4. Splines in Subspaces

In the previous chapters we have already discussed the main theoretical questions concerning characterization formulae and convergence of variational splines. It is obvious now that there are certain numerical difficulties which arise in the construction and applications of the variational splines (for example, of multi-dimensional D^m-splines on the scattered meshes).

To construct the variational spline in the analytical form, we need to know exactly the reproducing mapping or kernel of the corresponding Hilbert or semi-Hilbert space. In the particular case of D^m-splines we need to know the the Green function of the polyharmonic operator Δ^m in the multi-dimensional domain Ω. Usually, this function is known in the analytical form only for one-dimensional case.

If the reproducing kernel is known, then the second difficulty is the solution of the linear algebraic system (see Section 2.5) with a dense matrix. Only in one-dimensional case this matrix can be done a band matrix.

And the last, the representation formula with the help of the reproducing kernel is very complicated and often unstable for numerical calculations. But the main preferance of the splines in applications was simple representation formulae, like in the case of the piecewise polynomial splines.

All these reasons suggest us the following simple idea: instead of the complicated analytical solution of the variational spline-problem we need to find simple approximation of the exact solution using ideas of the finite element method. In this case we obtain sparse linear algebraic systems and simple representation formulae.

The aim of this chapter is to discuss in the general form the finite-dimensional analogs of the interpolating and smoothing splines, to obtain the corresponding convergence theorems and to give the error estimates. The finite element method will be illustrated for the multi-dimensional D^m-splines on the scattered meshes.

4.1. Interpolating and Pseudo-Interpolating Splines in Subspaces

4.1.1. Definitions, Algebraic Systems

Let X, Y and Z be some Hilbert spaces and $T : X \to Y$, $A : X \to Z$ be linear bounded operators. We assume that (T, A) is spline-pair. It means that spline interpolation problem

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2 \tag{4.1}$$

has the unique solution if $A^{-1}(z) \neq \emptyset$, $z \in Z$. The pseudo-interpolation problem

$$\sigma = \arg \min_{x \in (A^*A)^{-1}(z)} \|Tx\|_Y^2 \tag{4.2}$$

also has the unique solution for every $z \in Z$.

Let us consider in the space X the family of the finite dimensional subspaces $\{E_k\}_{k=1}^{\infty}$.

Definition 1. *The pseudo-interpolating spline σ_k in the subspace E_k is the solution of problem*

$$\sigma_k = \arg \min_{x \in \Re_k(z)} \|Tx_k\|_Y^2 \tag{4.3}$$

$$\Re_k(z) = \{x_k \in E_k : \|Ax_k - z\|_Z^2 = \min_{u_k \in E_k} \|Au_k - z\|_Z^2\}. \tag{4.4}$$

If the interpolation condition $A\sigma_k = z$ is non-contradictory in the space E_k the pseudo-interpolating spline $\sigma_k \in E_k$ is the interpolating spline in E_k.

It is a trivial fact that when (T, A) form a spline-pair, the pseudo-interpolating spline σ_k does always exist and is unique.

We obtain a linear algebraic system to find the pseudo-interpolating spline. Let $\omega_1, \omega_2, \cdots, \omega_{n(k)}$ be some basis of the space E_k . Then

$$\sigma_k = \sum_{i=1}^{n(k)} \sigma_k^i \omega_i, \tag{4.5}$$

where σ_k^i are any coefficients. Our aim is to find the minimum of the quadratic functional

$$\|T\sigma\|^2 = \sum_{i,j=1}^{n(k)} \sigma_k^i \sigma_k^j (T\omega_i, T\omega_j) \tag{4.6}$$

with respect to the variables $\sigma_k^1, \sigma_k^2, \cdots, \sigma_k^{n(k)}$ under the linear constraint

$$\sum_{i=1}^{n(k)} \sigma_k^i A_k^* A_k \omega_i = A_k^* z, \tag{4.7}$$

where A_k is the restriction of the operator A to the subspace E_k. We calculate the scalar products of both sides of (4.7) with $\omega_1, \omega_2, \cdots, \omega_{n(k)}$ and obtain

$$\sum_{i=1}^{n(k)} \sigma_k^i (A\omega_i, A\omega_j)_Z = (z, A\omega_j)_Z, \quad j = 1, 2, \cdots, n(k). \tag{4.8}$$

The index k disappears because A_k is equal to A in the subspace E_k. We introduce the vectors-colomns $\bar{\sigma}$ and $\bar{f}$ by the following formulae

$$\bar{\sigma} = (\sigma_k^1, \sigma_k^2, \cdots, \sigma_k^{n(k)})^T,$$

$$\bar{f} = \left((z, A\omega_1)_Z, ..., (z, A\omega_{n(k)})_Z\right)^T$$

and consider two $n(k) \times n(k)$ matrices $\bar{T}$ and $\bar{A}$ composed of the elements $(T\omega_i, T\omega_j)_Y$ and $(A\omega_i, A\omega_j)_Z$. Then our problem for the pseudo-interpolating spline in E_k can be written in the form

$$\bar{\sigma} = \arg\min_{\bar{x} \in M_f} (T\bar{x}, \bar{x}), \tag{4.9}$$

$$M_{\bar{f}} = \{\bar{x} \in R^{n(k)} : \bar{A}\bar{x} = \bar{f}\}, \tag{4.10}$$

where $(,)$ denote the usual scalar product of the vectors of the length $n(k)$. If $\bar{\Lambda} = (\Lambda_1, \Lambda_2, ..., \Lambda_{n(k)})^T$ is the vector of the Lagrangian parameters, then the corresponding Lagrange function can be written in the form:

$$\Phi(\bar{\sigma}, \bar{\Lambda}) = \frac{1}{2}(\bar{T}\bar{\sigma}, \bar{\sigma}) + (\bar{\Lambda}, \bar{A}\bar{\sigma} - \bar{f}). \tag{4.11}$$

The minimization of this function gives us the block system

$$\begin{pmatrix} \bar{T} & \bar{A} \\ \bar{A} & 0 \end{pmatrix} \begin{pmatrix} \bar{\sigma} \\ \bar{\Lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{f} \end{pmatrix}. \tag{4.12}$$

This system is symmetric, in the general case it is singular but always solvable, it has the unique solution with respect to the vector $\bar{\sigma}$ and a non-unique solution with respect to $\bar{\Lambda}$. The eigenvalues of this matrix have the different signs. To find the spline $\sigma_k \in E_k$ we need to find any solution of system (4.12).

If the interpolation condition $A\sigma_k = Z$ is non-contradictory in the subspace E_k and Z is the finite-dimensional vector space then instead of (4.12) the other system arises for the interpolating spline in E_k

$$\begin{pmatrix} \bar{T} & \bar{B}^* \\ \bar{B} & 0 \end{pmatrix} \begin{pmatrix} \bar{\sigma} \\ \bar{\Lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ z \end{pmatrix}, \tag{4.13}$$

where $\bar{B}$ is rectangular $n(z) \times n(k)$-matrix of the rows $A\omega_i \in Z, i = 1, 2, ..., n(k)$, $n(z) = \dim Z$.

4.1.2. Convergence

Let us consider in the space X the family of the finite-dimensional subspaces $\{E_\tau\}_{\tau>0}$ where $\tau > 0$ is the real parameter (for example, the size of the finite elements).

Definition 2. *We say that E_τ converges to the space X ($E_\tau \to X$) if for every element $x \in X$ the sequence $x_{\tau_k} \in E_{\tau_k}$ does exist such that $\|x - x_{\tau_k}\|_X \to 0$ when $\tau_k \to 0$.*

Definition 3. *We say that E_τ weakly converges to the space X ($E_\tau \overset{W}{\to} X$) if for every element $x \in X$ the sequence $x_{\tau_k} \in E_{\tau_k}$ does exist such that $x_{\tau_k} \overset{W}{\to} x$ in the weak sense when $\tau_k \to 0$.*

We assume that (T, A) is a spline-pair, $z \in Z, A^{-1}(z) \neq \emptyset$ and $\sigma \in X$ is the solution of the spline interpolation problem

$$\sigma = \arg \min_{x \in A^{-1}(z)} \|Tx\|_Y^2. \tag{4.14}$$

We suppose that the interpolation condition $Ax_\tau = z$ is also non-contradictory in the subspace E_τ for the sufficiently small $\tau \leq \tau_0$, and consider the corresponding spline interpolation problems in the subspace $E_\tau, \tau \leq \tau_0$,

$$\sigma_\tau = \arg \min_{x \in A_\tau^{-1}(z)} \|Tx_\tau\|_Y^2, \tag{4.15}$$

where

$$A_\tau^{-1}(z) = \{x_\tau \in E_\tau : Ax_\tau = z\}. \tag{4.16}$$

Assume now that $N(T) \subset E_\tau$ for $\tau \leq \tau_0$. In this case for $z \in AN(T)$ $\|T\sigma\|_Y^2 = \|T\sigma_\tau\|_Y^2 = 0$ and $\sigma = \sigma_\tau$. In other words the element of the null space $N(T)$ is reproduced exactly both in problems (4.14) and (4.15). In fact the non-trivial approximation process goes in the orthogonal complement of $N(T)$.

