3. General Convergence Techniques and Error Estimates for Interpolating Splines

3.1. General Convergence Theorem

Let X and Y be the Hilbert spaces and $T : X \to Y$ be the linear bounded operator. Consider some infinite system

$$\mathcal{A} = \{A_i : X \to Z_i,\ i = 1, 2, ...\} \tag{3.1}$$

of the linear bounded operators A_i, each of them acting from X to some Hilbert space Z_i, $i = 1, 2, \dots$.

Definition 3.1. *We say the sequence $x_n \in X$ converges to $x \in X$ by the system $\mathcal{A}$ ($x_n \xrightarrow{\mathcal{A}} x$) if for every $A_i \in \mathcal{A}$ we have*

$$\lim_{n\to\infty} \|A_i(x_n - x)\|_{Z_i} = 0. \tag{3.2}$$

Definition 3.2. *The system $\mathcal{A}$ is said to be correct if the convergence $x_n \xrightarrow{\mathcal{A}} x$ brings about the weak convergence of x_n to x on some set K which is dense in the space X. Symbolically,*

$$[x_n \xrightarrow{\mathcal{A}} x] \Rightarrow [\exists K : \bar{K} = X, \quad \forall k \in K \quad (k, x_n)_X \to (k, x)_X]. \tag{3.3}$$

Let us fix the element $\varphi_* \in X$ and approximate it by the solutions σ_N of the following spline interpolation problems

$$\begin{aligned} &A_i \sigma_N = A_i \varphi_*, \quad i = 1, 2, ..., N, \\ &\|T\sigma_N\|_Y^2 = \min . \end{aligned} \tag{3.4}$$

We assume that the ranges $R(A_i)$ are closed in Z_i and introduce the operator $B_N : X \to Z_1 \times Z_2 \times ... \times Z_N$ by the formulae

$$B_N x = [A_1 x, A_2 x, ..., A_N x].$$

Assume that for any N_0 (T, B_{N_0}) forms a spline-pair; then (T, B_N) is also a spline-pair for $N \geq N_0$, and the interpolating spline σ_N exists and is unique when $N \geq N_0$.

Theorem 3.1. If $\mathcal{A}$ is correct system then

$$\lim_{N\to\infty} \|\sigma_N - \varphi_*\|_X = 0. \tag{3.5}$$

Proof. It is obvious that $\sigma_N \overset{\mathcal{A}}{\to} \varphi_*$. Really, for every $A_i \in \mathcal{A}$ we have $A_i\sigma_N = A_i\varphi_*$ when $N \geq i$. Hence there exists set $K \subset X$ which is dense in the space X and for every $k \in K \quad (k, \sigma_N)_X \to (k, \varphi_*)_X$.

We prove now that the sequence σ_N is bounded in X-norm with the constant independently of N. Let us decompose the space X into the orthogonal sum of the null space $N(T)$ and its orthogonal complement

$$X = N(T) \oplus N(T)^\perp.$$

With respect to this decomposition we have

$$\sigma = \sigma_N^1 + \sigma_N^2, \quad \sigma_N^1 \in N(T), \quad \sigma_N^2 \in N(T)^\perp.$$

The restriction $\tilde{T}$ of the operator T to $N(T)^\perp$ is the bijective mapping from $N(T)^\perp$ to the closed range $R(T)$. Then by the Banach inversion theorem we have

$$\sigma_N^2 = \tilde{T}^{-1}T\sigma_N,$$
$$\|\sigma_N^2\|_X \leq \|\tilde{T}^{-1}\| \cdot \|T\sigma_N\|_Y \leq \|\tilde{T}^{-1}\| \cdot \|T\varphi_*\|_Y.$$

We use here the evident inequality $\|T\sigma_N\|_Y \leq \|T\varphi_*\|_Y$. Thus the sequence σ_N^2 is bounded in X-norm.

Every element $k \in K$ can be represented in the form

$$k = k^1 + k^2, \quad k^1 \in N(T), \quad k^2 \in N(T)^\perp.$$

Since K is dense in X we choose from the elements k^1 the finite basis $k_1^1, k_2^1, \ldots, k_q^1$ of the q-th dimensional null space $N(T)$. Then

$$(\sigma_N^1, k_i^1)_X = (\sigma_N^1, k_i)_X = (\sigma_N - \sigma_N^2, k_i)_X.$$

It is clear $(\sigma_N, k_i)_X \to (\varphi_*, k_i)_X$ and therefore $(\sigma_N, k_i)_X$ is bounded, and $(\sigma_N^2, k_i)_X$ is bounded because $\|\sigma_N^2\|_X$ is bounded. Finally, every projection of σ_N^1 to the basic element of the finite-dimensional space $N(T)$ is bounded, and as a result $\|\sigma_N^1\|_X$ is bounded.

We know now that the interpolating splines σ_N weakly converge to φ_* in the dense set K and $\|\sigma_N\|_X$ are bounded. It means that the sequence σ_N weakly converges to φ_* $(\sigma_N \overset{W}{\to} \varphi_*)$ in the whole space X (Appendix 1, Theorem 3).

