# 2. Reproducing Mappings and Characterization of Splines

In the previous chapter, you learned something about splines and their properties. Now you know some types of splines (interpolating, smoothing and mixed), some criteria of existence and uniqueness of such splines, some examples of spline-functions.

This chapter deals with such an important aspect in the spline theory as characterization. Here we introduce not a well-known notion of the reproducing mapping in semi-Hilbert spaces, which is the generalization of notions of the reproducing kernel in the functional Hilbert spaces, and carry out an investigation revealing its essence and properties. We do it successively because of the great importance of this notion with respect to the problem of spline characterization and, as we think, many other problems. In a sense, this investigation moves forward the papers (Aronszajn 1950; Atteia 1970; Duchon 1977).

Proving general theorems of characterization for various types of splines we draw your attention to particular cases, too. We successively simplify the abstract interpolating operator $A$ and obtain a simpler formulation of theorems becoming more constructive and practical. Another very important way to solve the problem of spline characterization, based on the finite-element approach, will be proposed in Chapter 4. Further, these two ways will be widely used in the following chapters.

Chapter 2 is well illustrated with examples. These are hyperbolic splines, periodic and non-periodic polynomial splines, splines in the space of mesh functions, splines on the sphere. Many other examples will be discussed in other Chapters, where necessary definitions will be introduced in a more natural and simple manner. We mean the chapters about splines on manifolds, vector and tensor splines, and so on.

## 2.1. Reproducing Mappings and Kernels

### 2.1.1. Definitions

Consider a real Hilbert space $X$ with a scalar product $(u, v)_X$ and a norm $\|u\|_X$. Let $X^*$ denote the space of linear continuous functionals, i.e. $X^* = \mathcal{L}(X, \mathbb{R})$. The Riesz theorem asserts that there exists the unique isomorphism of linear spaces

$$\pi : X^* \to X, \tag{2.1}$$

which for all $L \in X^*$ satisfies conditions

$$L(u) = (\pi(L), u)_X, \quad \forall u \in X. \tag{2.2}$$

**Definition 2.1.** *We will call the isomorphism $\pi$ a reproducing mapping of the Hilbert space $X$.*

So, by the Riesz theorem the reproducing mapping of the Hilbert space exists and is unique. Let $P$ be a Hilbert subspace in $X$.

**Definition 2.2.** *A symmetric bilinear form $(u,v)_P : X \times X \to \mathbb{R}$ is called a scalar semi-product and the induced functional $|u|_P = \sqrt{(u,u)_P}$ is called a semi-norm, if the last possess the following four properties:*

1) $|u|_P \geq 0, \quad \forall u \in X,$
2) $|u|_P = 0 \Leftrightarrow u \in P,$
3) $|\lambda u|_P = |\lambda||u|_P, \quad \forall u \in X, \quad \forall \lambda \in \mathbb{R},$
4) $|u+v|_P < |u|_P + |v|_P, \quad \forall u, v \in X.$

**Lemma 2.1.** The scalar semi-product and the semi-norm satisfy the following three properties

$$|(u,v)_P| \leq |u|_P |v|_P, \tag{2.3a}$$

$$(u+p_1, v+p_2)_P = (u,v)_P, \tag{2.3b}$$

$$|u+p_1|_P = |u|_P. \tag{2.3c}$$

for any $u, v \in X, \quad p_1, p_2 \in P$.

*Proof.* The first property is proved in the conventional way by considering the discriminant of the quadratic positive polynomial on the parameter $\lambda$ : $(u + \lambda v,\ u + \lambda v)_P$. The second is proved using linearity of the scalar semi-product:

$$(u+p_1, v+p_2)_P = (u,v)_P + (u,p_2)_P + (p_1,v)_P + (p_1,p_2)_P.$$

Applying the first property we have

$$|(u,p_2)_P| \leq |u|_P |p_2|_P = 0,$$

and, also, $(p_1,v)_P = 0$, $(p_1,p_2)_P = 0$. The third property trivially follows from the second one. □

Denote by $X/P$ the space of the factor-classes $u+P$, where $u$ is an arbitrary element from $X$.

**Definition 2.3.** *The space $X$ with an additionally introduced scalar semi-product and semi-norm is called the semi-Hilbert space, if*

1) *The seminorm is majorized by the norm, i.e.*

$$|u|_P \leq c\|u\|_X \; ; \tag{2.4}$$

2) *The factor-space $X/P$ is hilbertian relatively to the following scalar product and norm*

$$(u+P, v+P)_{X/P} = (u,v)_P \; , \tag{2.5}$$

$$\|u+P\|_{X/P} = |u|_P \; . \tag{2.6}$$

It is obvious from the previous Lemma that scalar product (2.5) and norm (2.6) are correctly introduced.

Let us consider the operator $T : X \to X/P$, putting in correspondence to the element $u \in X$ the factor class $u+P$. Then, owing to (2.4), $T$ is a continuous operator:

$$\|Tu\|_{X/P} = \|u+P\|_{X/P} = |u|_P \leq c\|u\|_X .$$

The linear space $P^0 = \{L \in X^* : \; L(u) = 0, \; \forall u \in P\}$ is called the annulator of the subspace $P$. Obviously, $\dim P^0 = \operatorname{codim} P$.

**Definition 2.4.** *The linear mapping $\pi_P : X^* \to X$ will be called a reproducing mapping of the semi-Hilbert space $X$, if*

$$L(u) = (\pi_P(L), u)_P , \quad \forall u \in X, \tag{2.7}$$

*for all $L$ from $P^0 \subset X^*$.*

The bilinear form $(u,v)_P$ is annihilated in $X$ on the elements from $P$, thus, it is clear why not all the functionals $L \in X^*$ are represented in form (2.7).

### 2.1.2. Basic Properties of Reproducing Mappings

Here we formulate (in the form of Theorems) and prove some basic properties of reproducing mappings, which will be used further.

**Theorem 2.1.** The reproducing mapping $\pi_P$ of the semi-Hilbert space $X$ always exists.

*Proof.* Let $X_P$ be the orthogonal complement to $P$ in the Hilbert space $X$, i.e. $X = P \oplus X_P$ . Then, any element $u \in X$ has unique expansion

$$u = u_1 + u_2, \quad u_1 \in P, \quad u_2 \in X_P, \tag{2.8}$$

where $(u_1, u_2)_P = 0, \quad \forall u \in X$. Let us use the extended notation $(X, \|\cdot\|_X)$ instead of $X$, if we need an exact indication of the norm we mean. Prove that

$(X_P, |\cdot|)_P$ is a Hilbert space, i.e. $|\cdot|_P$ is a Hilbert norm in $X_P$. Really, introduce the mapping

$$\Re : (X/P, \|\cdot\|_{X/P}) \to (X_P, |\cdot|_P),$$

which maps the element $u + P$ onto the unique element $u_2$ according to expansion (2.8). Clearly, $\Re$ is a linear isomorphism. Norms of the elements $u + P$ and $u_2$ coincide: $\|u + P\|_{X/P} = |u|_P = |u_2|_P$. It could be shown that $\Re$ preserves also the scalar product. Consequently, $(X_P, |\cdot|_P)$ is the Hilbert space because it is isomorphique to the Hilbert factor-space $X/P$.

According to property (2.4) the norm $|\cdot|_P$ is majorized by the norm $\|\cdot\|_X$ in the space $X_P$ . Owing to the Banach theorem, these norms are equivalent. This fact allows us to conclude that the conjugate spaces $(X_P, |\cdot|_P)^*$ and $(X_P, \|\cdot\|_X)^*$ coincide as sets. To this end we will write $X_P^*$ instead of these two.

Let us introduce the reproducing mappings of some Hilbert spaces:

$$\pi : X^* \to (X, \|\cdot\|_X), \quad \rho : X_P^* \to (X_P, \|\cdot\|_X), \quad \rho_P : X_P^* \to (X_P, |\cdot|_P).$$

They exist and are unique. Prove the following Lemma.

**Lemma 2.2.** If $L \in P^0$ , then $\pi_P(L) = \rho_P \rho^{-1} \pi(L)$, i.e. the mapping $\pi_P$ is uniquely defined in the space $P^0$.

*Proof.* Since $L \in P^0$ , then $L(u) = 0$ for all $u \in P$. According to equality (2.2) $\pi(L) \perp P$ or, which is the same, $\pi(L) \in X_P$. Utilizing the properties of reproducing mappings we have the following equalities

$$\begin{aligned} L(u) &= (\pi(L), u)_X = (\pi(L), u_2)_X = \rho^{-1}\pi(L)(u_2) = (\rho_P \rho^{-1}\pi(L), u_2)_P \\ &= (\rho_P \rho^{-1}\pi(L), u)_P, \end{aligned}$$

which prove the Lemma. □

To finish the proof of Theorem 2.1 we must extend the mapping $\rho_P \rho^{-1}\pi$ to the whole $X^*$ preserving linearity. It may be fulfilled, for example, with the help of the project operator $\mathrm{Pr}_P : X \to X_P$. Then the mapping

$$\pi_P \rightleftharpoons \rho_P \rho^{-1} \mathrm{Pr}_P \pi \tag{2.9}$$

will be reproducing for the semi-Hilbert space $X$. □

**Theorem 2.2.** Any reproducing mapping $\pi_P$ of the semi-Hilbert space $X$ is symmetric on the space $P^0$, i.e. for all $L_1, L_2 \in P^0$ it satisfies the condition

$$L_1 \pi_P(L_2) = L_2 \pi_P(L_1).$$

Moreover, there exists a reproducing mapping $\pi_P$, which is symmetric on the whole $X^*$.