Let us consider the maximal spline-pair $(T, \tilde{A})$ with respect to (T, A) and the corresponding scalar product

$$(x_1, x_2)_* = (\tilde{A}x_1, \tilde{A}x_2)_{\tilde{Z}} + (Tx_1, Tx_2) \tag{4.17}$$

and norm

$$\|x\|_* = \left(\|\tilde{A}x\|_{\tilde{Z}}^2 + \|Tx\|_Y^2\right)^{1/2}, \tag{4.18}$$

which is equivalent to the initial X-norm. In the orthogonal complement $N(T)_*^\perp = N(\tilde{A})$ the expressions $(Tx_1, Tx_2)_Y = (x_1, x_2)_*$ and $\|Tx\|_Y = \|x\|_*$ become the scalar product and the norm. For these reasons we can consider only the situations

$$z \in AN(T)_*^{\perp}, \quad E_\tau \to N(T)_*^{\perp} \quad \text{or} \quad E_\tau \xrightarrow{W} N(T)_*^{\perp}$$

and can reformulate problem (4.14), (4.15) in the following forms for the normal splines

$$\begin{aligned} &\sigma = \arg\min_{x\in\Re} \|x\|_*^2, \\ &\Re = \{x \in N(T)_*^{\perp} : Ax = z\} \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} &\sigma_\tau = \arg\min_{x\in\Re_\tau} \|x_\tau\|_*^2, \\ &\Re_\tau = \{x_\tau \in E_\tau : Ax_\tau = z\}. \end{aligned} \tag{4.20}$$

We have already obtained the resolvent formula for the solution of problem (4.19) (see Section 1.4.2)

$$\sigma = A^*(AA^*)^{-1}z. \tag{4.21}$$

To obtain the corresponding formula for problem (4.20) let us consider the following problem: find σ_τ from the conditions

$$\begin{aligned} &AB_\tau\hat{\sigma}_\tau = z, \\ &\|\hat{\sigma}_\tau\|_*^2 = \min. \end{aligned} \tag{4.22}$$

Here B_τ is the orthogonal projector of the space $N(T)_*^{\perp}$ to the subspace $E_\tau \subset N(T)_*^{\perp}$. It is obvious that B_τ is the self-adjoint operator. Then from the general consideration of Section 1.4.2 the solution of problem (4.22) can be written by the following way

$$\hat{\sigma}_\tau = B_\tau A^*(AB_\tau A^*)^{-1}z. \tag{4.23}$$

But it is clear $\hat{\sigma}_\tau \in E_\tau$ .Thus, the minimization of the functional $\|\hat{\sigma}_\tau\|_*^2$ can be done only in the subspace $N(T)_*^{\perp}$. Since the solutions of (4.22) and (4.20) are unique, $\hat{\sigma}_\tau = \sigma_\tau$. Finally, the solution of problem (4.20) for the interpolating spline in the subspace E_τ can be represented in the form

$$\sigma_\tau = B_\tau A^*(AB_\tau A^*)^{-1}z. \tag{4.24}$$

Theorem 4.1. Let the operator A have the finite dimensional range and $E_\tau \xrightarrow{W} N(T)_*^{\perp}$. Then $\|\sigma_\tau - \sigma\|_* \to 0$ when $\tau \to 0$.

Proof. Since the interpolating spline σ and interpolating spline σ_τ in the subspace E_τ justify to the interpolation condition $A\sigma = A\sigma_\tau = z$ then the orthogonal property takes place (see Chapter 1)

$$\|\sigma_\tau - \sigma\|_*^2 = \|\sigma_\tau\|_*^2 - \|\sigma\|_*^2. \tag{4.25}$$

In details

$$\|\sigma\|_*^2 = (A^*(AA^*)^{-1}z, A^*(AA^*)^{-1}z)_* = ((AA^*)^{-1}z, z)_Z z, \tag{4.26}$$

$$\begin{aligned}\|\sigma_\tau\|_*^2 &= (B_\tau A^*(AB_\tau A^*)^{-1}z, B_\tau A^*(AB_\tau A^*)^{-1}z)\\ &= ((AB_\tau A^*)^{-1}z, z)_Z,\end{aligned} \tag{4.27}$$

because B_τ is orthogonal (self-adjoint) projector. The mapping A can be described by the finite number of linear bounded functionals. In other words, the elements $k_i \in N(T)_*^\perp, i = 1, 2, ..., n,\ n = \dim R(A)$ do exist and are linear independent, such that

$$Ax = [(k_1, x)_*, \cdots, (k_n, x)_*].$$

The space $Z = E_n$ is n-dimensional Euclidian space. In this situation the adjoint operator $A^* : E_n \to N(T)_*^\perp$ acts by the formula

$$A^*\lambda = \sum_{i=1}^{n} \lambda_i k_i \tag{4.28}$$

because

$$\begin{aligned}\forall x \in N(T)_*^\perp \quad \forall \lambda \in E_n \quad (Ax, \lambda)_{E_n} &= \sum_{i=1}^{n} \lambda_i (k_i, x)_*\\ &= (x, \sum_{i=1}^{n} \lambda_i k_i)_* = (x, A^*\lambda)_*.\end{aligned}$$

On this account the operator $AA^* : E_n \to E_n$ is $n \times n$-Gram matrix of the elements

$$\alpha_{i,j} = (k_i, k_j)_*, \quad i, j = \overline{1, n}. \tag{4.29}$$

For the same reason the operator $AB_\tau A^* = (AB_\tau) \times (AB_\tau)^*$ is also the Gram matrix of the elements

$$\alpha_{i,j}^\tau = (B_\tau k_i, B_\tau k_j)_* = (B_\tau k_i, k_j)_*, \quad i, j = \overline{1, n}. \tag{4.30}$$

Since $E_\tau \stackrel{W}{\to} N(T)_*^\perp$, it means that

$$\forall x, y \in N(T)_*^\perp \quad (B_\tau x, y) \to (x, y)_*, \quad \tau \to 0, \tag{4.31}$$

and we have the convergence of the elements of the matrix $AB_\tau A^*$,

$$\alpha_{i,j}^\tau \to \alpha_{ij}, \quad i, j = \overline{1, n}, \quad \tau \to 0. \tag{4.32}$$

The solution of the linear algebraic system continiously depends on the elements of the matrix. Therefore

$$((AB_\tau A^*)^{-1}z, z)_{E_n} \to ((AA^*)^{-1}z, z)_{E_n}$$

or taking into account (4.25) to (4.27) we obtain $\|\sigma_\tau - \sigma\|_* \to 0$. □

4.1.3. Error Estimates

In this Section we obtain an error estimate in the general form for the interpolating spline in the subspaces E_τ in the situation when the operator A is not fixed, but depends on the mesh parameter h, and τ tends to zero, simultaniously.

Let $A_h : N(T)_*^\perp \to Z_h$ depend on the small parameter $h > 0$ and (A_h, T) form a spline-pair for $h \le h_0$. Let $\varphi_*^\perp$ be some element from $N(T)_*^\perp$. Denote by

$$\sigma^h = S^h \varphi_*^\perp \overset{\mathrm{df}}{=} A_h^*(A_h A_h^*)^{-1} A_h \varphi_*^\perp \tag{4.33}$$

its spline-interpolation and let

$$\sigma_\tau^h = S_\tau^h \varphi_*^\perp \overset{\mathrm{df}}{=} B_\tau A_h^*(A_h B_\tau A_h^*)^{-1} A_h \varphi_*^\perp \tag{4.34}$$

be its spline-interpolation in the subspace E_τ. We assume that the interpolation condition $A_h x_\tau = A_h \varphi_*^\perp$ is non-contradictory in the subspace E_τ, when $\tau \le \tau(h)$. Denote by $Sp(h)$ the space of the interpolating splines corresponding to the operator A_h, $Sp(h) = S^h N(T)_*^\perp$.

Let $D_\tau^h : Sp(h) \to N(T)_*^\perp$ be a restriction of the operator $I - B_\tau$ on the spline-space $Sp(h)$.

Lemma. If a family of operators $D_{\tau(h)}^h$ is uniformly bounded by the constant $C < 1$ independent of $h \le h_0$, then the corresponding splines $\sigma_{\tau(h)}^h$ are bounded in the X-norm by the constant which is independent of h.

Proof. Denote

$$M_\tau^h = A_h^*(A_h B_\tau A_h^*)^{-1} A_h. \tag{4.35}$$

Then

$$M_\tau^h - S^h = A_h^* \left[(A_h B_\tau A_h^*)^{-1} - (A_h A_h^*)^{-1}\right] A_h.$$

By the identity $C^{-1} - D^{-1} = D^{-1}(D - C)C^{-1}$ we obtain

$$M_\tau^h - S^h = A_h^*(A_h A_h^*)^{-1} A_h (I - B_\tau) A_h^* (A_h B_\tau A_h^*)^{-1} A_h = S^h (I - B_\tau) M_\tau^h.$$

Hence,

$$M_{\tau(h)}^h \varphi_*^\perp = S^h \varphi_*^\perp + S^h (I - B_{\tau(h)}) M_{\tau(h)}^h \varphi_*^\perp.$$

Taking into account $M_{\tau(h)}^h \varphi_*^\perp \in Sp(h)$, we have

$$\|M_{\tau(h)}^h \varphi_*^\perp\|_* \le \|S^h \varphi_*^\perp\|_* + \|S^h\| \cdot C \cdot \|M_{\tau(h)}^h \varphi_*^\perp\|_*.$$

Since $\|S^h\| = 1$, $\|S^h \varphi_*^\perp\|_* \le \|\varphi_*^\perp\|_*$, we finally obtain

$$\|\sigma_{\tau(h)}^h\|_* = \|B_{\tau(h)} M_{\tau(h)}^h \varphi_*^\perp\|_* \le \|M_{\tau(h)}^h \varphi_*^\perp\|_* \le (1 - C)^{-1} \|\varphi_*^\perp\|_*$$

and Lemma is proved. □

Theorem 4.2. Let V be a semi-normed space with the following embedding condition

$$\forall x \in X \quad \|x\|_V \leq C_1 \|x\|_X, \tag{4.36}$$

and the following error estimates are valid

$$\forall \varphi_* \in X \quad \|\varphi_* - S^h \varphi_*\|_V \leq C_2 \cdot g_1(h) \cdot \|T\varphi_*\|_Y, \tag{4.37}$$

$$\|\varphi_* - B_\tau \varphi_*\|_V \leq C_3 \cdot g_2(\tau) \cdot \|T\varphi_*\|_Y, \tag{4.38}$$

Then under the lemma constraints the error estimate takes place

$$\|\varphi_* - S^h_{\tau(h)}\varphi_*\|_* \leq [C_4 \cdot g_1(h) + C_5 \cdot g_2(\tau(h))] \cdot \|T\varphi_*\|_Y, \tag{4.39}$$

Here $C_1 \div C_5$ are any constants independent of h.