If $\sigma_N \overset{W}{\to} \varphi_*$ then $T\sigma_N \overset{W}{\to} T\varphi_*$ in the space Y. Using the orthogonal property of the interpolating spline (see Chapter 1) we have

$$\|T\sigma_N\|_Y^2 = (T\sigma_N, T\varphi_*)_Y.$$

Therefore $\|T\sigma_N\|_Y^2 \to \|T\varphi_*\|_Y^2$ and by the well-known theorem (Appendix 1, Theorem 4) we obtain

$$\|T\sigma_N - T\varphi_*\|_Y \to 0, \quad N \to \infty.$$

If we introduce in the space X the norm

$$\|u\|_* = (\|\tilde{A}u\|_Z^2 + \|Tu\|_Y^2)^{1/2} \tag{3.6}$$

where $(T, \tilde{A})$ is the maximal spline-pair with respect to (T, B_{N_0}) then $\tilde{A}$ has the finite-dimensional range and (3.6) is equivalent to the initial norm $\|u\|_X$. Therefore the conditions $T\sigma_N \overset{S}{\to} T\varphi_*$ in Y-norm and $\sigma_N \overset{W}{\to} \varphi_*$ in X bring about

$$\|\sigma_N - \varphi_*\|_* \to 0, \quad N \to \infty$$

and as the final result

$$\|\sigma_N - \varphi_*\|_X \to 0, \quad N \to \infty.$$

Theorem is proved. □

Remark. If you can prove by any way the existence of the dense set K in X such that we have the weak convergence of the sequence σ_N to φ_* at K, the proof of the strong convergence in X-norm can be automatically repeated without notion of the correct system of operators. We shall use this fact in the next Section.

3.2. General Convergence Theorem on ε-Nets

Let us consider the interpolation spline problem

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2$$

and suppose that the interpolation condition $A\sigma = z$ can be replaced by condition

$$k_p(\sigma) = k_p(\varphi_*), \quad p \in B, \tag{3.7}$$

where $\varphi_* \in X$ is the fixed element and k_p is the linear bounded functional over the space X while p is any parameter which runs in the compact $B \subset R^m$. In other words some nonlinear mapping $k : B \to X^*$ is given, X^* is a dual space with respect to X.

Let us assume that the interpolation problem

$$\begin{aligned} &\sigma = \arg \min_{x \in M_{B,\varphi_*}} \|Tx\|_Y^2, \\ &M_{B,\varphi_*} = \{x \in X : k_p(x) = k_p(\varphi_*), \quad p \in B\} \end{aligned} \tag{3.8}$$

is uniquely solvable for every $\varphi_* \in X$. This fact takes place (see Theorems 1.1, 1.2) if $N(T)$ is finite-dimensional, $R(T)$ is closed in the space Y and

$$N(T) \cap K^{\perp} = \{\Theta_X\}, \tag{3.9}$$

where

$$K^{\perp} = \{x \in X : k_p(x) = 0, \quad p \in B\}. \tag{3.10}$$

Denote by $Sp(B)$ a linear set in X which consists of the solutions of problems (3.8) for all $\varphi_* \in X$. It is easy to see that $Sp(B)$ is weakly closed in X. Actually the orthogonal property

$$(T\sigma, Tx)_Y = 0 \quad \forall x \in K^{\perp} \tag{3.11}$$

is necessary and sufficient condition to provide $\sigma \in Sp(B)$. If $\sigma_n \in Sp(B)$ and $\sigma_n \overset{W}{\rightarrow} \sigma_*$ then for every $x \in K^{\perp}$ we have

$$0 = (T\sigma_n, Tx)_Y = (\sigma_n, T^*Tx)_X \to (\sigma_*, T^*Tx)_X = (T\sigma, Tx)_Y = 0.$$

It means that $\sigma_* \in Sp(B)$, and $Sp(B)$ is closed in X-norm. If we introduce in $Sp(B)$ X-scalar product it becomes the Hilbert space.

For every $\varepsilon > 0$ we consider now the finite ε-net B_ε in the compact B and the following interpolation spline problem

$$\begin{aligned} &\sigma_\varepsilon = \arg \min_{x \in M_{B_\varepsilon, \varphi_*}} \|Tx\|_Y^2, \\ &M_{B_\varepsilon, \varphi_*} = \{x \in X : k_p(x) = k_p(\varphi_*), \quad p \in B_\varepsilon\}. \end{aligned} \tag{3.12}$$

Theorem 3.2. If the mapping $p \to k_p$ is continuous in the compact B then problem (3.12) is also uniquely solvable for sufficiently small $\varepsilon > 0$ and $\|\sigma_\varepsilon - \sigma\|_X \to 0$ when $\varepsilon \to 0$; here σ is the solution of problem (3.8).