*Proof.* Symmetry of the reproducing mapping on $P^0$ evidently follows from Definition 2.4 and from symmetry of the scalar semi-product:

$$L_1\pi_P(L_2) = (\pi_P(L_1), \pi_P(L_2))_P = (\pi_P(L_2), \pi_P(L_1))_P = L_2\pi_P(L_1).$$

Show that mapping (2.9) is symmetric on the whole $X^*$. Using the properties of the mappings $\pi$ and $\pi_P$ we have

$$L_1\pi_P(L_2) = L_1\rho_P\rho^{-1}\mathrm{Pr}_P\pi(L_2) = (\pi(L_1), \rho_P\rho^{-1}\mathrm{Pr}_P\pi(L_2))_X. \tag{2.10}$$

Note that the element $\rho_P\rho^{-1}\mathrm{Pr}_P\pi(L_2)$ is from $X_P$ and the element $\pi(L_1) - \mathrm{Pr}_P\pi(L_1)$ is from P, therefore, the last member in equalities (2.10) is equal to

$$(\mathrm{Pr}_P\pi(L_1), \rho_P\rho^{-1}\mathrm{Pr}_P\pi(L_2))_X .$$

Since both elements from the last scalar product belong to $X_P$, then it reduces to

$$\rho^{-1}\mathrm{Pr}_P\pi(L_1)(\rho_P\rho^{-1}\mathrm{Pr}_P\pi(L_2)) ,$$

and using the property of the reproducing mapping $\rho_P$ the latter one reduces to $(\rho_P\rho^{-1}\mathrm{Pr}_P\pi(L_1), \rho_P\rho^{-1}\mathrm{Pr}_P\pi(L_2))_P$ . This expression is symmetric relatively to $L_1$ and $L_2$. Thus, the proof of Theorem 2.2 is completed. □

Let $L = \{L_1, \ldots, L_n\}$ be a set of functionals from $X^*$ . Denote by $\mathbb{R}_L$ the subspace in $\mathbb{R}^n$ consisting of the vectors $(a_1, \ldots, a_n)$ satisfying the following condition

$$\sum_{j=1}^{n} a_j L_j \in P^0 .$$

**Theorem 2.3.** The matrix $G = \{L_i\pi_P(L_j)\}_{i=1,\ldots,n}^{j=1,\ldots,n}$ is symmetric and positive in the subspace $\mathbb{R}_L$. It is positively defined in the space $\mathbb{R}_L$ if the functionals $L_1, \ldots, L_n$ form a linear independent system.

*Proof.* If any functional from the set $\{L_1, \ldots, L_n\}$ is from $P^0$, then $\mathbb{R}_L = \mathbb{R}^n$ and symmetry in the subspace means ordinary matrix symmetry. In this case symmetry follows immediately from Theorem 2.2.

Symmetry of the matrix $G$ in the space $\mathbb{R}_L$ signifies that

$$(Ga, b) = (a, Gb), \quad \forall a \in \mathbb{R}_L,\ b \in \mathbb{R}_L,$$

where brackets denote the Euclidean scalar product in $\mathbb{R}^n$. Since the functionals $A = \sum_{j=1}^{n} a_j L_j$ and $B = \sum_{j=1}^{n} b_j L_j$ are from the space $P_0$, then we have

$$\begin{aligned}(Ga, b) &= \sum_{i=1}^{n}(\sum_{j=1}^{n} L_i\pi_P(L_j)a_j)b_i = (\sum_{i=1}^{n} b_i L_i)(\pi_P(\sum_{j=1}^{n} a_j L_j)) \\ &= B(\pi_P(A)) = (\pi_P(B), \pi_P(A))_P = (\pi_P(A), \pi_P(B))_P = (Gb, a)\end{aligned}$$

and symmetry of the matrix $G$ is proved.

The positiveness signifies that $(Ga, a) \geq 0,\ \forall a \in \mathbb{R}_L$. But from the previous formulas we have the equalities

$$(Ga, a) = (\pi_P(A), \pi_P(A))_P = |\pi_P(A)|_P^2,$$

which confirm the positiveness. The matrix $G$ is positive defined if $(Ga, a) > 0$ for all $a \in \mathbb{R}_L \backslash \{0\}$. Using the latter equalities we see that it is sufficient to prove that equality

$$|\pi_P(A)|_P^2 = 0 \tag{2.11}$$

holds only if $a = 0$. From equality (2.11) and Definition 2.2 it follows that $\pi_P(A) \in P$, but then $A(u) = (\pi_P(A), u)_P$ is equal to zero. Thus, $A = 0$ and, hence, $a = 0$, because the set $L = \{L_1, \ldots, L_n\}$ is linear independent. □

### 2.1.3. Basic Properties of Reproducing Kernels

Let us consider an abstract set of points $\Omega$ and a set of functions $X = X(\Omega)$ given on $\Omega$ and forming a Hilbert space with the scalar product $(u, v)_X = (u(s), v(s))_{X(\Omega)}$ and the norm $\|u\|_X = \sqrt{(u, u)_X}$. Assume that the functional $k_t(u) = u(t)$ is continuous for any point $t \in \Omega$.

**Definition 2.5.** *The function $G(s,t) : \Omega \times \Omega \to \mathbb{R}$ is said to be the* reproducing kernel *of the functional Hilbert space $X(\Omega)$ if*

*a) for any point $t \in \Omega$ the function $g_t(s) = G(s,t)$ belongs to $X(\Omega)$ as a function of the variable $s$;*

*b) for any function $u \in X(\Omega)$ and any point $t \in \Omega$, the following equality is valid:*

$$u(t) = (G(s,t), u(s))_{X(\Omega)} . \tag{2.12}$$

**Theorem 2.4.** The reproducing kernel $G(s,t)$ of the Hilbert space $X(\Omega)$ exists and is unique. It is symmetric with respect to the components $s$ and $t$.

*Proof.* First, check symmetry of the function $G(s,t)$. Since the function $g_t(s) = G(s,t)$ is from $X(\Omega)$, then $g_t(t_0) = (G(s,t_0), g_t(s))_X$. Further, we have the equalities

$$\begin{aligned} g_t(t_0) &= (G(s,t_0), g_t(s))_X = (G(s,t_0), G(s,t))_X \\ &= (G(s,t)), G(s,t_0))_X = (G(s,t), g_{t_0}(s))_X \\ &= g_{t_0}(t), \end{aligned}$$

which prove the required symmetry. Existence and uniqueness easily follow from the fact that the reproducing mapping $\pi$ of the Hilbert space $X(\Omega)$ exists and is unique. First, note that $G(s,t) = \pi(k_t)$ is the reproducing kernel of $X(\Omega)$. Really, since $k_t$ is from $X^*$ and $\pi$ is a mapping from $X^*$ onto $X$, then $\pi(k_t)$

is from $X$. From the definition of the reproducing kernel follows the equality $k_t(u) = (\pi(k_t), u)_X$, which is equivalent to (2.12). It only remains to prove that the function $G(s,t)$ is unique. Assume the existence of the two reproducing mappings $G(s,t)$ and $F(s,t)$. Let $g_t(s) = G(s,t)$ and $f_t(s) = F(s,t)$, then we have

$$\begin{aligned} g_t(t_0) - f_t(t_0) &= (G(s,t_0), g_t(s) - f_t(s))_X \\ &= (G(s,t), G(s,t_0))_X - (F(s,t), G(s,t_0))_X \\ &= G(t,t_0) - G(t,t_0) = 0. \end{aligned}$$

This completes the proof. □

**Theorem 2.5.** The reproducing kernel $\pi : X^* \to X$ can be determined as follows

$$\pi(L)(s) = L(G(s,\cdot)), \tag{2.13}$$

where the operator $L$ affects the function $G(s,t)$ with respect to the variable $t$.

*Proof.* We have to prove that $L(u) = (L(G(s,\cdot)), u(s))_X$. In the proof of the preceding theorem we noted that $G(s,t) = \pi(k_t)$. Since $G(s,t)$ is a symmetric function, then $\pi(k_t)(s) = \pi(k_s)(t)$. Taking in account these facts, equality (2.13) is reduced to the equality $\pi(L)(s) = L(\pi(k_s))$, and the latter is proved in the following way:

$$L(\pi(k_s)) = (\pi(L), \pi(k_s))_X = k_s(\pi(L)) = \pi(L)(s).$$

This completes the proof of the theorem. □

Let $P$ be a closed subspace in $X(\Omega)$, and let us equip the space $X(\Omega)$ with a semi-Hilbert structure $(X(\Omega), |\cdot|_P)$.

**Definition 2.6.** *The function $G_P(s,t)$ is said to be the reproducing kernel of the semi-Hilbert space $(X(\Omega), |\cdot|_P)$, if*

*a) for any functional $L \in X^*$ the function $f(s) = LG_P(s,\cdot)$ lies in $X(\Omega)$;*

*b) any functional $L \in X^*$ vanishing on the space $P$ can be represented by formula*

$$L(u) = (LG_P(s,\cdot), u(s))_P, \quad \forall u \in X. \tag{2.14}$$

**Theorem 2.6.** There exists the reproducing kernel $G_P(s,t)$ of the semi-Hilbert space of functions $X(\Omega)$, though, it is not unique in general. It may be chosen as follows

$$G_P(s,t) = \pi_P(k_s), \tag{2.15}$$

where $\pi_P$ is determined by (2.9).