Proof. It is clear from error estimates (4.37), (4.38) that the elements φ_* from the null space $N(T)$ are reproduced exactly in the spline-interpolation and projection to the subspace E_τ. Hence, the proof can be done only for $\varphi_* = \varphi_*^\perp \in N(T)_*^\perp$. We have

$$\begin{aligned}\|\varphi_* - S^h_{\tau(h)}\varphi_*\|_V &\leq \|\varphi_* - S^h\varphi_*\|_V + \|(I - B_{\tau(h)})S_h\varphi_*\|_V \\ &+ \|B_{\tau(h)}(S^h - M^h_{\tau(h)})\varphi_*\|_V \leq [C_2 g_1(h) + C_3 g_2(\tau(h))] \cdot \|T\varphi_*\|_Y \\ &+ \|B_{\tau(h)}S^h(I - B_{\tau(h)})M^h_{\tau(h)}\varphi_*\|_V.\end{aligned}$$

The latter term can be estimated in the following way

$$\begin{aligned}\|B_{\tau(h)}S^h(I - B_{\tau(h)})M^h_{\tau(h)}\varphi_*\|_V &\leq \|B_{\tau(h)}\|_{N(T)_*^\perp \to V} \cdot \|S^h\|_{N(T)_*^\perp \to V} \\ \times \|(I - B_{\tau(h)})M^h_{\tau(h)}\varphi_*\|_V &\leq C_1^2(1-C)^{-1} g_2(\tau(h)) \cdot \|T\varphi_*\|_Y.\end{aligned}$$

In this estimation we use the following simple fact: Since $B_{\tau(h)}$ and S^h are orthoprojectors we have

$$\|B_{\tau(h)}\|_{N(T)_*^\perp \to N(T)_*^\perp} = \|S^h\|_{N(T)_*^\perp \to N(T)_*^\perp} = 1.$$

However, from embedding condition (4.36) we obtain

$$\|S^h\|_{N(T)_*^\perp \to V} = \sup_{x \neq \Theta_X} \frac{\|S^h x\|}{\|x\|_*} V = C_1.$$

The same fact takes place for $B_{\tau(h)}$ and Theorem is proved. □

Remark. We are able to replace the requirement of Lemma to another stronger condition. Let E_1, E_2 be two subspaces in the space X, and B_1, B_2 be the corresponding projectors to E_1, E_2 . Then the angle $\Theta(E_1, E_2)$ between E_1 and E_2 is

$$\Theta(E_1, E_2) = \max\{\|(I - B_1)|_{E_2}\|, \quad \|(I - B_2)|_{E_1}\|\}, \tag{4.40}$$

where $(I - B_1)|_{E_2}$ and $(I - B_2)|_{E_1}$ mean the restrictions of the operators $I - B_1, I - B_2$ on the subspaces E_2, E_1 . Thus, the requirement of Lemma can be replaced by

$$\Theta(Sp(h), B_{\tau(h)}Sp(h)) \leq \Theta_0 < 1 \tag{4.41}$$

with the constant Θ_0 independent of h. In this situation if $Sp(h)$ is finite-dimensional, then $B_{\tau(h)}Sp(h)$ has the same dimension.

4.2. Smoothing Splines in the Subspaces

4.2.1. Definition, Algebraic System

Let X, Y and Z be some Hilbert space, and operators $T : X \to Y$, $A : X \to Z$ form a spline-pair (T, A). Consider $\alpha > 0, z \in Z$ and the corresponding smoothing spline $\sigma_\alpha \in X$,

$$\sigma_\alpha = \arg\min_{x \in X} \alpha\|Tx\|_Y^2 + \|Ax - z\|_Z^2. \tag{4.42}$$

The *smoothing spline σ_α^k in the subspace $E_K \subset X$* is the solution of the problem

$$\sigma_\alpha^k = \arg\min_{x_k \in E_k} \alpha\|Tx\|_Y^2 + \|Ax - z\|_Z^2. \tag{4.43}$$

If E_k is the finite-dimensional subspace, then σ_α^k does always exist and is unique. Let $\omega_1, \omega_2, \cdots, \omega_{n(k)}$ form the basis of E_k . Then

$$\sigma_\alpha^k = \sum_{i=0}^{n(k)} \sigma_{\alpha,i}^k \omega_i$$

and the variational functional of problem (4.43) can be written as the function of the coefficients $\sigma_{\alpha,i}^k$,

$$\begin{aligned}\Phi_\alpha(\sigma_\alpha^k) &= \alpha\|T\sigma_\alpha^k\|_Y^2 + \|A\sigma_\alpha^k - z\|_Z^2 \\ &= \sum_{i,j=1}^{n(k)} \sigma_{\alpha,i}^k \sigma_{\alpha,j}^k \left[\alpha(T\omega_i, T\omega_j)_Y + (A\omega_i, A\omega_j)_Z\right] \\ &- 2\sum_{i=1}^{n(k)} \sigma_{\alpha,i}^k (A\omega_i, z)_Z + \|z\|_Z^2.\end{aligned}$$

Using the notations of Section 4.1.1, we have

$$\Phi_\alpha(\sigma_\alpha^k) = \alpha(\bar{T}\bar{\sigma}, \bar{\sigma}) + (\bar{A}\bar{\sigma}, \bar{\sigma}) - 2(\bar{f}, \bar{\sigma}) + \|z\|_Z^2.$$

The minimization of this quadratic functional with respect to the variables $\sigma^k_{\alpha,i}$ results in the following algebraic system

$$(\alpha\bar{T} + \bar{A})\bar{\sigma} = \bar{f} \tag{4.44}$$

with the symmetric and positive defined matrix.

4.2.2. Convergence

Theorem 4.3. If $E_k \overset{W}{\to} X$ and for every k $E_{k+1} \supset E_k$, then $\sigma^k_\alpha \overset{W}{\to} \sigma_\alpha$ when $k \to \infty$.

Proof. At first we show that the sequence $\{\sigma^k_\alpha\}$ is bounded in the X-norm with respect to the index k. We represent σ^k_α in the form

$$\sigma^k_\alpha = \sigma^{k,1}_\alpha + \sigma^{k,2}_\alpha,$$

where $\sigma^{k,1}_\alpha \in N(T)^\perp$, $\sigma^{k,2} \in N(T)$. Every subspace E_k contains zero point Θ_X. As σ^k_α is the point of minimum of the functional

$$\Phi_\alpha(x) = \alpha\|Tx\|^2_Y + \|Ax - z\|^2_Z$$

in the subspace E_k , we have

$$\frac{1}{\alpha}\Phi_\alpha(\sigma^k_\alpha) = \|T\sigma^k_\alpha\|^2_Y + \frac{1}{\alpha}\|A\sigma^k_\alpha - z\|^2_Z \le \frac{1}{\alpha}\Phi(\Theta_X) = \frac{1}{\alpha}\|z\|^2_Z. \tag{4.45}$$

Thus, the sequence $\|T\sigma^k_\alpha\|^2_Y$ is bounded with respect to k. The sequence $\|A\sigma_\alpha\|_Z$ is also bounded because from (4.45) we have

$$\|A\sigma^k_\alpha\|_Z - \|z\|_Z \le \|A\sigma^k_\alpha - z\|^2_Z \le \|z\|_Z, \quad \|A\sigma^k_\alpha\|_Z \le 2\|z\|_Z.$$

By the norm equivalence theorem we have

$$\|\sigma^k_\alpha\|_X \le C(\|T\sigma^k_\alpha\|^2 + \|A\sigma^k_\alpha\|^2)^{1/2} \le (4 + 1/\alpha)^{1/2}\|z\|_Z.$$

Let us separate from the bounded sequence σ^k_α the subsequence $\sigma^{k'}_\alpha$ which weakly converges to any element $\sigma_* \in X$. By the orthogonal property of the smoothing splines we can write

$$a(T\sigma^{k'}_\alpha, T\omega_l)_Y + (A\sigma^{k'}_\alpha - z, A\omega_l - z)_Z = -(A\sigma^{k'}_\alpha - z, z)_Z. \tag{4.46}$$

Here $\omega_l \in X$ is the sequence such that $\omega_l \in E_l$ and $k' \ge l$. If $k' \to \infty$ then we obtain

$$\alpha(T\sigma_*, T\omega_l)_Y + (A\sigma_* - z, A\omega_l - z)_Z = -(A\sigma_* - z, z)_Z. \tag{4.47}$$

We can choose $\omega_l \overset{W}{\to} \sigma_\alpha$ because $E_k \overset{W}{\to} X$. Then from (4.47) we obtain

$$\alpha(T\sigma_*, T\sigma_\alpha)_Y + (A\sigma_* - z, A\sigma_\alpha - z)_Z = -(A\sigma_* - z, z)_Z. \tag{4.48}$$

On the other hand the left-hand side of (4.48) is equal to $-(A\sigma_\alpha - z, z)$ from the orthogonal property. If we use now in (4.47) the sequence $\omega_l \xrightarrow{W} \sigma_*$ we obtain

$$\alpha\|T\sigma_*\|_Y^2 + \|A\sigma_* - z\|_Z^2 = -(A\sigma_* - z, z)_Z. \tag{4.49}$$

