6. Splines on Manifolds

In the present chapter, we propose a method of solving approximation problems for functions defined on manifolds in $\mathbb{R}^n$ by using D^m-spline traces onto the manifolds. For the sake of simplicity, we confine ourselves to the case of $(n-1)$-dimensional smooth manifolds in $\mathbb{R}^n$, which are boundaries of simply connected bounded domains. In Section 6.1, an analysis is given of existence and uniqueness of traces of interpolating D^m-splines and, also, of their convergence (convergence orders) in the case of condensed grids of interpolation nodes on a manifold.

In Section 6.2 we propose the method of numerical realization on the basis of D^m-splines in $\mathbb{R}^n$, which brings about the presentation of the solution in terms of reproducing kernels of the semi-Hilbert spaces $D^{-m}L^2$, which are known in the explicit form (see Section 5.3.1). Section 6.2 contains three examples illustrating the algorithms for fitting 3-dimensional surfaces by using its prescribed points and normals to the surfaces at these points.

Another method of numerical implementation may be the finite element method. In this case, finite elements could be constructed in a fixed domain in $\mathbb{R}^n$, comprising a manifold, but not on a manifold, which is much simpler. So, in Section 6.3 we suggest method named "spline-approximation in thin layer", where this idea is further developed having advantages over the finite element approach.

6.1. Traces of D^m-Splines in Ω Onto a Manifold

We refer the reader for the definition of D^m-splines and attendant notations to Section 5.1.1.

6.1.1. Definitions

Let $\Omega_0 \subset \Omega$ be a simply connected bounded domain whose boundary Γ is an infinite differentiable $(n-1)$-dimensional manifold.

Definition 6.1. *Assume that* $f \in W_2^m(\Omega)$, $A \subset \Gamma$. *The restriction of the interpolating* D^m*-spline*

$$\sigma^A = \arg \min_{u \in A^{-1}(f)} \|D^m u\|_{L^2(\Omega)} \tag{6.1}$$

onto the boundary Γ is said to be a trace of D^m-spline onto the manifold Γ. Here, $A^{-1}(f) = \{u \in W_2^m(\Omega): \quad u(a) = f(a), \ \forall a \in A\}$.

Henceforth, we will find the conditions of existence and uniqueness for the trace of D^m-spline onto the manifold Γ and prove the convergence of the splines σ^A in the space of traces. To obtain the convergence orders, it is necessary to define the space of traces. The space of traces of the Sobolev functions from $W_2^m(\Omega)$ is known to be a space of the Sobolev functions $H^{m-1/2}(\Gamma)$ with a fractional index.

Let us give the definition of $H^s(\Gamma)$ according to (Lions, Magenes 1968). Let $\mathcal{Q}_j, \quad j = 1, ..., \nu$ be bounded domains in $\mathbb{R}^n$ covering Γ; $\varphi_j : x \to y = \varphi_j(x)$ be infinite differentiable mappings from $\mathcal{Q}_j$ into $\mathcal{Q} = B_{n-1} \times [-1, 1]$, where B_{n-1} is the unit ball in $\mathbb{R}^{n-1}$, which are such that the parts $\mathcal{Q}_j \cap \Gamma$ of the boundary Γ transform into the ball $\mathcal{Q} \cap \{y_n = 0\}$, and the inverse mappings $\varphi_j^{-1} : \mathcal{Q} \to \mathcal{Q}_j$ are also infinite differentiable.

If $\{\alpha_j\}$ is a partition of unity on Γ, which consists of infinite differentiable functions having compact supports in $\mathcal{Q}_j \cap \Gamma$, then for any summarized function u on Γ we determine the functions $\varphi_j^*(u)$ on the ball B_{n-1}:

$$\varphi_j^*(u)(y) = (\alpha_j u)(\varphi_j^{-1}(y, 0)) \tag{6.2}$$

and set them to zero outside the ball B_{n-1}. Assume s to be a real number, then the space of functions

$$H^s(\Gamma) = \{u : \varphi_j^*(u) \in H^s(\mathbb{R}^{n-1})\}$$

with the norm

$$\|u\|_{H^s(\Gamma)} = \left(\sum_{j=1}^{\nu} \|\varphi_j^*(u)\|^2_{H^s(\mathbb{R}^{n-1})} \right)^{1/2}$$

is the Hilbert space. Here, $H^s(\mathbb{R}^{n-1})$ denotes the Sobolev space with a fractional index.