*Proof.* Comparing Definitions 2.4 and 2.6 one can understand that for the mapping, defined by (2.15), it is sufficient to verify

$$LG_P(s,\cdot) = \pi_P(L)(s), \quad \forall L \in X^*, \tag{2.16}$$

and Theorem 2.6 will be proved. We have

$$LG_P(s,\cdot) = L\pi_P(k_s) = (\pi(L), \pi_P(k_s))_X = (\mathrm{Pr}_P\,\pi(L), \pi_P(k_s))_X.$$

The last equality is implied by the fact that $\pi_P(k_s) \in X_P$. Continuing the transformations

$$\begin{aligned} LG_P(s,\cdot) &= (\mathrm{Pr}_P\,\pi(L), \pi_P(k_s))_X = (\mathrm{Pr}_P\,\pi(L), \pi(k_s))_X \\ &= (\pi_P(L), \pi(k_s))_X = \pi_P(L)(s), \end{aligned}$$

we terminate the proof of the Theorem. □

### 2.1.4. Additional Properties of Reproducing Mappings and Kernels

Denote by $X(\Omega)$ the Hilbert space of functions, defined on some compact domain $\Omega \subset \mathbb{R}^n$. Assume that $X(\Omega)$ is compactly imbedded in the space of continuous functions $C(\Omega)$. The set of linear continuous functionals on $X(\Omega)$ we will note as $X^*$. By $\delta$-function we will call any functional $k$ of the form $k(u) = u(t)$, where $t$ is some point from $\Omega$. Let $P$ be a closed subspace in $X(\Omega)$, and let us equip the space $X(\Omega)$ with a semi-Hilbert structure $(X(\Omega), |\cdot|_P)$. Let, as formerly, $P^0 = \{L \in X^* : L(u) = 0, \quad \forall u \in P\}$ be the annulator of the subspace $P$.

**Definition 2.7.** *The function $G(s,t)$ will be called a reproducing mapping for $\delta$-functions if for any combination of $\delta$-functions $k = \sum \alpha_i \delta_i \in P^0$ the following equality*

$$k(u) = (kG(s,\cdot), u(s))_P$$

*is valid for all $u \in X$. Here the functional $k$ treats the function $G$ with respect to the second variable.*

Henceforth we show that if $G$ is reproducing for $\delta$-functions from $P^0$ then it is reproducing for all $L$ from $P^0$. But now we formulate an auxulary Lemma.

**Lemma 2.3.** For any functional $L \in P^0$ there exists a sequence of finite sums of $\delta$-functions from $P^0$, which strongly converges to $L$. In other words, a linear shell of $\delta$-functions from $P^0$ is dense in $P^0$.

*Proof.* Let $S = \dim P < \infty$, $\rho_1, \ldots, \rho_S$ be $L$-set of $\delta$-functions. If $p_1, \ldots, p_S$ is the basis of $P$, then the following system of linear equations

$$\begin{bmatrix} \rho_1(p_1) & \ldots & \rho_S(p_1) \\ \ldots & & \ldots \\ \rho_1(p_S) & \ldots & \rho_S(p_S) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \ldots \\ \beta_S \end{bmatrix} = \begin{bmatrix} g_1 \\ \ldots \\ g_S \end{bmatrix} \tag{2.17}$$

has the solution for any set of real numbers $g_1, \dots, g_S$. Let $\varepsilon > 0$ be an arbitrary value. Since $X(\Omega)$ is compactly embedded in $C(\Omega)$, then there exists a finite sum of $\delta$-functions $\sum \lambda_i \delta_i$ such that $\|L - \sum \lambda_i \delta_i\| \le \varepsilon$. Consider the finite sum of $\delta$-functions

$$k = \sum \lambda_i \delta_i + \sum_{i=1}^{S} \beta_i \rho_i \, , \tag{2.18}$$

where the coefficients $\beta_1, \dots, \beta_S$ are chosen from system (2.17) with $g_j = L(p_j) - \sum \lambda_i \delta_i(p_j)$, $j = 1, \dots, n$. Clearly, under this condition, sum (2.18) is from $P^0$. Let us estimate the difference $\|k - L\|$. We have

$$\|k - L\| \le \| \sum \lambda_i \delta_i - L\| + \| \sum_{i=1}^{S} \beta_i \rho_i \| \le \varepsilon + \| \sum_{i=1}^{S} \beta_i \rho_i \| \, .$$

But the latter term is estimated: $\| \sum_{i=1}^{S} \beta_i \rho_i \| \le C \|\beta\|$ . where $C = \max_{i=1,\dots,S} \|\rho_i\|$. If $R$ denotes the norm of the inverse matrix to (2.17), then $\beta \le R\|g\|$ and, hence,

$$\| \sum_{i=1}^{S} \beta_i \rho_i \| \le CR\|g\| \, .$$

We have $|g_j| \le \|L - \sum \lambda_i \delta_i\| \; \|p_j\|$ . Thus, compiling the previous inequalities we obtain

$$\|k - L\| \le (1 + CRC')\varepsilon \, ,$$

where $C' = \max_{j=1,\dots,S} \|p_j\|$. We proved that the difference $\|k - L\|$ can become as small as it is needed. □

**Theorem 2.7.** If $G$ is reproducing for $\delta$-functions from $P^0$, then $G$ is reproducing for all functionals $L$ from $P^0$.

*Proof.* From the previous Theorem it follows that there exists the sequence of the finite sums $\sum \lambda_i \delta_i$ from $P^0$ strongly converging to $L$. Consider the following valid inequalities

$$\begin{aligned} &|((L - \sum \lambda_i \delta_i) G(s, \cdot), u(s))| \\ &\quad \le |((L - \sum \lambda_i \delta_i) G(s, \cdot)|_P \, |u(s)|_P \\ &\quad \le \|L - \sum \lambda_i \delta_i)\| \; \|G(s,t)\|_{P \times P} |u(s)|_P \, . \end{aligned}$$

The latter expression converges to zero. This shows that $(LG(s, \cdot), u(s))_P$ is the limit of the sequence $(\sum \lambda_i \delta_i G(s, \cdot), u(s))_P$. But the latter expression is equal to $\sum \lambda_i \delta_i(u)$, because $G(s,t)$ is reproducing for $\delta$-functions from $P^0$. Since $\sum \lambda_i \delta_i$ converges to $L$ then $L(u)$ is the limit of the sequence $\sum \lambda_i \delta_i(u)$. Thus,

$$L(u) = (LG(s,\cdot), u(s))_P$$

for any $u \in X$ and $L \in P^0$. □

**Theorem 2.8.** Let $K(s,t)$ be a symmetric function such that

$$(K(s,t), v(s))_{X(\Omega)} = 0$$

for any point $t \in \Omega$ and any function $v \in X$, annihilated at the point $t$. Then, $K(s,t)$ is the reproducing kernel of the Hilbert space $X(\Omega)$ with accuracy to multiplication on some constant.

*Proof.* The function $v(s) = u(s) - u(t)$ is equal to zero at the point $t$. Thus, the equality

$$(K(s,t), u(s) - u(t))_X = 0$$

is valid for any function $u \in X$, or, which is the same, the equality

$$\left(\frac{K(s,t)}{(K(s,t),1)_X}, u(s)\right)_X = u(t)$$

is valid for any point $t \in \Omega$ and any function $u \in X(\Omega)$. Denote $c(t) = (K(s,t),1)_X$. Then, clearly, $G(s,t) = K(s,t)/c(t)$ is the reproducing kernel of $X(\Omega)$. Since $G(s,t)$ must be symmetric, then

$$\frac{K(s,t)}{c(t)} = \frac{K(t,s)}{c(s)}.$$

Finally, $c(s) = c(t) \equiv \text{const}$, because under conditions of the Theorem, $K(s,t)$ is symmetric. □

Let $G(s,t)$ be the reproducing mapping of the semi-Hilbert space $(X(\Omega), |\cdot|_P)$, $n$ be the dimension of $P$, $l_1, ..., l_n$ be $L$-solvable set of functionals. For such a set there exists the basis of the space $P$: $p_1(s), ..., p_n(s)$ satisfying the conditions

$$l_i(p_j) = \delta_{ij}, \quad i = 1, ..., n, \; j = 1, ..., n.$$

The expression

$$(u,v) = \sum_{i=1}^{n} l_i(u) l_i(v) + (u,v)_P$$

determines a Hilbert scalar product in $X(\Omega)$, hence, $(u,v)_P$ is a Hilbert scalar product in the space

$$X_l(\Omega) = \{u \in X(\Omega) : \; l_i(u) = 0, \; \forall i = 1, ..., n\}.$$

We assert that the reproducing kernel of this space is the function

$$F(s,t) = G(s,t) - \sum_{j=1}^{n} p_j(s) l_j^{(1)} G(\cdot,t) - \sum_{k=1}^{n} p_k(t) l_k^{(2)} G(s,t)$$
$$+ \sum_{j=1}^{n} \sum_{k=1}^{n} p_j(s) p_k(t) l_j^{(1)} l_k^{(2)} G(s,t).$$

Here the superscript with the functionals $l_j$ or $l_k$ mean the component of the function $G(s,t)$, which is treated by the functionals. Firstly, one can convince oneself that the function $F(s,t)$ is from $X_l(\Omega)$ for all fixed $t$. Secondly, in accordance with Theorem 2.7 one must prove that

$$u(t) = (F(s,t), u(s))_P$$

for all $u \in X_l(\Omega)$ and $t \in \Omega$. We have

$$(F(s,t), u(s))_P = (G(s,t) - \sum_{k=1}^{n} p_k(t) l_k^{(2)} G(s,t), u(s))_P$$
$$= u(t) - \sum_{k=1}^{n} p_k(t) l_k(u) = u(t).$$

The first equality follows from the representation of the function $F(s,t)$ and the obvious fact, that the expression

$$-\sum_{j=1}^{n} p_j(s) l_j^{(1)} G(s,t) + \sum_{j=1}^{n} \sum_{k=1}^{n} p_j(s) p_k(t) l_j^{(1)} l_k^{(2)} G(s,t)$$

is a function of the space $P$ when $t$ is fixed, and, consequently, it annihilates the scalar semi-product $(\cdot,\cdot)_P$. The second equality follows from the main property of the reproducing mapping $G$, and a simple remark that the functional $L_t(u) = u(t) - \sum_{k=1}^{n} p_k(t) l_k(u)$ is annihilated on the basis $p_1, ..., p_n$ of the space $P$. The third equality follows from definition of the space $X_l(\Omega)$, because $l_k(u) = 0, \ \forall l = 1, ..., n$.