Finally, $\Phi_\alpha(\sigma_*) = \Phi_\alpha(\sigma_\alpha)$ and by uniqueness of the solution for the initial spline-problem we have $\sigma_* = \sigma_\alpha$. So, every weak limit point of the sequence $\{\sigma_\alpha^k\}$ is σ_α, i.e. $\sigma_\alpha^k \xrightarrow{W} \sigma_\alpha$ when $k \to \infty$. □

Corollary. If the operator $A : X \dashrightarrow Z$ has the finite-dimensional range, then under the theorem conditions we have

$$\|\sigma_\alpha^k - \sigma_\alpha\|_X \to 0, \quad k \to \infty.$$

Proof. Really, the value

$$\Phi_\alpha(\sigma_\alpha^k) = \alpha\|T\sigma_\alpha^k\|_Y^2 + \|A\sigma_\alpha^k - z\|_Z^2 = -(A\sigma_\alpha^k - z, z)_Z$$

tends by the weak convergence to

$$-(A\sigma_\alpha - z, z)_Z = \alpha\|T\sigma_\alpha\|_Y^2 + \|A\sigma_\alpha - z\|_Z^2.$$

Since $R(A)$ is finite-dimensional, $\|A\sigma_\alpha^k - A\sigma_\alpha\|_Z \to 0$ and $\|A\sigma_\alpha^k - z\|_Z \to \|A\sigma_\alpha - z\|_Z$. Therefore $\|T\sigma_\alpha^k\|_Y \to \|T\sigma_\alpha\|_Y$, and by the well-known theorem $\|T\sigma_\alpha^k - T\sigma_\alpha\|_Y \to 0$. At last, by the norm equivalence theorem we obtain

$$\|\sigma_\alpha^k - \sigma_\alpha\|_X \le C\left[\|T(\sigma_\alpha^k - \sigma_\alpha)\|_Y^2 + \|A(\sigma_\alpha^k - \sigma_\alpha)\|_Z^2\right]^2 \to 0$$

and the corollary is proved. □

4.2.3. Error Estimates

As we have already shown in Chapter 1 (Section 1.4.3.), the non-trivial smoothing process goes in the subspace $N(T)_*^\perp$ because it is impossible to smooth the elements from the null-space $N(T)$. Then the initial smoothing problem

$$\sigma_{\alpha,h} = \arg\min_{x\in X} \alpha\|Tx\|_Y^2 + \|A_h x - z\|_Z^2$$

can be reformulated in the form

$$\sigma_{\alpha,h} = \arg\min_{x\in N(T)_*^\perp} \alpha\|x\|_*^2 + \|A_h x - z\|_Z^2 \tag{4.50}$$

where the subspace $N(T)_*^\perp$ is connected with the maximal spline-pair $(T, \tilde{A})$ with respect to the initial spline-pair (T, A_h), and

$$N(T)_*^\perp = N(\tilde{A}),$$

with the special scalar product

$$(x_1, x_2)_* = (Tx_1, Tx_2)_Y$$

and the corresponding Hilbert norm $\|x\|_* = (x,x)_*^{1/2}$. As we have already known from Chapter 1, the resolvent operator for problem (4.50) can be written in the form

$$\sigma_{\alpha,h} = S_{\alpha,h}\varphi_*^\perp = A_h^*(\alpha I + A_h A_h^*)^{-1} A_h \varphi_*^\perp, \tag{4.51}$$

where $A_h\varphi_*^\perp = z$, $\varphi_*^\perp \in N(T)_*^\perp$.

Let E_τ be the finite dimensional subspace in $N(T)_*^\perp$, and $B_\tau : N(T)_*^\perp \to E_\tau$ be the corresponding orthogonal projector onto E_τ . If we consider the smoothing problem

$$\sigma_* = \arg \min_{x \in N(T)_*^\perp} \alpha\|x\|_*^2 + \|AB_\tau x - z\|_Z^2,$$

its solution can be obtained by the formula

$$\sigma_* = B_\tau A_h^*(\alpha I + A_h B_\tau A_h^*)^{-1} A_h \varphi_*^\perp,$$

and σ_* belongs to E_τ . Therefore σ_* is the solution of the problem

$$\sigma_* = \arg \min_{x_\tau \in E_\tau} \alpha\|x_\tau\|_*^2 + \|Ax_\tau - z\|_Z^2$$

or in other words σ_* is equal to the smoothing spline $\sigma_{\alpha,h}^\tau$ in the subspace E_τ and can be represented in the form

$$\sigma_{\alpha,h}^\tau = S_{\alpha,h}^\tau \varphi_*^\perp = B_\tau A_h^*(\alpha I + A_h B_\tau A_h^*)^{-1} A_h \varphi_*^\perp. \tag{4.52}$$

Theorem 4.4. Let V be the semi-normed space with the embedding condition

$$\forall x \in X \quad \|x\|_V \le C_1 \|x\|_X \tag{4.53}$$

and the following error estimate be valid

$$\forall \varphi_* \in X \quad \|\varphi_* - B_\tau \varphi_*\| \le C_3 \cdot g_2(\tau) \|T\varphi_*\|_Y. \tag{4.54}$$

If the restriction of the operator $(I - B_{\tau(h)})$ on the space $Sp(h)$ of the interpolating splines is bounded by the constant $C < 1$ independent of h, then the error estimate takes place

$$\|\sigma_\alpha - \sigma_\alpha^{\tau(h)}\|_V \le C_* g_2(\tau(h)) \|T\varphi_*\|, \tag{4.55}$$

where $C_* = C_3(1 + C_1^2(1 - C)^{-1})$.

Proof. It is necessary to make an estimation only for $\varphi_* = \varphi_*^\perp \in N(T)_*^\perp$. Let us prove the identity

$$S_{\alpha,h}^\tau - B_\tau S_{\alpha,h} = B_\tau S_{\alpha,h}(I - B_\tau) M_{\alpha,h}^\tau,$$

where $M_{\alpha,h}^\tau = A_h^*(\alpha I + A_h A_h^*)^{-1} A_h$. Really

$$B_\tau A_h^*(\alpha I + A_h B_\tau A_h^*)^{-1} A_h - B_\tau A_h^*(\alpha I + A_h A_h^*)^{-1} A_h$$
$$= B_\tau A_h^*(\alpha I + A_h A_h^*)^{-1} A_h (I - B_\tau) A_h^* (\alpha I + A_h B_\tau A_h^*)^{-1} A_h.$$

Taking into account the obvious inequality (see Lemma preceding Theorem 4.2)

$$\|M_{\alpha,h}^{\tau(h)} \varphi_*^\perp\|_* = \|A_h^*(\alpha I + A_h B_\tau A_h^*)^{-1} A_h \varphi_*^\perp\|_*$$
$$\le \|A_h^*(\alpha I + A_h B_\tau A_h^*)^{-1} A_h \varphi_*^\perp\|_* \le (1-C)^{-1} \|\varphi_*^\perp\|_*$$

we obtain

$$\|\sigma_{\alpha,h} - \sigma_{\alpha,h}^{\tau(h)}\|_V \le \|\sigma_\alpha - B_{\tau(h)}\sigma_{\alpha,h}\|_V + \|B_{\tau(h)}\sigma_{\alpha,h} - \sigma_{\alpha,h}^{\tau(h)}\|_V$$
$$= \|\sigma_\alpha - B_{\tau(h)}\sigma_{\alpha,h}\|_V + \|B_{\tau(h)} S_{\alpha,h}(I - B_{\tau(h)}) M_{\alpha,h}^{\tau(h)} \varphi_*^\perp\|_V$$
$$\le C_3 \cdot g_2(\tau(h)) \cdot \|\varphi_*^\perp\|_* + C_1^2 C_3 (1-C)^{-1} \cdot \|\varphi_*^\perp\| \cdot g_2(\tau(h))$$

and Theorem is proved. □

Remark. In inequality (4.55) the constant C_* is independent of α. If $\alpha \to 0$, then the smoothing splines $\sigma_{\alpha,h}$ and $\sigma_{\alpha,h}^{\tau(h)}$ go to the interpolating splines σ_h and $\sigma_h^{\tau(h)}$ correspondingly, and we obtain the error estimate

$$\|\sigma_h - \sigma_h^{\tau(h)}\|_V \le C_* g_2(\tau(h)) \cdot \|T\varphi_*\|_Y. \tag{4.56}$$

There is the other way to prove the error estimate (4.39) in Theorem 4.2.

4.2.4. On Estimation of the Angle Between Subspaces

The main property which provides obtaining the error estimates both for interpolating and smoothing splines in the subspaces: the restriction of the operator $I - B_\tau$ to the spline-space $Sp(h)$ is uniformly bounded in the $X \to X$-norm by the constant $C < 1$. What does it mean?