Proof. At first we prove that problem (3.12) is uniquely solvable for the sufficiently small ε. We introduce the operators A and A_ε defined by formulae

$$\begin{aligned} &\forall x \in X \quad Ax = \{k_p(x), \quad p \in B\}, \\ &A_\varepsilon x = \{k_p(x), p \in B_\varepsilon\}. \end{aligned}$$

For every $\varepsilon > 0$ the solution of problem (3.12) exists. But assume that this solution is not unique even for small ε. It means that the sequence $\varepsilon_k \to 0$ does exist and the sequence $n_k \in N(A_{\varepsilon_k}) \cap N(T)$, $n_k \neq \Theta_X$ does also exist. Without loss of generality we suppose $\|n_k\|_X = 1$, $k = 1, 2, \ldots$. Since $N(T)$ is the finite-dimensional space it is possible to separate from n_k the subsequence $n_{k'} \to n_*$, and $\|n_*\|_X = 1$, $n_* \in N(T)$. The set $S = \bigcup_k B_{\varepsilon_k}$ is dense in B, and the set $\{k_p,\ p \in S\}$ is dense in $k(B)$ because the mapping $p \to k_p$ is continuous. Hence for every fixed $p \in B$ we can find the sequence k_{p_k}, $p_k \in B_{\varepsilon_k}$ such that $k_{p_k} \to k_p$ in X^*-norm. If in the trivial equality $(k_{p_k}, n_k)_X = 0$ we go to the limit we obtain

$$An_* = \{k_p(n_*) = 0, \quad p \in B\}.$$

Thus we find the nonzero element n_* belonging to $N(T) \cap N(A)$. This situation is impossible, and for the sufficiently small $\varepsilon \le \varepsilon_1$ the solution of (3.12) is unique.

We prove now the following

Lemma. For the sufficiently small $\varepsilon \le \varepsilon_2 \le \varepsilon_1$ inequality

$$\forall x \in N(A_\varepsilon) \quad \|x\|_X \le C\|Tx\|_Y \tag{3.13}$$

takes place with the constant C which is independent of x and ε; here

$$N(A_\varepsilon) = \{x \in X : k_p(x) = 0, \quad p \in B_\varepsilon\}. \tag{3.14}$$

Proof. Let $\varepsilon \le \varepsilon_1$ and problem (3.12) is uniquely solvable. Then by the theorem on norm equivalence

$$\forall x \in X \quad \|x\|_X \le C(B_{\varepsilon_1}) \cdot \left(\sum_{p \in B_{\varepsilon_1}} k_p^2(x) + \|Tx\|_Y^2 \right)^{1/2}. \tag{3.15}$$

Denote by $n_1, n_2, ..., n_q$ the basic elements of the null space $N(T)$. Since $N(A_{\varepsilon_1}) \cap N(T) = \{\Theta_X\}$ the matrix L constructed of the elements $k_p(n_k)$, $p \in B_{\varepsilon_1}$, $k = 1, 2, ..., q$ has the rank greater or equal to q. Using continuity of the mapping $p \to k_p$ we can see that this property will be preserved when the positions of the points p vary inside of small closed neighborhoods $G_p = B \cap \tilde{G}_p$, where $\tilde{G}_p$ is the closed ball with the origins at the points $p \in B_{\varepsilon_1}$. Let these neighbourhoods have no intersections. If we take the point p' from every G_p we obtain

$$\forall x \in X \quad \|x\|_X \le C(B'_{\varepsilon_1}) \left(\sum_{p' \in B'_{\varepsilon_1}} k_{p'}^2(x) + \|Tx\|_Y^2 \right)^{1/2} \tag{3.16}$$

where B'_{ε_1} is formed by the points $p' \in G_p$, $p \in B_{\varepsilon_1}$. We prove that $C(B'_{\varepsilon_1}) \le C$ where C is independent of the positions of p' in the neighbourhoods G_p, $p \in B_{\varepsilon_1}$. If this constant C does not exist then we have the sequence of elements $x_k \in X$, $\|x_k\| = 1$ and the sequence of points $p' \in G_p$, $p \in B_{\varepsilon_1}$ such that

$$\sum_{\substack{p'_k \in G_p \\ p \in B_{\varepsilon_1}}} k_{p'_k}^2(x_k) + \|Tx\|_Y^2 \le \frac{1}{k}.$$

Without loss of generality we are able to provide $p'_k \to p'_0$. Then for the points $p'_0 \in G_p$, $p \in B_{\varepsilon_1}$ the inequality is valid

$$C_1^2 \leq \sum_{\substack{p_0' \in G_p \\ p \in B_{\varepsilon_1}}} k_{p_0'}^2(x_k) + \|Tx_k\|_Y^2$$

with constant $C_1 > 0$. Then

$$J_k = \sum_{\substack{p_0' \in G_p \\ p \in B_{\varepsilon_1}}} k_{p_0'}^2(x_k) - \sum_{\substack{p_k' \in G_p \\ p \in B_{\varepsilon_1}}} k_{p_k'}^2(x_k) \geq C_1^2 - \frac{1}{k}.$$

But this inequality does not take place for large k because

$$|J_k| \leq 2 \max_{p \in B} \|k_p\|_{X^*} \cdot n_{\varepsilon_1} \cdot \max_{\substack{p_0', p_k' \in G_p, \\ p \in B_{\varepsilon_1}}} \|k_{p_0'} - k_{p_k'}\|_{X^*} \tag{3.17}$$

and the right-hand side of (3.17) tends to zero; here n_{ε_1} is a number of points in ε_1-net B_{ε_1}. Finally for the sufficiently small $\varepsilon \leq \varepsilon_0 \leq \varepsilon_1$ in every neighbourhood G_p of the points $p \in B_{\varepsilon_1}$ lies at least one point from ε-net B_ε. Therefore for the element $x \in N(A_\varepsilon)$ we have the inequality $\|x\|_X \leq C\|Tx\|_Y$. Lemma is proved. □