## 2.2. Examples of Reproducing Mappings

### 2.2.1. Hyperbolic Reproducing Kernels in the Sobolev Space $W_2^1[a,b]$

Let us consider the Sobolev space $W_2^1[a,b]$ of functions, defined on the segment [a,b], with the scalar product

$$(u,v) = \int_a^b u(x)v(x)dx + \int_a^b u'(x)v'(x)dx \ .$$

Prove that the reproducing mapping of this Hilbert space is of the following form

$$K(x,y) = \begin{cases} \dfrac{ch(x-b)ch(y-a)}{sh(b-a)} , & y < x \\ \dfrac{ch(x-a)ch(y-b)}{sh(b-a)} , & y \geq x \end{cases}$$

According to Theorem 2.7, it is sufficient to ascertain equalities

$$u(x) = \int_a^b K(x,y)u(y)dy + \int_a^b \frac{dK}{dy}(x,y)u'(y)dy \, . \tag{2.19}$$

for any $x \in [a,b]$ and any $u \in W_2^1[a,b]$. Applying integration by parts we have

$$\int_a^b \frac{dK}{dy}(x,y)u'(y)dy = \int_a^x \frac{ch(x-b)ch(y-a)}{sh(b-a)}u'(y)dy +$$

$$\int_x^b \frac{ch(x-a)ch(y-b)}{sh(b-a)}u'(y)dy$$

$$= \frac{ch(x-b)}{sh(b-a)}[sh(x-a)u(x) - sh(0)u(a)] - \int_a^x K(x,y)u(y)dy$$

$$+ \frac{ch(x-a)}{sh(b-a)}[sh(0)u(b) - sh(x-b)u(x)] - \int_x^b K(x,y)u(y)dy$$

$$= u(x) - \int_a^b K(x,y)u(y)dy \, .$$

Thus, (2.19) is proved.

### 2.2.2. Polynomial Reproducing Kernels in the Space $W_2^m[a,b]$

Consider the Sobolev space $W_2^m[a,b]$ with the scalar product

$$(u,v) = \int_a^b u(x)v(x)dx + \int_a^b u^{(m)}(x)v^{(m)}(x)dx.$$

Let $P = P_{m-1}$ be the space of polynomials whose degries are less than $m$. Then, the scalar semi-product and the semi-norm

$$(u,v)_P = \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \quad |u|_P = \left( \int_a^b (u^{(m)}(x))^2 dx \right)^{1/2}$$

form a semi-Hilbert structure in $W_2^m[a,b]$. Prove that the reproducing kernel of this semi-Hilbert space is the following

$$G_P(x,y) = \frac{|x-y|^{2m-1}}{(-1)^m 2(2m-1)!}.$$

To do this it is necessary to verify that for any $L \in P^0$ and $u \in W_2^m[a,b]$

$$L(u) = \int_a^b \frac{d^m(LG_P(x,y))}{dx^m} \cdot u^{(m)}(x)dx.$$

According to Theorem 2.7 it is sufficient to verify its validity on the functionals $L$ of the following form

$$L = \sum_{i=1}^N \lambda_i \delta_i,$$

where $\delta_i(u) = u(t_i)$, $i = 1, ..., N$, and $t_i$ are arbitrary different points lying in the interval $[a,b]$, the real coefficients $\lambda_i$ satisfy equalities

$$\sum_{i=1}^N \lambda_i t_i^k = 0, \quad \forall k = 0, ..., m-1. \tag{2.20}$$

The latter equalities provide the linear combination of $\delta$-functions to be from $P^0$, i.e. it is annihilated on each polynomial from $1, x, ..., x^{m-1}$. Thus, we have to prove

$$\sum_{i=1}^N \lambda_i u(t_i) = \sum_{i=1}^N \lambda_i \int_a^b \left(\frac{|x-t_i|^{2m-1}}{(-1)^m 2(2m-1)!}\right)^{(m)} u^{(m)}(x)dx, \tag{2.21}$$

or, which is the same,

$$\sum_{i=1}^N \lambda_i u(t_i) = \sum_{i=1}^N \lambda_i \int_a^{t_i} \frac{(t_i - x)^{m-1}}{2(m-1)!} u^{(m)}(x)dx$$
$$+ \sum_{i=1}^N \lambda_i \int_{t_i}^b \frac{(x-t_i)^{m-1}}{(-1)^m 2(m-1)!} u^{(m)}(x)dx.$$

Using the integration by parts replace the derivatives from the function $u$ to the polynomials in the following way

$$\sum_{i=1}^N \lambda_i \int_a^{t_i} \frac{(t_i - x)^{m-1}}{2(m-1)!} u^{(m)}(x)dx = \sum_{i=1}^N \lambda_i \frac{(t_i - a)^{m-1}}{2(m-1)!}$$
$$+ \sum_{i=1}^N \lambda_i \int_a^{t_i} \frac{(t_i - x)^{m-2}}{2(m-2)!} u^{(m-1)}(x)dx,$$

$$\sum_{i=1}^{N} \lambda_i \int_{t_i}^{b} \frac{(x-t_i)^{m-1}}{(-1)^m 2(m-1)!} u^{(m)}(x)dx = \sum_{i=1}^{N} \lambda_i \frac{(b-t_i)^{m-1}}{(-1)^m 2(m-1)!}$$

$$+ \sum_{i=1}^{N} \lambda_i \int_{t_i}^{b} \frac{(x-t_i)^{m-2}}{(-1)^{m-1} 2(m-2)!} u^{(m-1)}(x)dx.$$

From (2.20) it follows

$$\sum_{i=1}^{N} \lambda_i \frac{(t_i-a)^{m-1}}{2(m-1)!} = 0, \quad \sum_{i=1}^{N} \lambda_i \frac{(b-t_i)^{m-1}}{(-1)^m 2(m-1)!} = 0,$$

hence,

$$\sum_{i=1}^{N} \lambda_i u(t_i) = \sum_{i=1}^{N} \lambda_i \int_{a}^{t_i} \frac{(t_i-x)^{m-2}}{2(m-2)!} u^{(m-1)}(x)dx$$

$$+ \sum_{i=1}^{N} \lambda_i \int_{t_i}^{b} \frac{(x-t_i)^{m-2}}{(-1)^{m-1} 2(m-2)!} u^{(m-1)}(x)dx.$$

Continuing the integration by parts we obtain

$$\sum_{i=1}^{N} \lambda_i u(t_i) = \sum_{i=1}^{N} \lambda_i \left[ \int_{a}^{t_i} \frac{u'(x)}{2} dx - \int_{t_i}^{b} \frac{u'(x)}{2} dx \right]$$

$$= \sum_{i=1}^{N} \lambda_i u(t_i) - \sum_{i=1}^{N} \lambda_i \left[ \frac{u(a)+u(b)}{2} \right].$$

The latter sum is equal to zero, because of (2.20). So, (2.21) is proved, and the introduced function $G_P(x,y)$ is really the reproducing kernel.

### 2.2.3. Analog of the Space $W_2^1[a,b]$ for Mesh Functions

Let $X$ be the space of the mesh functions $\hat{u} = (u(t_1), \ldots, u(t_n))^T$, defined on the mesh $\Delta = \{t_1 < t_2 < \ldots < t_n\}$. Introduce the scalar product

$$(\hat{u}, \hat{v}) = \frac{1}{n} \sum_{i=1}^{n} u(t_i)v(t_i) + \sum_{i=1}^{n-1} \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} (v(t_{i+1}) - v(t_i))$$

and the norm

$$\|\hat{u}\| = \left( \frac{1}{n} \sum_{i=1}^{n} u^2(t_i) + \sum_{i=1}^{n-1} (t_{i+1} - t_i) \left[ \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \right]^2 \right)^{\frac{1}{2}}$$

in the space $X$. It is easy to see that $X$ is the Hilbert space, whose scalar product and norm approximate the scalar product and norm in the Sobolev space $W_2^1[a,b]$, $[a,b] \supset \Delta$.