Let a subspace E_τ have the basis $\omega_1, \omega_2, \cdots, \omega_{n(\tau)}$ and B_τ be the orthogonal projector from $N(T)_*^\perp$ onto E_τ. It means that $B_\tau \varphi = \sum_{i=1}^{n(\tau)} \mu_i^* \omega_i$ is the best approximation of the element φ,

$$\|\varphi - \sum_{i=1}^{n(\tau)} \mu_i^* \omega_i\|_*^2 = \min_{\mu_1,\cdots,\mu_{n(\tau)}} \|\varphi - \sum_{i=1}^{n(\tau)} \mu_i^* \omega_i\|_*^2. \tag{4.57}$$

Since (4.57) is the least square problem, the best coefficients can be found from the following linear algebraic system

$$\Omega_\tau \bar\mu = F, \quad \bar\mu = [\mu_1, \dots, \mu_{n(\tau)}]^T, \tag{4.58}$$

where $\Omega_\tau = \{(\omega_i, \omega_j)_*\}_{i,j=1}^{n(\tau)}$ is the Gram matrix and $F = [(\varphi, \omega_1)_*, \cdots, (\varphi, \omega_{n(\tau)})_*]^T$. If we denote by $B : N(T)_*^\perp \to E_{n(\tau)}$ the operator

$$B\varphi = [(\varphi, \omega_1)_*, ..., (\varphi, \omega_{n(\tau)})_*]^T, \tag{4.59}$$

system (4.58) can be written in the form

$$(BB^*)\mu = B\varphi.$$

It is easy to see that

$$B^*\bar{\mu} = \sum_{i=1}^{n(\tau)} \mu_i^* \omega_i.$$

Finally, the operator B_τ can be represented in the form

$$B_\tau \varphi = B^*(BB^*)^{-1} B\varphi. \tag{4.60}$$

The problem of estimating the operator $(I - B_\tau)|_{Sp(h)}$ is: how to find the constant $C < 1$ such that

$$\forall \varphi \in N(T)_*^\perp \quad \|S_h\varphi - B_\tau S_h \varphi\|_* \le C\|S_h\varphi\|_*, \tag{4.61}$$

where $S_h = A_h(A_h A_h^*)^{-1} A_h$ is the operator of the spline-interpolation. So we have

$$\|S_h\varphi - B_\tau S_h\varphi\|_*^2 = (S_h\varphi, S_h\varphi)_* - 2(S_h\varphi, B_\tau S_h\varphi)_*$$
$$+ (B_\tau S_h\varphi, B_\tau S_h\varphi)_* = (S_h\varphi, S_h\varphi)_* - (B_\tau S_h\varphi, S_h\varphi)_*,$$

or, in other words, condition (4.61) is equivalent to

$$\forall \varphi \in N(T)_*^\perp \quad (S_h B_\tau S_h \varphi, \varphi)_* \le (1 - C^2)(S_h\varphi, \varphi)_*. \tag{4.62}$$

To simplify the situation we assume that the operator A_h has the finite-dimensional range, i.e. the linear independent elements $k_1, ..., k_{n(h)}$ do exist such that

$$A_h \varphi = [(k_1, \varphi)_*, (k_2, \varphi)_*, \dots, (k_{n(h)}, \varphi)_*]^T. \tag{4.63}$$

Let us introduce $n(h) \times n(h)$-Gram matrix

$$K_h = \{(k_i, k_j)_*\}_{i,j=1}^{n(h)} \tag{4.64}$$

and the rectangular $n(h) \times m(\tau)$-mixed Gram matrix

$$M_{h,\tau} = \{(k_i, \omega_j)_*\}_{i=1,j=1}^{n(h),m(\tau)}. \tag{4.65}$$

As $S_h\varphi = \sum_{i=1}^{n(h)} \lambda_i k_i$, we write

$$\|S_h\varphi - B_\tau S_h\varphi\|_*^2 = (K_h\bar{\lambda}, \bar{\lambda})_{n(h)} - (M_{h,\tau}\Omega_\tau^{-1} M_{h,\tau}^* \bar{\lambda}, \bar{\lambda})_{n(h)},$$

where $\bar{\lambda} = [\lambda_1, \dots, \lambda_{n(\tau)}]^T$, $(,)_{n(h)}$ is the usual scalar product in $n(h)$-dimensional vector space. Thus, in the matrix form inequality (4.62) can be replaced by

$$M_{h,\tau}\Omega_\tau^{-1}M_{h,\tau}^* \geq (1-C^2)K_h. \tag{4.66}$$

Since the matrices $M_{h,\tau}\Omega_\tau^{-1}M_{h,\tau}^*$ and K_h are self-adjoint and K_h is positive defined, the best constant dependent on h and τ in (4.66) is the minimal eigenvalue in the following generalized eigenvalue problem

$$M_{h,\tau}\Omega_\tau^{-1}M_{h,\tau}^* x = \lambda K_h x \tag{4.67}$$

and

$$\lambda_{\min}^{h,\tau} = \inf_{x \neq 0} \frac{(\Omega_\tau^{-1}M_{h,\tau}^* x, M_{h,\tau}^* x)_{m(\tau)}}{(K_h x, x)_{n(h)}}, \tag{4.68}$$

or, in other words,

$$1 - c^2(h,\tau) = \lambda_{\min}^{h,\tau}(K_h^{-1/2}M_{h,\tau}\Omega_\tau^{-1}M_{\tau,h}^* K_h^{-1/2}) \geq 1 - c^2, \tag{4.69}$$

where $\lambda_{\min}^{h,\tau}(D)$ means the minimal eigenvalue of the matrix D, and $c(h,\tau)$ is exactly the norm of the restriction of the operator $I - B_\tau$ on the spline-space $Sp(h)$,

$$c(h,\tau) = (1 - \lambda_{\min}^{h,\tau})^{1/2}. \tag{4.70}$$

It is clear that $\lambda_{\min}^{h,\tau} \leq 1$ and the condition $c(h,\tau) \leq C < 1$ is equivalent to $\lambda_{\min}^{h,\tau} \geq \gamma_0 > 0$ with the constant γ_0 which is independent of h and τ.

The weaker condition $\lambda_{\min}^{h,\tau} > 0$ means that the interpolation condition $A\sigma_\tau^h = A\varphi_*$, $\varphi_* \in N(T)_*^\perp$ can be exactly realized in the subspace E_τ. Actually, the matrix

$$M_{h,\tau}\Omega_\tau^{-1}M_{\tau,h}^* = A_h B(BB^*)^{-1}BA_h^* = A_h B_\tau A_h^*$$

is positive definite iff $(A_h B_\tau A_h^*)^{-1}$ exists.

4.3. Finite Element D^m-Splines at the Scattered Meshes

Let $\Omega \subset R^n$ be any parallelepiped

$$\Omega = [a_1,b_1] \times [a_2,b_2] \times \ldots \times [a_n,b_n], \tag{4.71}$$

$a_i < b_i, i = 1,2,\ldots,n$, and ω_h be scattered mesh in Ω,

$$\omega_h = \{P_1^{(h)}, P_2^{(h)}, \ldots, P_{N(h)}^{(h)}\}. \tag{4.72}$$

We assume that ω_h forms h-net in Ω when $h \leq h_0$. Let $X = W_2^m(\Omega)$ be Sobolev space with the natural embedding condition $m > n/2$ to the space $C(\bar{\Omega})$. Then D^m-spline $\sigma^h(P)$, which interpolates in ω_h the function $\varphi_* \in W_2^m(\Omega)$ is the solution of variational problem

$$\sigma(P_i^{(h)}) = \varphi_*(P_i^{(h)}), \quad i = 1, 2, \ldots, N(h), \tag{4.73}$$

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha \sigma^h)^2 d\Omega = \min. \tag{4.74}$$

The coefficients $m!/\alpha!$ are introduced to energy functional (4.74) to provide the invariance of this functional with respect to rotations and shifts of the domain Ω. For the sufficiently small $h \le h_0$ the problem (4.73)-(4.74) is always uniquely solvable, and the following error estimate takes place

$$\|D^k(\varphi_* - \sigma^h)\|_{L_p} \le C h^{m-n/2+n/p-k} \|D^m \varphi_*\|_{L_2}. \tag{4.75}$$

Here $2 \le p \le \infty$, $k - n/p \le m - n/2$, except the $k = m - n/2$ & $p = \infty$.

Let us introduce in every interval $[a_j, b_j]$ the uniform mesh Δ_{τ_j} with the mesh size τ_j,

$$\Delta_{\tau_j} = \{x_j^{(k)} = a_j + k\tau_j,\ k = 0, ..., N,\ \tau_j = (b_j - a_j)/N_j\} \tag{4.76}$$

and consider the space $S^{k^j}(\Delta_{\tau_j})$ of the piecewise polynomial splines of the degree k_j with the defect 1 ($S^{k_j}(\Delta_{\tau_j}) \subset C^{k_j-1}[a_j, b_j]$). Suppose that $k_j \ge m$. In this situation $S^{k_j}(\Delta_{\tau_j}) \subset W_2^m[a_j, b_j]$. Let $\Delta_\tau = \Delta_{\tau_1} \times \Delta_{\tau_2} \times \ldots \times \Delta_{\tau_n}$ and $S^k(\Delta_\tau)$ be a tensor product of the spaces $S^{k_j}(\Delta_{\tau_j})$, $j = 1, 2, ..., n$. It is clear that every element σ_τ from $S^k(\Delta_\tau)$ belongs to the space $W_2^m(\Omega)$. The interpolating spline σ_τ^h in the subspace $S^k(\Delta_\tau)$ is element from this subspace which provides the interpolation conditions at the scattered mesh ω_h,

$$\sigma_\tau^h(P_i^{(h)}) = \varphi_*(P_i^{(h)}), \quad i = 1, 2, \ldots, N(h), \tag{4.77}$$

and minimizes the energy functional,

$$\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega (D^\alpha \sigma_\tau^h)^2 d\Omega = \min_{S^k(\Delta_\tau)}. \tag{4.78}$$

If interpolation conditions (4.77) are non-contradictory in the subspace $S^k(\Delta_\tau)$, then the solution of this problem does exist and is always unique for sufficienly small $h \le h_0$.