At last we point explicitly the dense set in the Hilbert space $Sp(B)$ where we have the weak convergence of σ_ε to σ. For this consideration we introduce the sequence $\varepsilon_k \to 0$ and the set $S = \bigcup_k B_{\varepsilon_k}$, which is dense in B. It is clear that the condition over $\tilde{\varphi} \in Sp(B)$

$$k_p(\tilde{\varphi}) = 0, \quad p \in S$$

brings about $\tilde{\varphi} = \Theta_X$. Therefore the set of finite linear combinations of the elements k_p forms in $Sp(B)$ the dense set K. In this set we have the weak convergence of σ_ε to σ. Really, for $k_p \in K$ we have

$$\begin{aligned} k_p(\sigma_{\varepsilon_k} - \sigma) &= (k_p - k_{p_k}, \sigma_{\varepsilon_k} - \sigma) \leq \|k_p - k_{p_k}\|_{X^*} \cdot \|\sigma_{\varepsilon_k} - \sigma\|_X \\ &\leq C\|k_p - k_{p_k}\|_X \cdot \|T(\sigma_{\varepsilon_k} - \sigma)\|_Y \\ &\leq 2C\|T\varphi_*\|_Y \cdot \|k_p - k_{p_k}\|_{X^*}. \end{aligned}$$

If p_k are the points of ε_k-net B_{ε_k} which tend to p, we have the weak convergence of σ_{ε_k} to σ in the dense set K. Using remark 3.1 we obtain the final result

$$\|\sigma_\varepsilon - \sigma\|_X \to 0, \quad \varepsilon \to 0.$$

Theorem is proved. □

3.3. Convergence of D^m-Splines on Scattered Meshes

Let Ω be some bounded domain in R^n. Consider in Ω the sequence of the scattered meshes

$$\omega_k = \{P_1, P_2, ..., P_{S(k)}\}, \quad S(k+1) > S(k), \quad k = 1, 2, ... \tag{3.18}$$

and let the set $S = \bigcup_k \omega_k$ be dense in $\bar{\Omega}$. It is clear that $\omega_k \subset \omega_{k+1}$. We formulate the problem of spline-interpolation on every mesh ω_k by D^m-spline σ_k from the conditions: find $\sigma_k \in W_2^m(\Omega), \quad m > n/2$ such that

$$\begin{aligned} &\sigma_k(P_j) = \varphi_*(P_j), \quad j = 1, 2, ..., S(k). \\ &\|D^m \sigma_k\|_Y^2 = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha \sigma_k)^2 d\Omega = \min. \end{aligned} \tag{3.19}$$

We will prove now that the system of linear bounded functionals $k_p(u) = u(P),\ P \in S$ is the correct system (functional is the case of operator which maps to R). Let $\mathcal{A} = \{k_p : W_2{}^m \to R,\ p \in S\}$ and $u_n \xrightarrow{\mathcal{A}} u$ (see Definition 3.1). It means that

$$\forall k_p \in \mathcal{A} \quad k_p(u_n) \to k_p(u), \quad n \to \infty. \tag{3.20}$$

Let $\pi : X^* \to X$ be the reproducing mapping of the space W_2^m, then

$$\forall u \in X \quad k_p(u) = (\pi(k_p), u)_{W_2^m}.$$

If some element $u \in W_2^m$ is orthogonal to every $\pi(k_p), \quad p \in S$; i.e.

$$(\pi(k_p), u)_{W_2^m} = u(P) = 0 \quad \forall p \in S.$$

It means that $u = 0$ because $u(P)$ vanishes at the dense set S in Ω and $u(P)$ is continuous (note that $m > n/2$). Therefore the set K of the finite linear combinations of the element $\pi(k_p),\ p \in S$ is dense in the space W_2^m, and by (3.20) we have the weak convergence of the sequence u_n to u on the set K. Finally, by Theorem 3.1 we have

$$\|\sigma_k - \varphi_*\|_{W_2^m(\Omega)} \to 0, \quad k \to \infty. \tag{3.21}$$

In this situation we have the only trouble: $\omega_n \subset \omega_{n+1}$. We refuse this condition with the help of Theorem 3.2.

Let $B \subset \bar{\Omega}$ be some a compact set, $\varphi_* \in W_2^m(\Omega)$ be some fixed function. We consider interpolation spline problem

$$\begin{aligned} &\forall p \in B \quad \sigma(P) = \varphi_*(P), \\ &\|D^m \sigma\|_Y^2 = \min \end{aligned} \tag{3.22}$$

and suppose that this problem is uniquely solvable. Let $B_\varepsilon \subset B$ be ε-net in B and $\sigma_\varepsilon \in W_2^m(\Omega)$ be the solution of the interpolation problem

$$\forall p \in B_\varepsilon \quad \sigma_\varepsilon(P) = \varphi_*(P)$$
$$\|D^m \sigma_\varepsilon\|_Y^2 = \min .$$

Since $m > n/2$ the space $W_2^m(\Omega)$ is continuously embedded to the space $C(\bar{\Omega})$ of the continuous functions with the uniform norm. We introduce the mapping $p \to k_p$ from B to the dual space X^* by the following formula

$$\forall u \in W_2^m(\Omega) \quad \forall p \in B \quad u(P) = k_p(u). \tag{3.23}$$

We prove now that $p \to k_p$ is a continuous mapping. Actually, the space $W_2^m(\Omega)$ is continuously embedded to the Hölder space with the power $\alpha \in (0,\ m-n/2)$. Then

$$\|k_p - k_{p'}\|_{X^*} = \sup_{\|u\|_{W_2^m}=1} | u(P) - u(P') | \leq C \cdot \|P - P'\|_2^\alpha ,$$

the constant C is independent of u, $\|P - P'\|_2$ is the Euclidean distance between P and P'.