Any functional from $X^*$ has the representation

$$L(\hat{u}) = \sum_{i=1}^{n} a_i u(t_i) \tag{2.22}$$

with an appropriate vector $(a_1, \dots, a_n) \in \mathbb{R}^n$. Conversely, any functional (2.22) is a linear continuous functional in $X$. It is easy to see, because $X$ is finite-dimensional and, consequently, $X^*$ is the same. For this reason, let us identify $X$ and $X^*$ with $\mathbb{R}^n$, when the matter concerns their elements, but not their internal structures.

Let $P$ be the space of constant mesh functions. Clearly, $\dim P = 1$. Let us now introduce the scalar semi-product

$$(\hat{u}, \hat{v})_P = \sum_{i=1}^{n-1} \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} (v(t_{i+1}) - v(t_i))$$

and the semi-norm

$$|\hat{u}|_P = \left( \sum_{i=1}^{n-1} (t_{i+1} - t_i) \left[ \frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} \right]^2 \right)^{\frac{1}{2}}$$

in the space $X$, annihilated in the space $P$. It is easy to verify that they define a semi-Hilbert structure in $X$. How the reproducing mapping $\pi_P$ could be found in this space? The answer is in the following statement.

The reproducing mapping is determined by matrix

$$G = \{-\frac{1}{2}|t_i - t_j|\}_{i=1,\dots,n}^{j=1,\dots,n},$$

so that

$$\pi_P(L) = Ga, \tag{2.23}$$

where $a$ is a vector satisfying (2.22). To prove this fact, first, let us find the space $P^0 \subset X^*$, which is the annulator of the space $P$. Verify that

$$P^0 = \{(a_1, \dots, a_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} a_i = 0\} . \tag{2.24}$$

Indeed, if $u(t_i) = c$, $\forall i \in 1, \dots, n$, then

$$L(u) = \sum_{i=1}^{n} a_i u(t_i) = c \sum_{i=1}^{n} a_i = 0, \quad \forall L \in P^0.$$

Since there exist nonzero functionals from $X^*\backslash P^0$ , for example, $L(u) = u(t_1)$, then $\dim P^0 \leq n-1$. Hence, we have correctly found $P^0$, because the dimension of space (2.24) is equal to $n-1$.

Expression (2.23) will be a reproducing mapping, if

$$L(\hat{u}) = (Ga, \hat{u})_P$$

for all $\hat{u} \in X$ and $L \in P^0$, or, which is the same,

$$\sum_{i=1}^{n} a_i u(t_i) = \sum_{i=1}^{n-1} \frac{(Ga)_{i+1} - (Ga)_i}{t_{i+1} - t_i}(u(t_{i+1}) - u(t_i)) \tag{2.25}$$

for $u \in X^0$ and the vectors $a$, satisfying equality (2.24).

The basis in the space of vectors $a$, satisfying equality (2.24), may be chosen as $e_1 = (-1, 1, 0, \dots, 0)^T$, $e_2 = (0, -1, 1, 0, \dots, 0)^T, \dots,$ $e_{n-1} = (0, .., 0, -1, 1)^T$. It is sufficient to verify equality (2.25) on these vectors, i.e. to check

$$u(t_{l+1}) - u(t_l) = \sum_{i=1}^{n-1} \frac{(Ge_l)_{i+1} - (Ge_l)_i}{t_{i+1} - t_i}(u(t_{i+1}) - u(t_i)) \, , \tag{2.26}$$

for all $l = 1, ..., n-1$. We have

$$2Ge_l =$$

$$\begin{bmatrix} 0 & t_1 - t_2 & \dots & t_1 - t_l & t_1 - t_{l+1} & \dots \\ t_1 - t_2 & 0 & \dots & t_2 - t_l & t_2 - t_{l+1} & \dots \\ \dots & \dots & & \dots & \dots & \\ t_1 - t_l & t_2 - t_l & & 0 & t_l - t_{l+1} & \dots \\ t_1 - t_{l+1} & \dots & & t_l - t_{l+1} & \dots & \dots \\ \dots & & & \dots & \dots & \\ t_1 - t_n & \dots & & t_l - t_n & t_{l+1} - t_n & \dots \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dots \\ -1 \\ 1 \\ \dots \\ 0 \end{bmatrix} = \begin{bmatrix} t_l - t_{l+1} \\ t_l - t_{l+1} \\ \dots \\ t_l - t_{l+1} \\ t_{l+1} - t_l \\ \dots \\ t_{l+1} - t_l \end{bmatrix}$$

and, thus,

$$(Ge_l)_{i+1} - (Ge_l)_i = \begin{cases} t_{l+1} - t_l \quad , & \text{if } \ i = l \\ \quad 0 \quad , & \text{otherwise.} \end{cases}$$

From here immediately follow equalities (2.26) and the statement is proved.

### 2.2.4. Space $\tilde{W}_2^m[0, 2\pi]$ of Periodic Functions and Bernully Functions

Let us consider the integer $m \geq 1$ and the Sobolev space $\tilde{W}_2^m[0, 2\pi]$ of periodic functions. More exactly, the functions $u \in \tilde{W}_2^m[0, 2\pi]$ are from the space $C^{m-1}[0, 2\pi]$ and

$$u^{(r)}(0) = u^{(r)}(2\pi), \quad \forall r = 0, ..., m-1.$$

the $m$-th derivative is summable in $L_2$-norm:

$$|u|_m = \left( \int_0^{2\pi} (u^{(m)})^2(x)dx \right)^{1/2} < \infty.$$

The scalar product, additional scalar semi-product and semi-norm are defined by the same formulas as in Section 2.2.2. The only difference is that the kernel of the semi-norm $|\cdot|_m$ consists of constants only, i.e. the polynomials of zero order.

To determine the reproducing kernel in this semi-Hilbert space of periodic functions, let us introduce the Bernully functions (Korneichuk 1976):

$$D_r(x) = \sum_{k=1}^{\infty} \frac{\cos(kx - \pi r/2)}{k^r} \quad (r = 1, 2, ...).$$

Indicate some important properties of the Bernully functions:

1. $D'_r(x) = D_{r-1}(x), \quad (r = 2, 3, ...)$,

2. $D_1(x) = \begin{cases} \pi(1/2 - \{x/2\pi\}), & x \neq 2\pi k \\ 0, & x = 2\pi k \end{cases}$

3. $D_r$ are $2\pi$ - periodic functions.

Here $\{x/2\pi\}$ stands for the fractional part of the ratio $x/2\pi$. It is interesting that the Bernully function $D_r(x)$ is the $r$-th order polynomial on the period $[0, 2\pi)$.

Now the reproducing kernel in the semi-Hilbert space $\tilde{W}_2^m[0, 2\pi]$ may be determined as follows

$$G_p(x, y) = \frac{D_{2m}(y - x)}{(-1)^m \pi}.$$

To prove this fact it is necessary to verify that for any $L \in P^0$ and $u \in \tilde{W}_2^m[0, 2\pi]$

$$L(u) = \int_0^{2\pi} \frac{d^m LG_p(x, y)}{dx^m} u^{(m)}(x)dx.$$

According to Theorem 2.7 it is sufficient to establish this on the $\delta$-functions

$$L = \sum_{i=1}^{N} \lambda_i \delta_i,$$

where $\delta_i(u) = u(t_i)$, $i = 1, ..., N$, and $t_i$ are arbitrary different points lying in the interval $[0, 2\pi)$. The real coefficients $\lambda_i$ satisfy equalities

$$\sum_{i=1}^{N} \lambda_i = 0, \tag{2.27}$$

because only the constants belong to $P$.

Thus, we have to prove

$$\sum_{i=1}^{N} \lambda_i u(t_i) = \sum_{i=1}^{N} \lambda_i \int_0^{2\pi} \left( \frac{D_{2m}(t_i - x)}{(-1)^m \pi} \right)^{(m)} u^{(m)}(x) dx. \tag{2.28}$$

Using the first property of the Bernully functions one can reduce the latter term to the following

$$\sum_{i=1}^{N} \lambda_i / \pi \int_0^{2\pi} D_m(t_i - x) u^{(m)}(x) dx.$$

Produce the integration of the expression by parts

$$\sum_{i=1}^{N} \lambda_i / \pi \Big( D_m(t_i - 2\pi) u^{(m-1)}(2\pi) - D_m(t_i) u^{(m-1)}(0) \Big)$$
$$+ \sum_{i=1}^{N} \lambda_i / \pi \int_0^{2\pi} D_{m-1}(t_i - x) u^{(m-1)}(x) dx.$$

The first sum is equal to zero for any $m \geq 2$, because the Bernully function $D_k$ and the derivatives of the functions $u \in W_2^m[0, 2\pi]$ up to the $(m-1)$-th order are $2\pi$ - periodic functions. In the same manner we can reduce the right sum of (2.28) to the following

$$\sum_{i=1}^{N} \lambda_i / \pi \int_0^{2\pi} D_1(t_i - x) u'(x) dx.$$

Now applying the implicit form of the Bernully function $D_1$ (the second property) we reduce the latter expression to the following

$$\sum_{i=1}^{N} \lambda_i \left[ \int_0^{t_i} \left(\frac{1}{2} - \frac{t_i - x}{2\pi}\right) u'(x) dx + \int_{t_i}^{2\pi} \left(\frac{x - t_i}{2\pi} - \frac{1}{2}\right) u'(x) dx \right]$$
$$= \sum_{i=1}^{N} \lambda_i \left[ \frac{1}{2} u(t_i) - \left(\frac{1}{2} - \frac{t_i}{2\pi}\right) u(0) - \int_0^{t_i} \frac{u(x) dx}{2\pi} \right.$$
$$\left. + \left(\frac{1}{2} - \frac{t_i}{2\pi}\right) u(2\pi) + \frac{1}{2} u(t_i) - \int_{t_i}^{2\pi} \frac{u(x) dx}{2\pi} \right]$$
$$= \sum_{i=1}^{N} \lambda_i u(t_i) - \int_0^{2\pi} \frac{u(x) dx}{2\pi} \sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \lambda_i u(t_i).$$

Here, in the first equality we used the integration by parts, in the second one - "$2\pi$-periodic" of the function $u$, in the third one - condition (2.27). Thus, we obtain required expression (2.28).