According to the general theory (De Boor 1978) every one-dimensional spline $\sigma_{\tau_j} \in S^{k_j}(\Delta_{\tau_j})$ can be represented with the basic B-splines,

$$\sigma_{\tau_j}^h(x_j) = \sum_{l_j=1}^{N_j+k_j} C_{l_j} B_{l_j}^{k_j}(x_j). \tag{4.79}$$

Every B-spline is the local piecewise polynomial function concentrated at k_j+1 mesh intervals (or less near end points a_j and b_j). Thus, the basis of the tensor product $S^k(\Delta_\tau)$ consists of the local functions

$$\omega_l^k(P) = \omega_{l_1 l_2 \ldots l_n}(x_1, x_2, ..., x_n) = \prod_{j=1}^n B_{l_j}^{k_j}(x_j), \tag{4.80}$$

and $\dim S^k(\Delta_\tau) = \prod_{j=1}^{n}(N_j + K_j)$. The support of the function $\omega_l^k(P)$ is any parallelepiped Ω_l^k. To find the finite-dimensional analog $\sigma_\tau^h \in S^k(\Delta_\tau)$ we need to solve system (4.12) (or system (4.44) for the smoothing spline). The matrices $\bar{T}$ and $\bar{A}$ arising here are fairly sparse because the elements

$$t_{l,s} = (D^m\omega_l^k, D^m\omega_s^k)_{L_2} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\Omega_l^k \cap \Omega_s^k} D^\alpha\omega_l^k \cdot D^\alpha\omega_s^k d\Omega, \tag{4.81}$$

$$a_{l,s} = (A_h\omega_l^k, A_h\omega_s^k)_{E_{N(h)}} = \sum_{P\in\omega_h\cap\Omega_l^k\cap\Omega_s^k} \omega_l^k(P)\cdot\omega_s^k(P) \tag{4.82}$$

of these matrices are not zeros if the intersection of two supports Ω_l^k and Ω_s^k is not empty. The structure of non-zero elements is the same for both matrices. The problems of fast computations for the matrices $\bar{T}$ and $\bar{A}$ including the special decomposition and fast multiplication on the vector to organize the iterative process for the solution of the corresponding linear algebraic system will be discussed in detail in Section 5.2.

An application of B-splines as a finite element method for the data interpolation at the scattered meshes is not only possibility. Many other finite element constructions can be applied here. But the main property to provide the convergence of the finite element spline to the exact function is $S^k(\Delta_\tau) \to W_2^m(\Omega)$, when $\tau \to 0$. For the spline-spaces on the rectangular grids this fact is well-known.

And the last but important question is: how to provide the preservation of error estimates (4.75) for the finite element analogs of splines? What is the connection between the condensation laws of the scattered mesh w_h and the grid Δ_τ?

It is easy to show (Vasilenko 1976) that the analytical D^m-spline σ^h (solution of problem (4.73), (4.74)) belongs to the Sobolev space $W_2^{m+\beta}(\Omega)$ for $0 \le \beta < m - n/2$. Let B be the orthoprojector of the space $W_2^m(\Omega)$ onto the B-spline subspace $S^k(\Delta_\tau)$ connected with the special scalar product, and the following approximative property takes place

$$\forall u \in W_2^{m+\beta}(\Omega) \quad \|(I - B_\tau)u\|_{W_2^m} \le C_1\tau^\beta\|u\|_{W_2^{m+\beta}}. \tag{4.83}$$

For D^m-splines σ^h it was shown (Matveev 1991) that

$$\|\sigma^h\|_{W_2^{m+\beta}} \le \frac{C_2}{h_{\min}^\beta}\|\sigma^h\|_{W_2^{m+\beta}}, \tag{4.84}$$

where

$$h_{\min} = \min_{\substack{P,Q\in\omega_h \\ P\ne Q}} \|P - Q\|_2. \tag{4.85}$$

Let D_h^τ be the restriction of the operator $I - B_\tau$ on the space $Sp(h)$ of the interpolating D^m-splines on the mesh w_h. Then

$$\|D_h^\tau\| = \sup_{\sigma_h \in Sp(h)} \frac{\|(I - B_\tau)\sigma_h\|_{W_2^m}}{\|\sigma_h\|_{W_2^m}} \leq C_1 \cdot C_2 \cdot \left(\frac{\tau}{h_{\min}}\right)^\beta . \tag{4.86}$$

Finally, to provide error estimates (4.75) for the finite element spline σ_τ^h it is necessary to have

$$C_1 C_2 (\tau / h_{\min})^\beta \leq \Theta_0 < 1, \tag{4.87}$$

where the constant Θ_0 is independent of τ and h; in other words

$$\tau / h_{\min} \leq C_3 = (\Theta_0 / (C_1 C_2))^{1/\beta}. \tag{4.88}$$

If the scattered mesh w_h is quasi-uniform ($h_{\min} \geq C_4 h, C_4 = \text{const}$), then it means that the grid step τ is proportional to the scattered mesh parameter h.

4.4. Discontinuous Finite Element D^m-splines

Here we consider the problem of spline interpolation of a discontinuous function of two (or more) variables which is sufficiently smooth everywhere except for the separate lines (or surfaces), where this function has discontinuities of the first type (limits on both sides of discontinuity are finite). There are two aspects in this problem. The first is as follows: we know only values of the function on the scattered mesh and the first problem is localization of discontinuity lines, which probably have complicated geometry in the domain where the function is defined. It may be that the discontinuity line begins at the boundary of the domain and finishes also at the boundary (but possibly inside) or this line is a closed curve. How are various discontinuity lines situated with respect to one anothers? This problem have to be solved in some practical sense. And the second part of the problem is spline interpolation, when the positions of discontinuity lines are already given.

4.4.1. Discrete Localization of Discontinuities

Let A be a finite set of points in R^n and some metrics $\rho(P, Q)$ in R^n be given. We say that A is *ε-connected* set if for every point $P \in A$ and for every point $Q \in A$ the sequence $P_1, P_2, ..., P_N$ of points from A does exist such that

$$P_1 = P,\ P_N = Q,\ \rho(P_i, P_{i+1}) \leq \varepsilon, \quad i = 1, 2, ..., N-1. \tag{4.89}$$

Let A be ε-connected set and at every point of it some real valued function $f(P)$ be given. We say that two points P and Q from A are *(R, ε)-connected* if the sequence $P_1, P_2, ..., P_N$ of points from A does exist such that

$$P_1 = P,\ P_N = Q,\ \rho(P_i, P_{i+1}) \leq \varepsilon, \quad |f(P_i) - f(P_{i+1})| \leq R. \tag{4.90}$$

It is obvious that (R, ε)-connectiveness is the relation of equivalence in *ε-connected* set A and A is divided into classes of the equivalence $A_1, A_2, ..., A_S$.

Remark. If we have the values of discontinuous function on the scattered mesh, then only the user knows the "level of discontinuity" $R > 0$. If the distance between two measurement points P and Q is Smaller than the measurement step $\varepsilon > 0$ but the "jump" $|f(P) - f(Q)|$ is greater than $R > 0$, there is "discontinuity" here; in other situation the mesh function is already "continuous".

We say that two points $P \in A$ and $Q \in A$ are (R, ε)*-separated* if $\rho(P, Q) \leq \varepsilon$, but $|f(P) - f(Q)| > R$. Note that two points belonging to (R, ε)-connected subset may be (R, ε)-separated.

Let us consider (R, ε)-connected class A_i. We say that $P \in A_i$ is the *boundary point* of A_i (or $P \in \partial A_i$) if its ε-neibourhood $M_\varepsilon(P) = \{Q \in R^n : \rho(P, Q) \leq \varepsilon\}$ contains the point $Q \in A$ which is (R, ε)-separated from P. If the point $P \in \partial A_i$ contains in its neibourhood only the points from A_i, we say that P belongs to the *inner boundary* ∂A_i^{in} of the class A_i, in the other case P belong to the *external boundary* ∂A_i^{ex}. Thus, $\partial A_i = \partial A_i^{\text{in}} \cup \partial A_i^{ex}$.

Every point $P \in \partial A^i$ contains in its ε-neibourhood one or few points from A_i. The union of these additional points and the set A_i is the *closure* $\bar{A}_i$ of the class A_i . We call the set

$$\overline{\partial A_i} = \bar{A}_i \backslash (A_i \backslash \partial A_i)$$

full boundary of the class A_i. Finally the set

$$\overline{\partial A} = \bigcup_{i=1}^{S} \overline{\partial A_i}$$

we call the *full boundary of whole set A*.

If we consider now ε-connectiveness relation only at the full boundary $\overline{\partial A}$, then $\overline{\partial A}$ is divided into classes of the equivalence $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ (see Fig. 4.1., where we have 3 classes of equivalence and 4 components of the full boundary).

4.4.2. Accuracy of Localizations

Remember one of the possible definitions of Hausdorff's distance between two sets in a metric space. Let $\rho(P, Q)$ be some metrics in R^n and A be some set in R^n. For every $\varepsilon \geq 0$ we define ε-neibourhood A_ε of the set A by formula

$$A_\varepsilon = \bigcup_{P \in A} B_\varepsilon(P),$$

where $B_\varepsilon(P)$ is the ball of the radius ε with the center P. Let A and B be two sets in R^n. We introduce the values

$$\begin{aligned} \varepsilon(A, B) &= \inf_{\varepsilon \geq 0} \{\varepsilon : A_\varepsilon \supset B\}, \\ \varepsilon(B, A) &= \inf_{\varepsilon \geq 0} \{\varepsilon : B_\varepsilon \supset A\}. \end{aligned} \tag{4.91}$$

The essence of these values is very simple especially for the bounded sets A and B. For sufficiently large ε the neibourhood A_ε of A "absorbs" the set B,

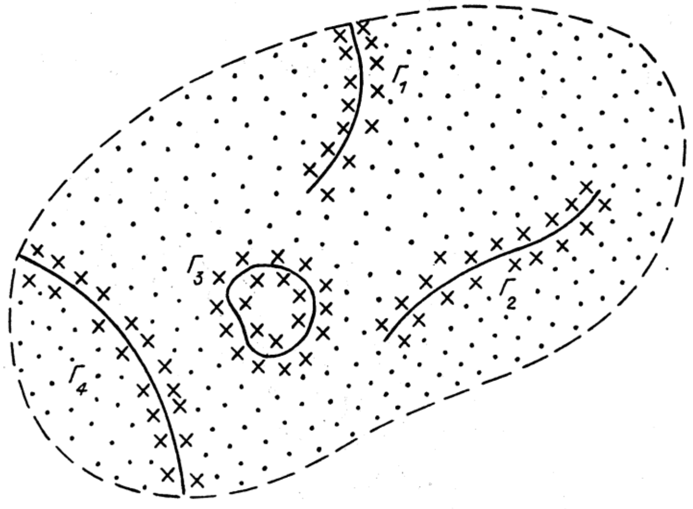

Fig. 4.1.

and $\varepsilon(A, B)$ is the minimal among $\varepsilon \geq 0$ with this property. So the Hausdorff's distance between A and B is the value

$$\rho(A, B) = \max\{\varepsilon(A, B), \varepsilon(B, A)\}. \tag{4.92}$$

It is clear that $\rho(A, A) = 0$, $\rho(A, B) = \rho(B, A)$. If a_ε is ε-net for A, then $\rho(A, a_\varepsilon) \leq \varepsilon$.