Thus, by Theorem 3.2 we obtain

$$\|\sigma_\varepsilon - \sigma\|_{W_2^m(\Omega)} \to 0, \quad \varepsilon \to 0, \tag{3.24}$$

or in the particular case $B = \bar{\Omega}$

$$\|\sigma_\varepsilon - \varphi_*\|_{W_2^m(\Omega)} \to 0, \quad \varepsilon \to 0. \tag{3.25}$$

3.4. Error Estimates for the Interpolating Splines

Let $\Omega \subset R^n$ be some bounded domain. Denote by $X(\Omega)$ some Hilbert space of the functions defined on Ω. We assume that the space $X(\Omega)$ is continuously embedded to $C(\Omega)$,

$$\forall u \in X(\Omega) \quad \|u\|_{C(\Omega)} \leq C_1 \|u\|_{X(\Omega)} \tag{3.26}$$

and every $u \in X(\Omega)$ can be prolonged by Pu to the functional space $X(\hat{\Omega})$, $\hat{\Omega} \supset \Omega$ and the distance between the boundary $\hat{\Gamma}$ of $\hat{\Omega}$ and Ω is greater than fixed $\delta > 0$,

$$\forall u \in X(\Omega) \quad \|Pu\|_{X(\hat{\Omega})} \leq C_2 \|u\|_{X(\Omega)} \tag{3.27}$$

We consider in Ω the family of the finite ε-nets ω_ε, $\varepsilon \to 0$, and introduce the spline interpolation problem

$$\begin{aligned} &\sigma_\varepsilon = \arg \min_{u \in M_{\omega_\varepsilon, \varphi_*}} \|Tu\|_{Y(\hat{\Omega})}^2 , \\ &M_{\omega_\varepsilon, \varphi_*} = \{u \in X(\hat{\Omega}) : u \lfloor_{\omega_\varepsilon} = \varphi_* \vert_{\omega_\varepsilon} \}, \end{aligned} \tag{3.28}$$

where $\varphi_* \in X(\hat{\Omega})$, $T : X(\hat{\Omega}) \to Y(\hat{\Omega})$ is a linear bounded operator acting to the Hilbert space $Y(\hat{\Omega})$. Under the standard constraints (see Chapter 1) this problem is uniquely solvable and by the Theorem 3.2

$$\|\sigma_\varepsilon - \sigma\|_{X(\hat{\Omega})} \to 0, \quad \varepsilon \to 0,$$

where $\sigma \in X(\hat{\Omega})$ is the solution of the continuation spline problem

$$\sigma = \arg \min_{u \in M_{\Omega,\varphi_*}} \|Tu\|_{Y(\hat{\Omega})},$$
$$M_{\Omega,\varphi_*} = \{u \in X(\hat{\Omega}) : u\,|_\Omega = \varphi_*\,|_\Omega\}. \tag{3.29}$$

Since the trace operator from $X(\hat{\Omega})$ to $X(\Omega)$ is continuous, we have

$$\|\sigma_\varepsilon - \varphi_*\|_{X(\Omega)} \to 0, \quad \varepsilon \to 0.$$

Our aim is to obtain error estimates in other weaker norms or seminorms.

3.4.1. Error Estimates for the Generalized Lagrangian Interpolation

Denote by $n_1(P), ..., n_q(P)$ the basic elements of the null space $N(T)$. Let $\bar{P} = \{P_1, P_2, ..., P_q\}$ be L-set, i.e. the problem of finding the element $n(P) \in N(T)$ from the conditions

$$n(P_i) = r_i, \quad i = 1, 2, ..., q$$

has the unique solution for arbitrary $r_1, r_2, ..., r_q$. We define the operator $\pi_p : X(\Omega) \to N(T)$ of the generalized Lagrangian interpolation in the following way: for every $u \in X(\Omega)$ its Lagrangian interpolation $\pi_p u \in N(T)$ is the solution of problem

$$(\pi_p u)(P_i), \quad i = 1, 2, ..., q. \tag{3.30}$$

Since $\pi_p u$ can be represented in the form

$$\pi_p u = \sum_{j=1}^{q} C_j n_j(P),$$

conditions (3.30) form a linear algebraic system with the nonsingular $q \times q$ - matrix D of the elements $n_j(P_i)$. The functions $n_j(P)$ are continuous and it is possible to vary positions of the points P_i in some small neighbourhoods to provide the non-singularity of the matrix D, in other words, the totality of the L-sets is open, and there exists compact set $B \subset \hat{\Omega}^q$ of L-sets.