For the integers $s = k \geq 0$ we can give another definition

$$H^k(\Gamma) = \{u : \varphi_j^*(u) \in W_2^k(B_{n-1})\}$$

$$\|u\|_{H^k(\Gamma)} = \left(\sum_{j=1}^{\nu} \|\varphi_j^*(u)\|^2_{W_2^k(B_{n-1})} \right)^{1/2}, \tag{6.3}$$

since the norms in the spaces $W_2^k(\mathbb{R}^{n-1})$ and $H^k(\mathbb{R}^{n-1})$ are equivalent (see Appendix 1), and the supports of the functions $\varphi_j^*(u)$ are located on the compact in the ball B_{n-1}. Theorems on traces and continuation (Lions, Magenes 1968) directly imply

Lemma 6.1. The trace of a function of the space $W_2^m(\Omega)$ on Γ belongs to $H^{m-1/2}(\Gamma)$, and the operator of the trace is continuous

$$\|f\|_{H^{m-1/2}(\Gamma)} \lessapprox \mu\|f\|_{W_2^m(\Omega)}. \tag{6.4}$$

There exists a continuous linear operator G of prolongation onto the domain Ω of the functions of the space $H^{m-1/2}(\Gamma)$:

$$\|Gf\|_{W_2^m(\Omega)} \leq k\|f\|_{H^{m-1/2}(\Gamma)}. \tag{6.5}$$

6.1.2. Sobolev Functions with Condensed Zeros on Manifold

Now we formulate and prove the Lemma on the Sobolev functions with condensed zeros, which is an analog of Lemma 5.6 for the case of a manifold. Like previously, we say that a set $A \subset \Gamma$ forms h-net, if for every $t \in \Gamma$ there exist $a \in A$ such, that $\|a - t\| \leq h$. The distance $\|\cdot\|$ is the ordinary metric in $\mathbb{R}^n$ (not connected with Γ).

Lemma 6.2. There exist the constants $c, h_0 > 0$ such that

$$\|u\|_{H^s(\Gamma)} \leq ch^{m-s-1/2}\|u\|_{H^{m-1/2}(\Gamma)} \tag{6.6}$$

for any function $u \in W_2^m(\Omega)$ which has an h-net of zeros in the manifold Γ for $h < h_0$. The constant C depends on the manifold Γ and the parameter $s \leq m - 1$.

Proof. Let $B = B_{n-1}$ be the unit ball in $\mathbb{R}^{n-1}$, then Lemma 5.6 implies that $h_0 > 0$ exists such that any function $\nu \in W_2^{m-1}(B)$ having an h-net of zeros in the domain B for $h \leq h_0$ satisfies inequalities

$$\|\nu\|_{W_2^k(B)} \leq c'_k h^{m-k-1}\|\nu\|_{W_2^{m-1}(B)}. \tag{6.7}$$

Assume that the function u has an h-net of zeros on Γ. The functions α_i of partitioning unity have compact supports in $Q_j \cap \Gamma$, and the functions φ_j are infinite differentiable and, therefore, satisfy the Lipschitz condition $\|\varphi_j(x) - \varphi_j(y)\| \leq M\|x - y\|, \quad \forall j \in 1, ..., \nu, \quad \forall x, y \in \operatorname{supp}(\alpha_j)$. Equality (6.2) and the Lipschitz condition imply that the functions $\varphi_j^*(u)$ have a Mh-net of zeros in the ball B. Therefore, inequalities (6.7) imply the relations

$$\|\varphi_j^*(u)\|_{W_2^k(B)} \leq c_k h^{m-k-1}\|\varphi_j^*(u)\|_{W_2^{m-1}(B)}$$

which are valid for any function $u \in H^{m-1}(\Gamma)$ having an h-net of zeros on Γ for $h \leq h_0$. Finally, definition of norm (6.3) implies the inequalities

$$\|u\|_{H^k(\Gamma)} \leq c_k h^{m-k-1}\|u\|_{H^{m-1}(\Gamma)}.$$

For fractional s, we make use of the fact that the spaces $H^s(\Gamma)$ are *intermediate* between $H^{[s]}(\Gamma)$ and $H^{[s]+1}(\Gamma)$, and therefore any function $u \in H^{[s]+1}(\Gamma)$ satisfies the interpolation inequality

$$\|u\|_{H^s(\Gamma)} \le c\|u\|^{[s]+1-s}_{H^{[s]}(\Gamma)}\|u\|^{s-[s]}_{H^{[s]+1}(\Gamma)}$$

where $[s]$ is the integer part of s. Estimating the factors $\|u\|_{H^{[s]}(\Gamma)}$ and $\|u\|_{H^{[s]+1}(\Gamma)}$ in terms of inequalities (6.8), we obtain the inequalities

$$\|u\|_{H^s(\Gamma)} \le c_s h^{m-s-1}\|u\|_{H^{m-1}(\Gamma)}$$

which is a generalization of inequalities (3.7). Concluding inequalities (6.6) are easily obtained from the latter one on the basis of the fact that the space $H^{m-1}(\Gamma)$ is intermediate between $H^s(\Gamma)$ and $H^{m-1/2}(\Gamma)$. □