Let $f(x, y)$ be the function defined in the domain Ω on the plane and Γ be some curve, where $f(x, y)$ has the discontinuity of the 1-st type. Denote by Γ^R its "R-visable" part, i.e. the "jump" at every point of Γ^R is greater than $R > 0$. Let us assume two properties of Γ^R.

1. Some fixed neibourhood Γ^R_δ of Γ^R is included to the domain Ω, and there are no other curves of discontinuity in Γ^R_δ.

2. Γ^R is connected curve without self-intersections with the finite length and bounded curvature.

In this situation the following condition of ε-regularity can be ensured: for sufficiently small $\varepsilon > 0$ the finite cover of Γ^R with the balls $B_\varepsilon(P_1), ..., B_\varepsilon(P_{N(\varepsilon)})$ does exist such that the centers of ε-balls lie on Γ^R, and every ball is divided by Γ^R into two parts with the squares of the order ε^2. The exception here may be the end balls $B_\varepsilon(P_1)$ and $B_\varepsilon(P_{N(\varepsilon)})$ (see Fig. 4.2.), when Γ^R is unclosed curve. We assume now that the curve Γ^R is ε-regular for $\varepsilon \leq \varepsilon_1$. Then for sufficienty small $\varepsilon_2 = C\varepsilon$ ($C =$ const) at least one point of the set A_{ε_2} lies in the ball $B_{\varepsilon_2}(P_i)$ from one side of the curve Γ^R and at least one lies from the other side of it. It is clear that both these points belong to the full boundary of A_{ε_2}, because the distance between them is not greater than 2ε and they are (R, ε)-separated. Using a simple consideration connected with the finite cover, we obtain

$$\rho(\Gamma^R, \Gamma^{R,\varepsilon}) \leq C\varepsilon, \tag{4.93}$$

where $\Gamma^{R,\varepsilon}$ is ε-connected component of the full boundary, which approximates the curve Γ^R.

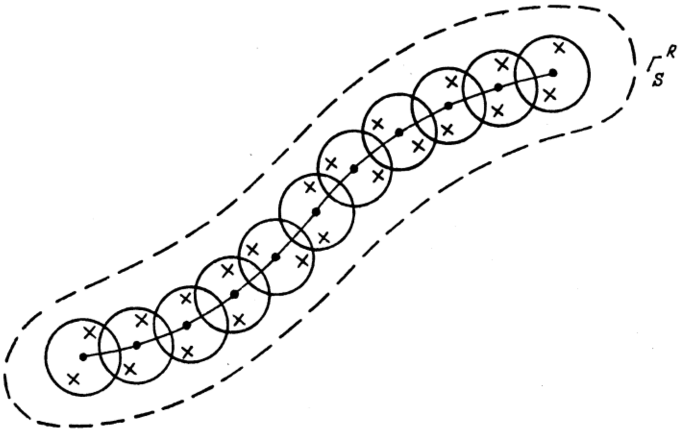

Fig. 4.2.

Thus, if R-visable fragments of discontinuity lines are sufficiently smooth and distances between various fragments are sufficiently great, than it is possible to localize them in Hausdorff's sense by points of the discrete full boundary of the scattered ε-mesh.

If we want to construct now some continuous curve which separates points of two classes, then it is possible to execute the following procedure. Because for every fragment the points from the discrete full boundary are naturally separated into two sets (from one side of the curve and from the other side) let us give the mesh values $(+1)$ on the 1-st set and (-1) on the second. If we construct now some simple interpolant in the "layer" near Γ^R (for example with the help of linear finite elements) then the isoline of zero level is approximation of the discontinuity line.

4.4.3. Special Finite Element Method for Discontinuous D^m-Splines

Let us consider some bounded domain Ω on the plane and assume that Ω is divided into elements Ω_h^i (triangular, rectangular etc.), $\Omega = \cup_i \Omega_h^i$. Every element has the linear size of the order h and the square of the order h^2 . Let us connect with this division some finite element space H_h^m, which is a subspace of the Sobolev space $W_2^m(\Omega), m \geq 2$. Denote by $\gamma_1, \gamma_2, ..., \gamma_{N(h)}$ the basic functions in H_h^m, and $S_1, S_2, ..., S_{N(h)}$ will be their local supports.

We want to solve the problem of interpolation of the function $f(x, y)$, which has discontinuity lines $\Gamma_1, \Gamma_2, ..., \Gamma_s$ of the 1-st type, and discontinuity curves are *a priori* given. We assume that:

1. every curve Γ_i is sufficiently smooth and has no self-intersections;
2. different curves have no intersections;
3. every element Ω_h^k may be cut by the curve Γ_i into not more than two connected parts;
4. if some element Ω_h^k is cut by curve Γ_i then there is no other curve which cuts it;
5. end points of every curve Γ_i are situated only at the boundaries of elements Ω_h^k.

If some curve starts (or finishes) strictly inside the element, then its first (or last) part has no influence on algorithm which presented here.

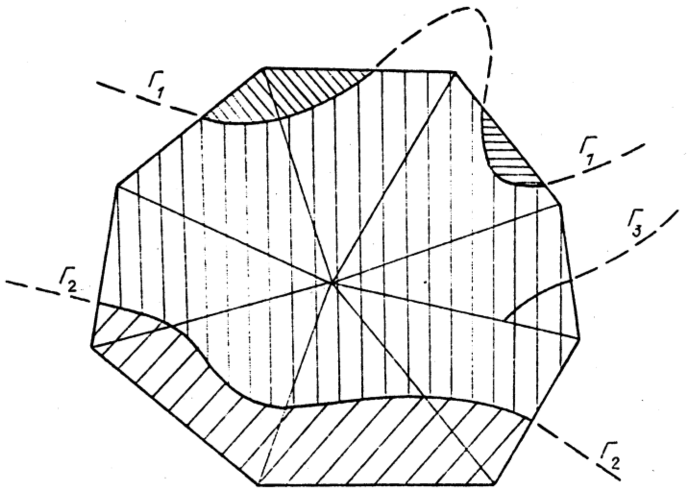

Fig. 4.3.

Let us consider the basic function φ_k and its support S_k. Let S_k be divided into a few parts $S_k^1, S_k^2, ..., S_k^{R_k}$ by the curves Γ_i . Then we correspond to the function φ_k a few functions $\varphi_k^1, \varphi_k^2, \dots \varphi_k^{R_k}$ by the rule

$$\varphi_k^j(x,y) = \begin{cases} \varphi_k(x,y), \ (x,y) \in S_k^j, & j = 1,2,...,R_k \\ 0 \quad \text{otherwise.} \end{cases} \tag{4.94}$$

If S_k is not divided by the curves Γ_i (it does not mean that S_k has no intersections with curves, see the curve Γ_3 in Figure 4.3.), then we put $R_k = 1, \varphi_k^1 = \varphi_k$ everywhere in S^k. For example, in Figure 4.3 the support is divided by the curves Γ_1 and Γ_2 into 4 parts and 4 functions $\varphi^1, \varphi^2, \varphi^3, \varphi^4$ appear, but the curve Γ_3 does not generate any function because it does not divide the support into two (or a few) connected parts; it divides only an element and may play its role in other supports.

Let us denote by $H^m_{h,\Gamma}$ the linear span of the new basic functions $\varphi^j_k, j = 1, 2, ..., R_k, k = 1, 2, ..., N(h)$. It is evident that every function of this subspace belongs to the space $W^m_2(\Omega_\Gamma)$, where $\Omega_\Gamma = \Omega \backslash (\Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_s)$.