Let the initial space $X(\Omega)$ be continuously embedded to some seminormed space $V(\Omega)$, i.e.

$$\forall u \in X(\Omega) \quad \|u\|_{V(\Omega)} \le C_3 \|u\|_{X(\Omega)}. \tag{3.31}$$

The following lemma is valid.

Lemma 3.1. The constant C_4 does exist independent of $u \in X(\Omega)$ and $\bar{P} \in B$ such that

$$\|u - \pi_{\bar{P}}u\|_{V(\Omega)} \leq C_4\|Tu\|_{Y(\Omega)}. \tag{3.32}$$

Proof. Using Theorem 1.5 on the norm equivalence with the trace operator A connected with the mesh $\bar{P} \in B$ we have

$$\|u\|_{X(\Omega)} \leq C(\bar{P}) \left[\sum_{i=1}^{q} u^2(P_i) + \|Tu\|_Y^2 \right]^{1/2}.$$

If we substitute instead of u the difference $u - \pi_{\bar{P}}u$ to this inequality and use embedding condition (3.31), then we obtain

$$\|u - \pi_{\bar{P}}u\|_{V(\Omega)} \leq C_4(\bar{P}) \cdot \|Tu\|_{Y(\Omega)}, \quad C_4(\bar{P}) = C_3 \cdot C(\bar{P}). \tag{3.33}$$

We will show that $C_4(\bar{P})$ can be taken independent of $\bar{P} \in B$. To prove it, we show that the family of the operators $\{\pi_{\bar{P}},\ \bar{P} \in B\}$ is uniformly bounded in the $X \to X$-operator norm. Actually, to find $\pi_{\bar{P}}u$ we need to solve a nonsingular linear algebraic system with the elements which are continuous with respect to $\bar{P} \in B$. Since the matrix inversion is also a continuous operation by the Weierstrass theorem we obtain that for every $u \in X(\Omega)$ the family of the functions $\{\pi_{\bar{P}}u,\ \bar{P} \in B\}$ is uniformly bounded in X-norm. Then by the Banach-Steinhaus (see Appendix 1, Theorem 9) theorem we have

$$\|\pi_{\bar{P}}\|_{X \to X} \leq C_5$$

with the constant C_5 independent of $\bar{P} \in B$. Then

$$\|I - \pi_{\bar{P}}\|_{X \to X} \leq 1 + C_5 = C_6.$$

Let us fix $\bar{P}_* \in B$ and represent the function u in the form

$$u = \pi_{\bar{P}}u + u^{\perp},$$

where $u^{\perp}$ belongs to the subspace $N(T)^{\perp} \subset X$ which consists of the functions vanishing at L-set $\bar{P}_*$. Since $\pi_{\bar{P}}\pi_{\bar{P}_*}u = \pi_{\bar{P}_*}u$ we have $\pi_{\bar{P}}u = \pi_{\bar{P}_*}u + \pi_{\bar{P}}u^{\perp}$. In the subspace $N(T)^{\perp}$ the expression $\|Tu^{\perp}\|_{Y(\Omega)}$ is the norm. Therefore

$$\|u - \pi_{\bar{P}}u\|_X \leq C_6\|Tu^{\perp}\|_Y = C_6\|Tu\|_Y.$$

Using embedding condition (3.31) we obtain

$$\|u - \pi_{\bar{P}}u\|_V \leq C_3\|u - \pi_{\bar{P}}u\|_X \leq C_3 \cdot C_6\|Tu\|_Y.$$

Finally, $C_4(\bar{P}) \leq C_3 \cdot C_6 = C_4$ and lemma is proved. □

Let the function $u(x)$ be defined in the small n-dimensional ball S_h with the center point $P_0 \in R^n$ of the radius h. Then the function $\bar{u}(t) = u(P_0 + th)$ is defined in the unit ball S_1 with zero center.

Assume that the seminorm $\|\cdot\|_V$ and the norm $\|\cdot\|_Y$ have the following properties of "homogeneity"

$$\|u(P_0+th)\|_{V(S_1)} \geq f_1(h)\cdot\|u\|_{V(S_h)} \tag{3.34}$$

$$\|Tu(P_0+th)\|_{Y(S_1)} \leq f_2(h)\cdot\|Tu\|_{Y(S_h)}, \tag{3.35}$$

where $f_1(h), f_2(h)$ are the positive functions of the argument $h>0$.

Theorem 3.3. For every function $u\in(\Omega)$ and for every L-solvable set $\bar{P}\in S_h\times\ldots\times S_h$ the following error estimate for the generalized Lagrangian approximation $\pi_{\bar{P}}u$ is valid

$$\|u-\pi_{\bar{P}}u\|_{V(S_h)} \leq Cf_2(h)/f_1(h)\cdot\|Tu\|_{Y(S_h)} \tag{3.36}$$

with the constant C independent of u and $\bar{P}_h$ from the compact B of L-sets, which is the closed neighborhood of the fixed L-set $\bar{P}_{*,h}\in S_h\times\ldots\times S_h$.