### 2.2.5. Reproducing Kernels in Hilbert Space of Spherical Functions

This Section is composed on the basis of the papers (Freeden 1980, 1984) and (Wahba 1981). Here the precise proofs are absent, but the brief results are formulated.

Let $\Omega$ denote the unit sphere in the Euclidean space $\mathbb{R}^3$. The rectangular coordinates in $\mathbb{R}^3$ are related to the polar coordinates by the equations

$$x = r\xi, \quad \xi = te_3 + \sqrt{1-t^2}(e_1 \cos\varphi + e_2 \cos\varphi)$$
$$t = \cos\theta, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi$$

($\theta$ is the polar distance, $\varphi$ is the geocentric longitade).

If by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

one denotes the Laplace-operator, then in terms of the polar coordinates one has the representation

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*.$$

Here $\Delta^*$ denotes the (Laplace-) Beltrami-operator of the unit sphere

$$\Delta^* = (1-t^2)\frac{\partial^2}{\partial t^2} - 2t\frac{\partial}{\partial t} + \frac{1}{1-t^2}\frac{\partial^2}{\partial \varphi^2}$$

As usual, the spherical harmonics $S_n$ of order $n$ are defined as everywhere on $\Omega$ infinitely differentiable eigenfunctions, corresponding to the eigenvalues $\lambda_n = n(n+1)$ for $n = 0, 1, 2, \dots$ of the (Beltrami) differential equation

$$\partial_n S_n = (\Delta^* + \lambda_n)S_n = 0.$$

Spherical harmonics of different order are orthogonal in the sense of the $L^2$-inner product:

$$\int_\Omega S_n(\xi)S_m(\xi)d\omega = 0, \quad n \ne m.$$

There exist $2n+1$ linearly independent spherical harmonics $S_{n,1}, \dots, S_{n,2n+1}$ of order $n$. We assume this system to be orthonormalized in the sense of the $L^2$-inner product. Then, for any $\xi, \eta \in \Omega$ the additional theorem gives

$$\sum_{j=1}^{2n+1} S_{n,j}(\xi)S_{n,j}(\eta) = \frac{2n+1}{4\pi}P_n((\xi,\eta)),$$

where $P_n$ is the Legendre polynomial

$$P_n(t) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n(n-2k)!(n-k)!k!} t^{n-2k}, \quad t \in [-1,1],$$

$(\xi, \eta)$ is the cosine of the angle beetween the vectors $\xi$ and $\eta$. The additional theorem allows us further to obtain the reproducing kernels in a simple form of series.

Let $\mathcal{H}$ be the space of distributions $F$ on $\Omega$ for which the partial derivatives $\partial_0^{\alpha_0}...\partial_m^{\alpha_m}F$ are square integrable ($\alpha_0, ..., \alpha_m$ are fixed positive integer constants):

$$\mathcal{H} = \{F \in \mathcal{E}' : \quad \partial_0^{\alpha_0}...\partial_m^{\alpha_m}F = f \in L^2(\Omega)\}. \tag{2.29}$$

$\mathcal{H}$ is naturally equipped with the scalar semi-product $(\cdot,\cdot)_m$ corresponding to the semi-norm

$$|F|_m = \left( \int_\Omega |\partial_0^{\alpha_0}...\partial_m^{\alpha_m}F|^2 d\omega \right)^{1/2}. \tag{2.30}$$

Freeden has thoroughly investigated the reproducing kernels of such spaces. We only give one particular result.

Let $\alpha_0 = 1, ..., \alpha_m = 1$. Then, the semi-normed space $\mathcal{H}$ is a functional semi-Hilbert space embedded in the space $C(\Omega)$ of continuous functions on $\Omega$. The kernel of semi-norm $|\cdot|_m$ is the linear space $\mathcal{P} = \mathcal{P}_{m-1}$ of the dimension $\dim \mathcal{P} = M' = (m+1)^2$ of all spherical harmonics of degree $m$ or less. The reproducing kernel of the semi-Hilbert space $\mathcal{H}$ is given by the formula

$$G(\xi,\eta) = \sum_{n=m+1}^{\infty} \frac{2n+1}{4\pi} \sigma_n^2 P_n((\xi,\eta)),$$

where $\sigma_n = [(\lambda_0 - \lambda_n)...(\lambda_m - \lambda_n)]^{-1}$, $n = m+1, ....$

Another result we take from (Wahba 1981). Let the semi-norm of the space $\mathcal{H}$ be defined by the formula

$$|F|_m = \begin{cases} \left( \int_\Omega (\partial_0^{m/2} F)^2 d\omega \right)^{1/2}, & m \quad \text{even} \\ \left( \int_\Omega \frac{\left( \frac{\partial}{\partial\varphi} \partial_0^{(m-1)/2} F \right)^2}{\sin^2\theta} + (\frac{\partial}{\partial\theta} \partial_0^{(m-1)/2} F)^2 d\omega \right)^{1/2}, & m \quad \text{odd.} \end{cases}$$

The kernel of the space $F$ consists of the constants only , and the reproducing kernel of such a space is the following:

$$G(\xi,\eta) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n^m(n+1)^m} P_n((\xi,\eta)).$$

The next step of Wahba was a simplification of $G(\xi,\eta)$. To this end she has considered the equivalent functional in the form of a series, instead of $|F|_m$. Thus, she obtained the kernel in the following form

$$G(\xi,\eta) = \frac{1}{4\pi}\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)...(n+2m-1)} P_n((\xi,\eta)),$$

and showed how to simplify the latter series and reduce it to the function consisting of logarithms and polynomials only.

## 2.3. Spline Characterization

First,, give the definition of the interpolating spline, which is equivalent to one given in Chapter 1, but which is more convenient for our aims of characterization.

Let $X, Z$ be real Hilbert spaces, $A : X \to Z$ be a surjective continuous linear operator, $P$ be a closed subspace in $X$ and $(X, |\cdot|_P)$ be a semi-Hilbert space.

**Definition 2.8.** *Take an element $z \in Z$. The element $\sigma \in X$ will be called the interpolating spline if it satisfies equality*

$$\sigma = \arg \min_{u \in A^{-1}(z)} |u|_P. \tag{2.31}$$

Let $Y = X/P$ be a Hilbert factor-space corresponding to Definition 2.3. Set the operator $T : X \to Y$ of canonical embedding as the energy operator introduced in Section 1.1.1. Then, setting $P = N(T)$, $(u,v)_P = (Tu, Tv)_Y$, $|u|_P = \|Tu\|_Y$ one can be convinced that this Definition coincides with the definition of interpolating spline given in Chapter 1. We give also the orthogonal property in new notations

$$(\sigma, u)_P = 0, \quad \forall u \in N(A). \tag{2.32}$$

If will be recalled that it is the necessary and sufficient condition for $\sigma$ to be an interpolating spline. We can assume to fulfil some of the sufficient conditions of existence and uniqueness of the spline among the proposed ones in Chapter 1.

### 2.3.1. General Characterization Theorems

Before the main characterization theorem will be formulated, we introduce some new notations and prove two Lemmas. Define the conjugate operator $A^* : Z \to X$ with the help of the equalities

$$(A^*\lambda, u)_X = (\lambda, Au)_Z, \quad \forall\lambda \in Z, \quad \forall u \in X.$$

We assume

$$N(A)^0 = \{k \in X^* : \quad k(u) = 0, \quad \forall u \in N(A)\}$$

to be the annulator of the space $N(A)$, $Z_P = (AP)^{\perp}$ be the orthogonal complement of the image $AP = \{Ap \in Z : \quad p \in P\}$. Let $\pi$ and $\pi_P$ be the reproducing mappings of the Hilbert space $X$ and of the semi-Hilbert space $(X, |\cdot|_P)$, respectively.

**Lemma 2.4.** If $k \in N(A)^0$, then there exists the unique element $\lambda \in Z$, which satisfies equalities

$$k(u) = (\lambda, Au)_Z, \quad \forall u \in X. \tag{2.32}$$

In other words, $k = \pi^{-1}(A^*\lambda)$.

