimensional subspaces $T_{m_x}^N(0,1)$ and $T_{m_y}^M(0,1)$ of trigonometric polynomials. The first of them is linear span of the functions

$$\{\sin k\pi x,\ \cos k\pi x\}, \quad k = 1,2,...,N \tag{11.29}$$

and the second is the span of the functions

$$\{\sin l\pi y,\ \cos l\pi y\}, \quad l = 1,2,...,M. \tag{11.30}$$

After a suitable normalization of system (11.29) by multiplying the constants $C_k = (2/(1+(k\pi)^{2m_x}))^{1/2}$, they form the orthonormal system in $W_2^{m_x}$-scalar product. Correspondingly functions (11.30) are also orthonormal in $W_2^{m_y}$-scalar product after multiplying the constants $D_k = (2/(1+(l\pi)^{2m_y}))^{1/2}$. Certainly, in this case calculation of the optimal $\Sigma\Pi$-apprtoximation is simpler because the Gram matrices A and B in (11.17) are units and generalized eigen value problem becomes usual. The optimal approximation of the function $f(x,y)$ can be expressed in the form

$$\sum_{k=1}^{s} T_{N,k}(x) \cdot T_{M,k}(y),$$

where $T_{N,k}(x)$ and $T_{M,k}(y)$ are trigonometrical polynomials of the orders N and M.

Example 2. Let $\Omega_1 = \{1,2,...,N\}$ and $\Omega_2 = \{1,2,...,M\}$ be the sets of integers. Let us introduce the spaces $H(\Omega_1)$ and $H(\Omega_2)$ of the real valued mesh functions $u(i)$ and $\nu(j)$, which are defined on Ω_1 and Ω_2 with the following norms

$$\|u\|_{H(\Omega_1)}^2 = \sum_{i=1}^{N} u^2(i), \quad \|\nu\|_{H(\Omega_2)}^2 = \sum_{j=1}^{M} \nu^2(j). \tag{11.31}$$

The tensor product $H(\Omega_1 \times \Omega_2)$ of these spaces is the space of two-variable mesh functions $f(i,j)$ with the simple cross-norm

$$\|f\|_{H(\Omega_1\times\Omega_2)}^2 = \sum_{i=1}^{N} \sum_{j=1}^{M} f^2(i,j). \tag{11.32}$$

There is a well-known fact that the vectors (or mesh functions) of the type

$$W_k^N(i) = \sqrt{2/(N+1)}\ \sin\frac{k\pi i}{N+1}, \quad i = 1,2,...,N$$

form for $k = 1, 2, ..., N$ the orthonormal basis in N-dimensional Euclidean space E_N. Let us fix the integers $k_1, k_2, ..., k_n$ from Ω_1 and $l_1, l_2, ..., l_m$ from Ω_2 and define the subspaces

$$H_n(\Omega_1) = \text{span}\{W_{k_1}^N, W_{k_2}^N, ..., W_{k_n}^N\}, \tag{11.33}$$

$$H_m(\Omega_2) = \text{span}\{W_{l_1}^M, W_{l_2}^M, ..., W_{l_m}^M\} \tag{11.34}$$

in the space $H(\Omega_1)$, $H(\Omega_2)$ respectively. In practical computations the fast Fourier transform can be used for the fast calculation of the coefficients

$$f_{sr} = \big(f(i,j),\, W_{k_s}^N(i) \cdot W_{l_r}^M(j)\big)_{H(\Omega_1 \times \Omega_2)}$$

and the matrix F in eigenvalue problem (11.17) is easily calculated. The optimal $\Sigma\Pi$-approximation

$$\sum_{r=1}^{s} U_r(i)V_r(j), \quad U_r(i) = \sum_{q=1}^{n} \alpha_q^{(r)} W_{k_q}^N(i), \quad V_r(j) = \sum_{q=1}^{m} \beta_q^{(r)} W_{l_q}^M(j)$$

can be used here not only for compression of the image $f(i,j)$, but also for filtration because only separate frequencies are presented in $\Sigma\Pi$-approximation.

It is clear that other kinds of orthogonal or non-orthogonal mesh functions can be used in this algorithm (see discrete Haar, Walsh, Hadamard and other transforms).