Proof. Consider in the ball S_h any fixed L-set $\bar{P}_{*,h}$ and L-set $\bar{P}_h\in B$. After the linear transformation $x=P_0+th$ the L-set $\bar{P}_h$ is transformed to the L-set $\bar{P}\in S_1\times\ldots\times S_1$. Let us apply lemma 3.1 to the function $\bar{u}(t)=u(P_0+th)$. Then we obtain

$$\|\bar{u}(t)-\pi_{\bar{P}}u(t)\|_{V(S_1)} \leq C\cdot\|T\bar{u}\|_{Y(S_1)}.$$

Using inequalities (3.34), (3.35) we have

$$\|u-\pi_{\bar{P}_h}u\|_{V(S_h)} \leq Cf_2(h)/f_1(h)\cdot\|Tu\|_{Y(S_h)},$$

and Theorem is proved. □

If the function u is equal to zero at the points of L-solvable set $\bar{P}$ then we have $\pi_{\bar{P}_h}u\equiv 0$ and

$$\|u\|_{V(S_h)} \leq Cf_2(h)/f_1(h)\cdot\|Tu\|_{Y(S_h)}. \tag{3.37}$$

This inequality is basic in obtaining the error estimates for the splines at h-nets.

3.4.2. Special Covers and Error Estimates for the Splines at h-Nets

Let us consider a bounded domain $\Omega\subset R^n$ and another domain $\hat{\Omega}\supset\Omega$ such that the distance between the boundary $\hat{\Gamma}$ of $\hat{\Omega}$ and the boundary Γ of Ω is greater than fixed $\delta>0$. For sufficiently small $h\leq h_0$ it is possible to cover the closed domain Ω with the finite number of balls $B_i^h(h)$ with the radius h, each of these balls lying inside $\hat{\Omega}$.

Definition. *We say that the family of the finite covers with the balls* $\{B_i^h(h)\}_{h>0}$ *is special if for every* $u\in X(\Omega)$ *inequalities*

$$\|u\|_{V(\Omega)} \leq K_1\sum_i\|u\|_{V(B_i^h(h))} \tag{3.38}$$

$$\sum_i \|Tu\|_{Y(B_i^h(h))} \le K_2 \|Tu\|_{Y(\hat{\Omega})} \tag{3.39}$$

take place with constant K_1, K_2 which are independent of $h \le h_0$.

We say briefly in this situation that the domain Ω can be *covered in a special way*.

Lemma 3.2. Let Ω be able to be covered in a special way and $\omega_h \subset \Omega$ be h-net in Ω. If some function $u \in X(\hat{\Omega})$ is equal to zero at ω_h then inequality

$$\|u\|_{V(\Omega)} \le C \cdot f_2(h)/f_1(h) \cdot \|Tu\|_{Y(\hat{\Omega})} \tag{3.40}$$

is valid with the constant C which is independent of u and h.

Proof. Let $\{B_i^h(h)\}$ be a special cover. Let us form inside every ball of the cover q closed balls such that their Cartesian product consists of L-sets. Since ω_h is the condensating h-net we are able (by increasing h into the finite number of times) to provide the following property: each of these balls contains the point from ω_h. Then by inequality (3.37) we have

$$\|u\|_{V(\Omega)} \le K_1 \sum_i \|u\|_{V(B_i^h(h))} \le C \cdot f_2(h)/f_1(h) \cdot K_1 \sum_i \|Tu\|_{Y(B_i^h(h)}$$
$$\le C \cdot K_1 \cdot K_2 f_2(h)/f_1(h) \cdot \|Tu\|_{Y(\hat{\Omega})}$$

and inequality (3.40) takes place with constant $C \cdot K_1 \cdot K_2$. □

We apply this lemma to obtain error estimates for the splines at h-nets. Let $\sigma_h(P) \in X(\hat{\Omega})$ be the solution of the interpolation problem

$$\sigma_h = \arg \min_{u \in M_{\omega_h,\varphi_*}} \|Tu\|^2_{Y(\hat{\Omega})}$$
$$M_{\omega_h,\varphi_*} = \{u \in X(\hat{\Omega}) : u|_{\omega_h} = \varphi_*|_{\omega_h}\} \tag{3.41}$$

and $\sigma(P) \in X(\hat{\Omega})$ be the solution of the prolongation problem

$$\sigma = \arg \min_{u \in M_{\Omega,\varphi_*}} \|Tu\|^2_{Y(\hat{\Omega})}$$
$$M_{\Omega,\varphi_*} = \{u \in X(\hat{\Omega}) : u|_{\Omega} = \varphi_*|_{\Omega}\}. \tag{3.42}$$

The difference $u(P) = \sigma_h(P) - \sigma(P)$ is equal to zero at h-net ω_h. Therefore by lemma 3.2 we have

$$\|\sigma_h - \sigma\|_{V(\Omega)} \le C f_2(h)/f_1(h) \cdot \|T(\sigma_h - \sigma)\|_{Y(\hat{\Omega})}. \tag{3.43}$$

If we take into account $\sigma = \varphi_*$ in Ω we obtain

$$\|\sigma_h - \varphi_*\|_{V(\Omega)} \le C f_2(h)/f_1(h) \cdot \|T(\sigma_h - \sigma)\|_{Y(\hat{\Omega})}. \tag{3.44}$$