*Proof.* According to the Riesz theorem, there exists the unique element $f \in X$ such that $k(u) = (f, u)_X$, $\forall u \in X$. Since $k \in N(A)^0$, then $f \perp N(A)$ and, thus, the equation $A^*\lambda = f$ has a solution. Since A is a surjective operator, and, consequently, $N(A^*) = \{0\}$, then this solution is unique. Finally, we have the equalities

$$k(u) = (f, u)_X = (A^*\lambda, u)_X = (\lambda, Au)_Z,$$

which prove the Lemma. □

**Lemma 2.5.** If $k \in N(A)^0 \cap P^0$, then there exists the unique element $\lambda \in Z_P$, which satisfies equalities

$$k(u) = (\pi_P \pi^{-1}(A^*\lambda), u)_P, \quad \forall u \in X. \tag{2.33}$$

*Proof.* Making use of the preceding Lemma we have $k = \pi^{-1}(A^*\lambda)$, and, then, utilizing the property of the reproducing mapping $\pi_P$ we obtain

$$k(u) = (\pi_P(k), u)_P = (\pi_P \pi^{-1}(A^*\lambda), u)_P. \tag{2.34}$$

The condition $\lambda \in Z_P$ follows from representation (2.32) and from the condition $k \in P^0$. Uniqueness follows from Lemma 2.4. □

**Theorem 2.9.** For all $\lambda \in Z_P$ and $p \in P$ the element

$$\sigma = \pi_P \pi^{-1}(A^*\lambda) + p \tag{2.35}$$

is an interpolating spline. Any interpolating spline is represented in form (2.35), where the elements $\lambda$ and $p$ are uniquely defined from the interpolating conditions $A\sigma = z$.

*Proof.* If $\lambda \in Z_P$, then $\pi^{-1}(A^*\lambda) \in N(A)^0$ and, consequently,

$$(\sigma, u)_P = (\pi_P \pi^{-1}(A^*\lambda), u)_P = \pi^{-1}(A^*\lambda)(u) = (A^*\lambda, u)_X$$
$$= (\lambda, Au)_Z = 0, \quad \forall u \in N(A).$$

Thus, the orthogonal property is fulfilled and $\sigma$ is, naturally, a spline. To prove the second proposition of the Theorem, assume $\sigma$ to be an interpolating spline and consider the linear functional $k_\sigma \in X^*$ of the following form

$$k_\sigma(u) = (\sigma, u)_P. \tag{2.36}$$

Orthogonal property (2.32) and scalar semi-product properties bring about to the condition $k_\sigma \in N(A)^0 \cap P^0$. With the help of Lemma 2.5 we have

$$k_\sigma(u) = (\pi_P \pi^{-1}(A^*\lambda), u)_P. \tag{2.37}$$

From equalities (2.36) and (2.37) we obtain the following equality

$$(\sigma - \pi_P \pi^{-1}(A^*\lambda), u)_P = 0, \quad \forall u \in X.$$

Hence, $|\sigma - \pi_P \pi^{-1}(A^*\lambda)|_P = 0$ and representation (2.35) is established. The uniqueness of (2.35) can be proved by the proof by contradiction. Let $(\lambda_1, p_1)$ and $(\lambda_2, p_2)$ be two different solutions. Then,

$$\pi_P \pi^{-1} A^*(\lambda_1 - \lambda_2) = p_2 - p_1.$$

Considering the scalar semi-product $(\cdot,\cdot)_P$ we have

$$(\pi_P \pi^{-1} A^*(\lambda_1 - \lambda_2), u)_P = (p_2 - p_1, u)_P = 0 \quad \forall u \in X$$

and, hence, $A^*(\lambda_1 - \lambda_2) = 0$. Since $N(A^*) = \{0\}$, then $\lambda_1 = \lambda_2$, and, certainly, $p_1 = p_2$. □

Apply the proved Theorem to the characterization of splines in the functional space $X(\Omega)$. To this end use two following lemmas.

**Lemma 2.6.** Let $\lambda \in Z_P$, $k^\lambda(u) = (\lambda, Au)_Z$ be a linear continuous functional. Then,

$$k^\lambda(u) = ((\lambda, AG_P(s,\cdot))_Z, u(s))_P, \quad \forall u \in X(\Omega), \tag{2.38}$$

where $G_P(s,t)$ is the reproducing kernel of the semi-Hilbert space $(X(\Omega), |\cdot|_P)$.

*Proof.* Since $k^\lambda \in P^0$, then

$$k^\lambda(u) = (k^\lambda G_P(s,\cdot), u(s))_P, \quad \forall u \in X(\Omega).$$

From the condition of the Lemma we have $k^\lambda G_P(s,\cdot) = (\lambda, AG_P(s,\cdot))_Z$, which implies (2.38). □

**Lemma 2.7.** $\pi_P \pi^{-1}(A^*\lambda) = (\lambda, AG_P(s,\cdot))_Z$, $\forall \lambda \in Z_P$.

*Proof.* The Lemma follows from two equalities $\pi_P(k^\lambda) = (\lambda, AG_P(s,\cdot))_P$ and $k^\lambda = \pi^{-1}(A^*\lambda)$. The first one is implied from (2.38) and the second one arises from Lemma 2.4. □

Now the following characterization theorem is a trivial consequence of Theorem 2.9 and Lemma 2.7.

**Theorem 2.10.** For all $\lambda \in Z_P$ and $p \in P$, function

$$\sigma(s) = (\lambda, AG_P(s,\cdot))_Z + p(s) \tag{2.39}$$

is an interpolating spline in the space of functions $X(\Omega)$. Any interpolating spline is represented in form (2.39), where the elements $\lambda$ and $p$ are uniquely defined from the interpolating conditions $A\sigma = z$.

### 2.3.2. Characterization for the Interpolation with Composite Interpolating Operator

Assume that the Hilbert space $Z$ is composed of the Hilbert spaces $Z_1, ..., Z_N$ with the help of the direct sum, i.e. the space $Z = \oplus_{i=1}^N Z_i$ consists of the vectors $(u_1, ..., u_N)$, $u_i \in Z_i$, $i = 1, ..., N$. The scalar product in such a space is introduced in the following form

$$(u,v)_Z = (u_1, v_1)_{Z_1} + ... + (u_N, v_N)_{Z_N} \tag{2.40}$$

**Definition 2.9.** *We consider $A : X \to Z$ a composite operator, if $A = \oplus_{i=1}^N A_i \to X$ is composed of the linear continuous operators $A_i : X \to Z_i$, $i = 1, ..., N$, and maps in accordance with the following rule*

$$Au = (A_1 u, ..., A_N u). \tag{2.41}$$

If $A$ is a composite operator and $z = (z_1, ..., z_N)$ is an element from $Z$, then Definition 2.8 about the interpolating spline is modified to the following form

$$\sigma = \arg \min_{\substack{u \in X \\ A_i u = z_i,\ i=1,...,N}} |u|_P.$$

**Theorem 2.11.** If the operator $A$ is composite, then the interpolating spline $\sigma$ is represented in the following form

$$\sigma = \sum_{i=1}^{N} \pi_P \pi^{-1}(A_i^* \lambda_i) + p, \tag{2.42}$$

and for the case of the functional space $X(\Omega)$ -

$$\sigma = \sum_{i=1}^{N} (\lambda_i, A_i G_p(s,\cdot))_{Z_i} + p(s) \tag{2.43}$$

where the elements $\lambda = (\lambda_1, ..., \lambda_N) \in Z_p$ and $p \in P$ are uniquely defined from the interpolating conditions $A_i \sigma = z_i$, $i = 1, ..., N$.

*Proof.* Representation (2.43) readily follows from Theorem 2.11 and equalities (2.40 - 2.41), and representation (2.42) follows from Theorem 2.10 and the equality $A^*\lambda = \sum_{i=1}^{N} A_i^*\lambda_i$. The latter is verified in the following manner

$$(A^*\lambda, u)_X = (\lambda, Au)_Z = \sum_{i=1}^{N}(\lambda_i, A_i u)_{Z_i} = \sum_{i=1}^{N}(A_i^*\lambda_i, u)_X.$$

□

**Theorem 2.12.** If the composite interpolating operator $A = k_1 \oplus ... \oplus k_N$ is formed by the functionals $k_i \in X^*$, $i = 1, ..., N$, then the interpolating spline $\sigma$ is represented as

$$\sigma = \sum_{i=1}^{N} \lambda_i \pi_p(k_i) + p, \tag{2.44}$$

and for the case of the functional space $X(\Omega)$ -

$$\sigma = \sum_{i=1}^{N} \lambda_i k_i G_p(s, \cdot) + p(s). \tag{2.45}$$

The vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_N) \in Z_p$ and the element $p \in P$ are uniquely defined from the interpolating conditions $k_1(\sigma) = z_1, ..., k_N(\sigma) = z_n$.

*Proof.* The proof follows from previous Theorem 2.11, where it is necessary to assume $Z_i = \mathbb{R}$, $\forall i$, and $Z = \mathbb{R}^N$. Then, we have the following equalities

$$\lambda_i(\pi(k_i), u)_X = \lambda_i k_i(u) = (\lambda_i, k_i(u))_{\mathbb{R}} = (k_i^*\lambda_i, u)_X,$$

from which it follows that $k_i^*\lambda_i = \lambda_i \pi(k_i)$. Thus, from (2.42) follows (2.44). Equality (2.45) trivially follows from (2.44) and the following equalities

$$\sum_{i=1}^{N} \lambda_i \pi_p(k_i) = \pi_P \sum_{i=1}^{N} \lambda_i k_i = (\sum_{i=1}^{N} \lambda_i k_i) G_p(s, \cdot) = \sum_{i=1}^{N} \lambda_i k_i G_p(s, \cdot).$$

The Theorem is proved. □

The next Theorem is the most important theorem for applications. It deals with spline interpolating problem

$$\sigma = \arg \min_{\substack{u \in X \\ k_1(u)=z_1, ..., k_N(u)=z_N}} |u|_P, \tag{2.46}$$

when $P$ is finite-dimensional space. Let $p_1, ..., p_S$ be a basis of the space $P$.