11.3.3. Numerical Tests

Example 1. Let us consider two-dimensional discrete function

$$f(i,j) = \sin(4\pi \cdot \frac{i-1}{129} \cdot \frac{j-1}{129}) \tag{11.35}$$

of two integer arguments i and j, every of them varies from 1 to 130. It means that we have instead of continuous function

$$f(x,y) = \sin(4\pi xy) \tag{11.36}$$

its discrete 130×130-image (see Fig. 11.1). Let us fix $w = 3$ and consider the space $S_w^{3,3} = S_w^3 \otimes S_w^3$ which consists of discrete analigs of bicubic splines over rectangular mesh with the discrete mesh step w. Space S_w^3 of one-dimensional cubic discrete splines has the usual basis of local discrete B-splines obtained by three convolutions and by corresponding shifts (see Remark in Section 3.1).

In two-variable discrete space of mesh functions the usual discrete analog of $W_{2,2}^{2,2}$ - cross-norm is introduced ($m_x = m_y = 2$, $N_x = N_y = 130$) and optimal $\Sigma\Pi$-approximations are calculated for the various accuracy levels. For example if $\varepsilon = 0.1\%$ then we have only 7 non-trivial components in $\Sigma\Pi$-approximation and corresponding eigen values are

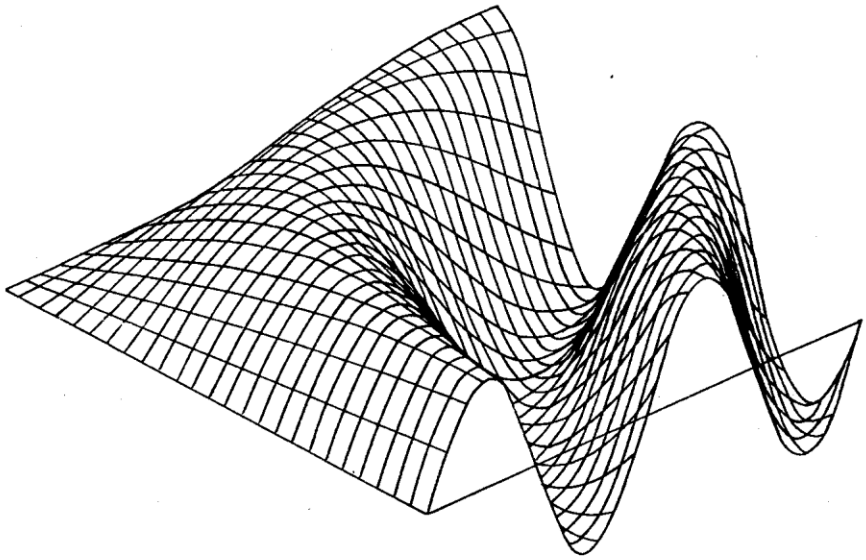

Fig. 11.1.

$\lambda_1 = 0.350834$ $\quad\lambda_5 = 0.094213$
$\lambda_2 = 0.350832$ $\quad\lambda_6 = 0.010491$
$\lambda_3 = 0.349698$ $\quad\lambda_7 = 0.000721$
$\lambda_4 = 0.297875$

The rest eigen values are less then 10^{-5}. Thus the compression coefficient in this case is 2786.

The universal program for data compression using discrete analogs of polynomial splines of various degrees and various kinds of cross-norms were created by Olga Baklanova in Computing Center of Russian Academy of Science, Novosibirsk (see also Appendix 2, subpackage SIGPI).

Example 2. Let us consider the frequency response of two-dimensional filter

$$f(x,y) = \frac{\sin(10r)}{r}, \quad r = [(x-0.5)^2 + (y-0.5)^2]^{1/2} \tag{11.37}$$

on the unit square $[0,1] \times [0,1]$ and replace this function by piecewise constant function $f_h(x,y)$ on 31×31 - iniform grid. The value at the elementary mesh square is equal to the value of $f(x,y)$ at the middle point of square. With the help of $\Sigma\Pi$-decomposition of this function we reduce two-dimensional filtration process to one-dimensional row-colomn filtrations. In L_2 - cross-norm only 5 non-trivial eigen values aries, and two of them are small,

$\lambda_1 = 2.937978, \quad \lambda_2 = 0.951244, \quad \lambda_3 = 0.079201,$
$\lambda_4 = 0.000189, \quad \lambda_5 = 0.115592 \times 10^{-6},$

and relative errors for corresponding $\Sigma\Pi$-approximations are

$$e_1 = 0.309000, \quad e_2 = 0.025639, \quad e_3 = 0.000139,$$
$$e_4 \leq 10^{-8}, \qquad e_5 \leq 10^{-11}$$

Thus, the action of complicated two-dimensional filter with 900 coefficients can be efficiently replaced by 2 (or 3) one-dimensional filters with 30 coefficients with respect to rows and the same number of one-dimensional filters with respect to columnes.