Moreover, by Theorem 3.2 $\|\sigma_h - \sigma\|_{X(\hat{\Omega})} \to 0, \quad h \to 0$, hence, $\|T(\sigma_h - \sigma)\|_{Y(\hat{\Omega})} \to 0$. Finally,

$$\|\sigma_h - \varphi_*\|_{V(\Omega)} = o(f_2(h)/f_1(h)). \tag{3.45}$$

3.4.3. Error Estimates for D^m-Splines in L_p-Norms

Let $\Omega \subset R^n$ be some bounded domain with the cone condition. It means that every function $u \in W_2^m(\Omega)$ can be prolonged by the function $\hat{u} \in W_2^m(\hat{\Omega})$, $\hat{\Omega} \supset \Omega$ where $\hat{\Omega}$ is the domain with the arbitrary smooth boundary and

$$\|\hat{u}\|_{W_2^m(\hat{\Omega})} \leq C\|u\|_{W_2^m(\Omega)} \tag{3.46}$$

where $C < \infty$ is the norm of the prolongation operator. In a particular case, the distance between the boundaries $\hat{\Gamma}$ and Γ of the domains $\hat{\Omega}$, Ω correspondingly can be done greater than fixed $\delta > 0$. For simplicity we assume that domain $\hat{\Omega}$ is n-dimensional parallelepiped. Consider in $\hat{\Omega}$ n-dimensional grid Δ_h with the mesh size $h > 0$. At every point of the grid we construct the ball with the center at this point. It is easy to understand that we can choose the radius of these balls in such a way that every point of $\hat{\Omega}$ (and every point of Ω) is covered with not greater than 2^n times with balls, and this number is independent of h. We denote these balls by $B_i(h)$.

Let $X(\Omega) = W_2^m(\Omega)$, $m > n/2$ and $V(\Omega) = W_p^k(\Omega)$, $2 \leq p \leq \infty$. If the inequality $k - n/p \leq m - n/2$ takes place (except the case $k = m - n/2$ & $p = \infty$), then the space W_2^m is continuously embedded to the space W_2^m. Introduce in W_q^k the seminorm $\|u\|_{V(\Omega)} = \|D^k u\|_{L_p(\Omega)}$ by formula

$$\|D^k u\|_{L_p(\Omega)} = \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\Omega} (\frac{\partial^k u}{\partial t^\alpha})^p d\Omega \right)^{1/p} . \tag{3.47}$$

Let us find for this seminorm the function $f_1(h)$, defined in (3.33). It is easy to see that

$$\|D^k u(P_0 + th)\|_{L_p(S_1)} = h^{k-n/p}\|D^k u\|_{L_p(S_h)} \tag{3.48}$$

and $f_1(h) = h^{k-n/p}$. In a similar way if $T = D^m$, then we have

$$\|D^m u(P_0 + th)\|_{L_2(S_1)} = h^{m-n/2}\|D^m u\|_{L_2(S_h)} \tag{3.49}$$

and $f_2(h) = h^{m-n/2}$. Thus, by Theorem 3.3 we have the error estimate for the Lagrangian polynomial in the small ball S_h

$$\|D^k(u - \pi_{\bar{P}_h} u)\|_{L_p(S_h)} \leq C h^{m-n/2-k+n/q}\|D^m u\|_{L_2(S_h)}. \tag{3.50}$$

It is easy to show that the cover of Ω, constructed with the grid in the parallelepiped $\hat{\Omega}$ is special. Actually,

$$\|D^k u\|_{L_p(\Omega)} \leq \sum_i \|D^k u\|_{L_p(B_i(h))} \tag{3.51}$$

and inequality (3.37) takes place with $K_1 = 1$. Inequality (3.38) is also valid

$$\sum_i \|D^m u\|_{L_2(B_i(h))} \leq 2^{n/2}\|D^m u\|_{L_2(\hat{\Omega})} \tag{3.52}$$

because every point of Ω is covered with the balls $B_i(h)$ not more than 2^n times and every elementary square participates in the integration not more than 2^n times.

Finally, if ω_h is h-net in Ω, we estimate the following error for the suitable D^m-splines σ_h which interpolate the function $\varphi_* \in W_2^m(\Omega)$

$$\|D^k(\sigma_h - \varphi_*)\|_{L_p(\Omega)} \leq Ch^{m-n/2-k+n/p}\|D^m(\sigma_h - \sigma)\|_{L_2(\Omega)} \tag{3.53}$$

where σ is the prolongation of φ_* to $\hat{\Omega}$ with the minimal D^m-seminorm, and $\|D^m(\sigma_h - \sigma)\|_{L_2(\Omega)}$ also tends to zero.

Remark. The error estimates technique is the same (and error estimates are the same) if instead of the usual interpolation conditions $\sigma_h|_{\omega_h} = \varphi_*|_{\omega_h}$ we have the interpolation conditions in the sense of the local integrals like these

$$\int_{B_i(h)} \sigma_h d\Omega = \int_{B_i(h)} \varphi_* d\Omega, \tag{3.54}$$

where $B_i(h)$ are any balls of the size h, whose centers form h-net in Ω. In this situation, the difference $\sigma_h - \varphi_*$ is also equal to zero at any Ch-net (C=const) because $\int_{B_i(h)} (\sigma - \varphi_*)d\Omega = 0$ and $\sigma - \varphi_*$ change the sign in every $B_i(h)$, and a standard technique can be applied.