Let $f \in W^m_2(\Omega_\Gamma)$ and its values $f(P_i)$ be given at the scattered mesh points $P_i \in \Omega_\Gamma, i = 1, 2, ..., M$. We formulate the problem of interpolation in the following way: find the finite element discontinuous D^m-spline $\sigma^h_\Gamma \in H^m_{h,\Gamma}$, which provides interpolating conditions

$$\sigma^h_\Gamma(P_i) = f(P_i), \quad i = 1, 2, ..., M \tag{4.95}$$

and simultaneously minimizes the energy functional

$$\|D^m \sigma^h_\Gamma\|^2_{L_2(\Omega_\Gamma)} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\Omega_\Gamma} (D^\alpha \sigma^h_\Gamma)^2 d\Omega_\Gamma. \tag{4.96}$$

Here the integration is executed only in Ω_Γ without discontinuity lines $\Gamma = \bigcup_{i=1}^{S} \Gamma_i$. Using the expansion

$$\sigma^h_\Gamma = \sum_{k=1}^{N(h)} \sum_{j=1}^{R_k} \lambda^j_k \varphi^j_k,$$

with undetermined coefficients λ^i_k and the usual Lagrange coefficients $\nu_1, \nu_2, ..., \nu_M$, we obtain a linear algebraic system of the order $R_1 + R_2 + ... + R_{N(h)} + M$

$$\begin{pmatrix} T & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} \bar{\lambda} \\ \bar{\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{f} \end{pmatrix}, \tag{4.97}$$

where T is the square sparse matrix of the order $n = R_1 + ... + R_{N(h)}$ with the elements (without reordering of the basic functions φ^j_k)

$$t^{ij}_{kl} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{S^i_k \cap S^j_l} \frac{\partial^m \varphi^i_k}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \cdot \frac{\partial^m \varphi^j_l}{\partial x^{\alpha_1} \partial y^{\alpha_2}} dx dy, \tag{4.98}$$

A is a sparse rectangular $n \times M$-matrix with the elements

$$a^j_{kj} = \varphi^i_k(P_j),$$

$$\bar{\lambda} = \left(\lambda^j_k, j = 1, ..., R_k, k = 1, ..., N(h)\right)^T, \quad \bar{\nu} = (\nu_1, \nu_2, ..., \nu_M)^T,$$

$$f = (f(P_1), f(P_2), ..., f(P_M))^T.$$

If the interpolation conditions are contradictory, the least square method can be used and a system arises:

$$\begin{pmatrix} T & A^*A \\ A^*A & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \bar{\nu} \end{pmatrix} = \begin{pmatrix} 0 \\ A^* f \end{pmatrix}. \tag{4.99}$$

This matrix consists of square $n \times n$-blocks and $\bar{\nu}$ has also n components.

Remark. In calculation of the coefficients $t_{kl}^{i,j}$ in (4.98) some difficulties arise because integrations are executed in the domains $S_k^i \cap S_l^j$, which have a complicated geometry. In this situation the following trick is possible: except for t_{kl}^{ij} we calculate the other element

$$\bar{t}_{kl}^{ij} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int\limits_{M_{kl}^{ij}} \frac{\partial^m \varphi_k}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \cdot \frac{\partial^m \varphi_l}{\partial x^{\alpha_1} \partial y^{\alpha_2}} dxdy, \tag{4.100}$$

where M_{kl}^{ij} is the union of elements which have non-empty intersections with the set $S_k^i \cap S_l^j$. In other words, except for the initial subspace $H_{h,\Gamma}^m$ we consider the other finite element space $\bar{H}_{h,\Gamma}^m$. Function from this space is doubled near the discontinuity line (in the general case some multi-valued function arises). Certainly, in calculation of values for the spline we need to ignore prolongations of the finite elements to obtain one-valued spline.

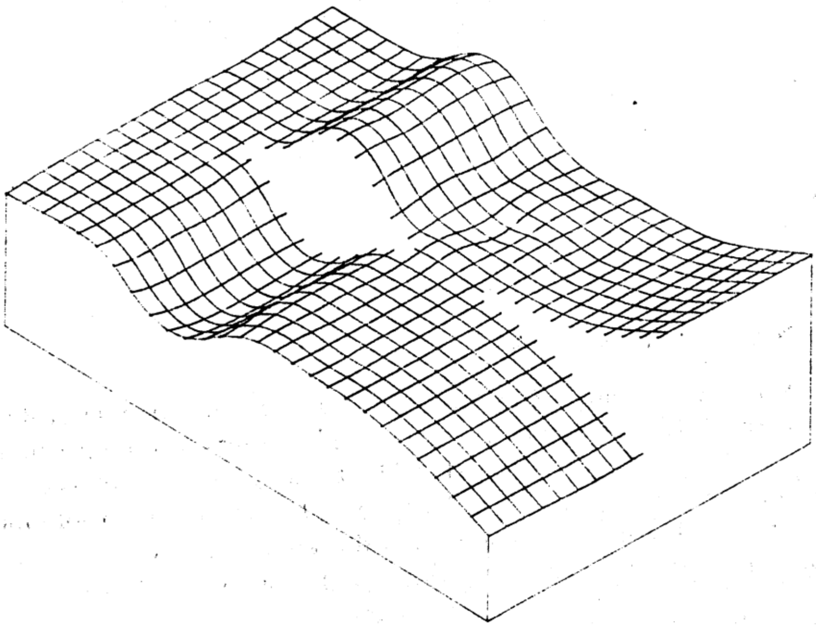

Fig. 4.4.

The error estimation techniques for discontinuons finite element D^m-spline are the same as in the continuous case. The main difference is only the following: to provide the error estimates in L_P -norm for analytical D^m-spline (see Chapter 3), it is necessary to have the cone condition for the boundary of the domain Ω. It is not true for the domains with discontinuity lines with the end points inside Ω. In this case it is possible to consider domain $\Omega_\delta = \Omega \backslash \Gamma_\delta$ where Γ_δ is δ-neibourhood of a set of discontinuity lines and δ is fixed. Using the results of Chapters 3 and 4 we obtain the same error estimates like for continuous case, but only in Ω_δ.

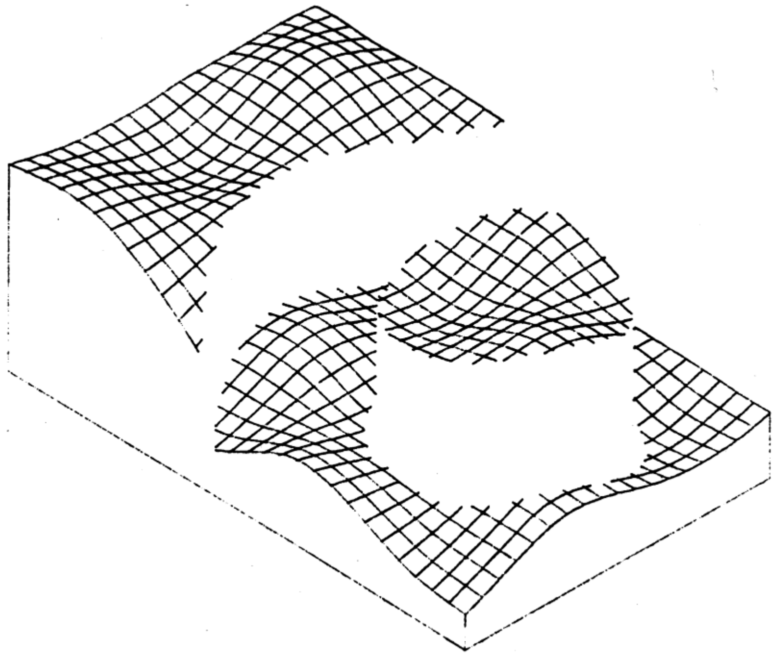

Fig. 4.5.

4.4.4. Numerical Examples

Let us consider a rectangle $\Omega = [0,1] \times [0,1]$ with the rectangular grid (uniform meshes with 12 points in the direction x and 8 points in the direction y) and connect with it the space of biparabolic finite elements of the class C^1. Dimension of this space is $13 \times 9 = 117$. There are 100 scattered interpolation points in our domain obtained by some standard randomizor. We want to construct the finite element D^2-spline σ_Γ by the minimization of the functional

$$\|D^2\sigma_\Gamma\|^2_{L_2(\Omega\backslash\Gamma)} = \int\limits_{\Omega\backslash\Gamma} (\sigma_\Gamma)^2_{xx} + 2(\sigma_\Gamma)^2_{xy} + (\sigma_\Gamma)^2_{yy}$$

under interpolation constraints, where Γ is the union of a few broken lines. For a few functions $f(x,y)$ with the given discontinuity lines the results of interpolation are presented in Fig. 4.4-4.6.

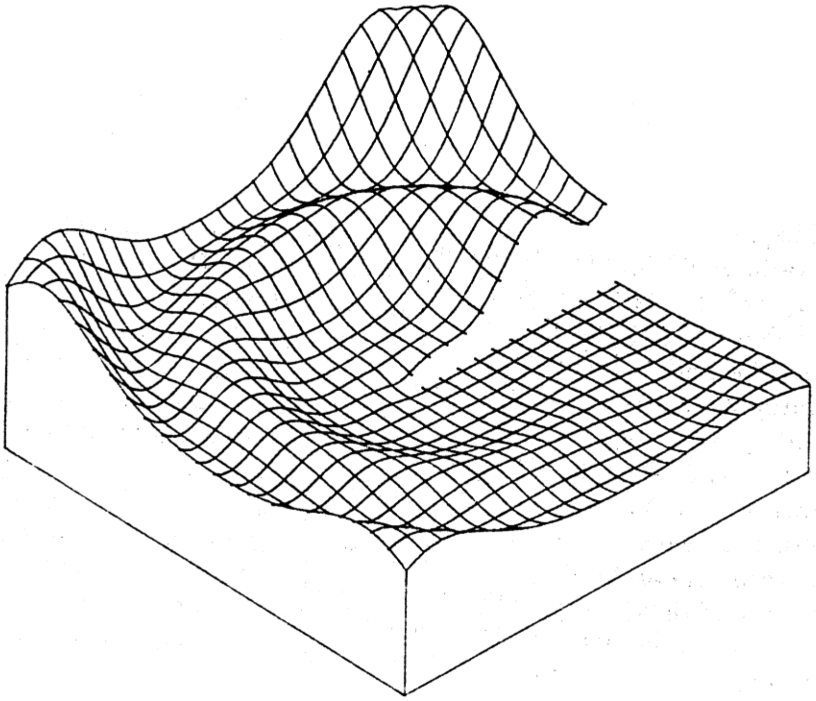

Fig. 4.6.