**Theorem 2.13.** The solution to problem (2.46) in the functional Hilbert space $X(\Omega)$ is represented in form

$$\sigma = \sum_{i=1}^{N} \lambda_i k_i G_p(s,\cdot) + \sum_{j=1}^{S} \mu_j p_j(s). \tag{2.47}$$

The vectors $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_N)$, $\boldsymbol{\mu} = (\mu_1, ..., \mu_S)$ form the solution to the system of linear algebraic equations

$$\begin{bmatrix} G & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \boldsymbol{z} \\ \mathbf{0} \end{bmatrix}. \tag{2.48}$$

Here $\boldsymbol{z} = (z_1, ..., z_N)^T$ is the unput vector, $\mathbf{0}$ is the null vector of the dimension $S$. $G = \{k_i k_j G_p(s,t)\}_{i=1,...,N}^{j=1,...,N}$ is a square matrix, $B = \{k_i(p_j)\}_{i=1,...,N}^{j=1,...,S}$ is a rectangular matrix, $B^T$ is the transposed matrix to $B$.

*Proof.* Expansion (2.47) as have already shown, coincides with (2.45) for the finite-dimensional $P$. Consequently, we must prove only (2.48). It is not difficult to see that the first block of the equations $G\boldsymbol{\lambda}+B\boldsymbol{\mu} = \boldsymbol{z}$ means the interpolating conditions. Let us that the condition $B^T\boldsymbol{\lambda} = 0$ is equivalent to the condition $\boldsymbol{\lambda} \in Z_p$. Really,

$$\begin{aligned} \boldsymbol{\lambda} \in Z_p \Leftrightarrow \boldsymbol{\lambda} \perp AP &\Leftrightarrow \forall p \in P \quad (\lambda, Ap)_{\mathbb{R}^N} = 0 \\ &\Leftrightarrow \forall j \in 1, ..., S \quad (\lambda, Ap_j)_{\mathbb{R}^N} = 0 \\ &\Leftrightarrow \forall j \in 1, ..., S \quad \sum_{i=1}^{N} k_i(p_j) \cdot \lambda_i = 0 \Leftrightarrow B^T\boldsymbol{\lambda} = \mathbf{0}. \end{aligned}$$

The condition $\boldsymbol{\lambda} \in Z_p$ is satisfied, and the Theorem is proved. □

### 2.3.3. Smoothing and Mixed Splines

Here we give the definition which includes both the interpolating and smoothing splines as particular cases. Instead of the space $Z$ and the operator $A : X \to Z$, introduced in the preamble to Section 2.3, define two Hilbert spaces $Z_1, Z_2$ and two linear continuous surjective operators $A_1 : X \to Z_1$, $A_2 : X \to Z_2$. Set $Z = Z_1 \oplus Z_2$, $A = A_1 \oplus A_2$.

**Definition 2.8.** *Take elements $z_1 \in Z_1$, $z_2 \in Z_2$ and a parameter $\alpha > 0$. The element $\sigma_\alpha \in X$ is called the mixed spline if it satisfies the condition*

$$\sigma_\alpha = \arg \min_{u \in A_1^{-1}(z_1)} \alpha |u|_p^2 + \|A_2 u - z_2\|_{Z_2}^2 \tag{2.49}$$

Clearly, the case $A_2 = 0$, $z_2 = 0$ corresponds to the interpolation with the interpolating operator $A_1$, and the case $A_1 = 0$, $z_1 = 0$ corresponds to the smoothing with the smoothing operator $A_2$. In Chapter 1 we have already formulated the orthogonal property for the mixed splines. Taking into account the discussions in the preamble to Section 2.3, concerning the orthogonal property, it accepts the following form

$$\alpha(\sigma_\alpha, u)_P + (A_2\sigma_\alpha - z_2, A_2 u)_{Z_2} = 0, \ \forall u \in A_1^{-1}(0). \tag{2.50}$$

Assume that the operators $A_1, A_2$ are independent, i.e. $A_2(A_1^{-1}(0)) = Z_2$ and $A_1(A_2^{-1}(0)) = Z_1$. In other words, the restriction of the operator $A_2$ to the kernel of the operator $A_1$ is a surjective operator and vice versa.

**Theorem 2.14.** The mixed spline $\sigma_\alpha$ is of the following form

$$\sigma_\alpha = \pi_p \pi^{-1}(A_1^* \lambda_\alpha) + \pi_p \pi^{-1}(A_2^* \rho_\alpha) + p \tag{2.51}$$

Where $(\lambda_\alpha, \rho_\alpha) \in Z_p$, $p \in P$ are uniquely defined from two interpolating conditions

$$\begin{cases} A_1\sigma_\alpha = z_1, \\ A_2\sigma_\alpha = z_2 - \alpha\rho_\alpha. \end{cases} \tag{2.52}$$

*Proof.* In Chapter 1, any smoothing spline was proved to be some interpolating one. One could prove more general assertion that any mixed spline with the operators $A_1$ and $A_2$ is some interpolating spline with composite interpolating operator $A_1 \oplus A_2$. Thus, representation (2.51) may be considered to be proved, because of the Theorem 2.11. It remains only to check conditions (2.52). Let us use properties of reproducing mappings, to do the following transformations

$$\begin{aligned}(\sigma_\alpha, u)_P &= (\pi_p\pi^{-1}(A_1^*\lambda_\alpha + A_2^*\rho_\alpha), u)_P = \pi^{-1}(A_1^*\lambda_\alpha + A_2^*\rho_\alpha)(u) \\ &= (A_1^*\lambda_\alpha + A_2^*\rho_\alpha, u)_X = (\rho_\alpha, A_2 u)_{Z_2}, \quad \forall u \in A_1^{-1}(0).\end{aligned}$$

Then, orthogonal property (2.50) is equivalent to

$$(\alpha\rho_\alpha + A_2\sigma_\alpha - z_2, A_2 u)_{Z_2} = 0, \quad \forall u \in A_1^{-1}(0). \tag{2.53}$$

Since the operators $A_1$ and $A_2$ are independent, i.e. $A_2(A_1^{-1}(0)) = Z_2$, then (2.53) is equivalent to $\alpha\rho_\alpha + A_2\sigma_\alpha - z_2 = 0$, which coincides with the second interpolating condition in (2.52). The Theorem is proved. □

Finally, let us formulate the analog of Theorem 2.13 for mixed splines. Consider $N_1 + N_2$ linear independent functionals $k_1, ..., k_{N_1}$, $l_1, ..., l_{N_2}$ in the space $X(\Omega)$ and the mixed spline

$$\sigma_\alpha = \arg \min_{\substack{u \in X \\ k_1(u)=z_1,\ldots,k_{N_1}(u)=z_{N_1}}} \alpha|u|_P^2 + \sum_{i=1}^{N_2}(l_i(u) - r_i)^2.$$

**Theorem 2.15.** The mixed spline $\sigma_\alpha$ in the functional Hilbert space $X(\Omega)$ is represented in the form

$$\sigma_\alpha = \sum_{i=1}^{N_1} \lambda_i k_i G_p(s,\cdot) + \sum_{i=1}^{N_2} \rho_i l_i G_p(s,\cdot) + \sum_{j=1}^{S} \mu_j p_j(s). \tag{2.54}$$

The vectors $\boldsymbol{\lambda}_\alpha = (\lambda_1, ..., \lambda_{N_1})^T$, $\boldsymbol{\rho}_\alpha = (\rho_1, ..., \rho_{N_2})^T$, $\boldsymbol{\mu}_\alpha = (\mu_1, ..., \mu_S)^T$ form the solution to the system of linear algebraic equations

$$\begin{bmatrix} G_{11} & G_{12} & B_1 \\ G_{21} & G_{22} + \alpha I & B_2 \\ B_1^T & B_2^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_\alpha \\ \boldsymbol{\rho}_\alpha \\ \boldsymbol{\mu}_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{r} \\ \boldsymbol{0} \end{bmatrix}. \tag{2.55}$$

Here $\boldsymbol{z} = (z_1, ..., z_{N_1})^T$, $\boldsymbol{r} = (r_1, ..., r_{N_2})^T$ are the input vectors, $\boldsymbol{0}$ is the null vector of the dimension coinciding with the dimension of the vector $\boldsymbol{\mu}$. The matrices forming system (2.55) are the following

$$G_{11} = \{k_i k_j G_p(s,t)\}_{i=1,...,N_1}^{j=1,...,N_1}, \quad G_{12} = \{k_i l_j G_p(s,t)\}_{i=1,...,N_1}^{j=1,...,N_2},$$
$$G_{21} = \{l_i k_j G_p(s,t)\}_{i=1,...,N_2}^{j=1,...,N_1}, \quad G_{22} = \{l_i l_j G_p(s,t)\}_{i=1,...,N_2}^{j=1,...,N_2},$$
$$B_1 = \{k_i(p_j)\}_{i=1,...,N_1}^{j=1,...,S}, \quad B_2 = \{l_i(p_j)\}_{i=1,...,N_2}^{j=1,...,S},$$

$I$ is the identity matrix of the order $N_2$.

*Proof.* We will not completely prove the Theorem, but only explain the sense of (2.54) and (2.55) and their sources. Expansion (2.54) is reduced from (2.51) like expansion (2.47) is successively obtained from Theorems 2.10 - 2.13.

The first two groups of equations (2.55) bring about from interpolating conditions (2.52), has applied to expansion (2.54). The third group of equations follows from the condition $(\boldsymbol{\lambda}_\alpha, \boldsymbol{\rho}_\alpha) \in Z_P$ of Theorem 2.13. □