6.1.3. Existence and Uniqueness

We call Γ an *algebraic manifold* (of degree $m-1$), if there exists a non-zero polynomial $p(x) \in P_{m-1}$, which vanishes at any point $x \in \Gamma$, i.e. $p(x) = 0$, $\forall x \in \Gamma$. Let P_Γ be the space of traces of the polynomials $\pi \in P_{m-1}$. The space P_Γ may be identified with the subspace in P_{m-1}, which are not degenerated on Γ (except for the zero polynomial). It is easy to understand that $P_\Gamma = P_{m-1}$, if Γ is not an algebraic manifold, because $P_{m-1} = P_\Gamma \oplus P_\Gamma^\perp$, where $P_\Gamma^\perp$ is the set of polynomials, annihilated on Γ. We leave it to the reader to prove the following simple

Lemma 6.3. There exists $h_0(\Gamma) > 0$ such that any set $A \subset \Gamma$ forming $h-$net for $h \le h_0$ contains $L-$solvable set for polynomials P_Γ.

Theorem 6.1. There exists $h_0(\Gamma)$ such that for any set A which is an h-net on Γ for $h \le h_0$ and any function $f \in H^{m-1/2}(\Gamma)$, the trace of the interpolating D^m-spline σ^A onto Γ exists and is unique.

Proof. When Γ is not an algebraic manifold, P_Γ is equal to P_{m-1} and the spline σ^A is really uniquely defined owing to Lemma 6.3 and the general theorem on D^m-spline existence (see Section 5.1.1.)

Let Γ be an algebraic manifold. Prove that if the set A contains L-solvable set for the space P_Γ, then all the solutions to spline problem (6.1) coincide on Γ. Let the points $Q_1, Q_2, ..., Q_L$ from the set A form L-solvable set for P_Γ. Supplement it with points $R_1, R_2, ..., R_{N-L}$ lying outside of Γ up to the L-solvable set for the space P_{m-1} in such a way that the points $R_1, ..., R_{N-L}$ form L-solvable set for the space $P_\Gamma^\perp$. Assume that $s_1, s_2, ..., s_{N-L}$ are arbitrary values. By Theorem 6.1, the solution to problem

$$\sigma_s^A = \arg \min_{u\in A^{-1}(f),\, u(R_i)=s_i} \|D^m u\|_{L^2(\Omega)} \tag{6.9}$$

is uniquely defined as the set $A \cup \{R_1, ..., R_{N-L}\}$ contains the L-solvable set for the space P_{m-1}. To prove existence, it is necessary to show that there exists a function $f_s \in W_2^m(\Omega)$ coinciding with the function f on the set A and taking the values $s_1, s_2, ..., s_{N-L}$ at the points $R_1, R_2, ..., R_{N-L}$. To this end, in accordance with Lemma 6.1, consider the prolongation function Gf for f onto the domain

Ω. Since the points $R_1, R_2, ..., R_{N-L}$ form the L-solvable set for the space $P_\Gamma^\perp$, there exists a polynomial $\mathcal{Q}(x) \in P_\Gamma^\perp$ satisfying the conditions $\mathcal{Q}(R_i) = s_i - Gf(R_i), \quad i = 1, ..., N - L$. It is obvious that the function $f_s = Gf + \mathcal{Q}$ meets the above-indicated requirements. Thus, the existence and uniqueness of problem (6.1) have been proved.

Prove that the traces of the solutions to problem (6.1) on Γ coincide for any choice of the values $s_1, s_2, ..., s_{N-L}$. Consider two problems

$$\sigma_1 = \arg \min_{u \in A^{-1}(f),\, u(R_i)=0} \|D^m u\|_{L^2(\Omega)} \tag{6.10}$$

$$\sigma_2 = \arg \min_{u \in A^{-1}(0),\, u(R_i)=s_i} \|D^m u\|_{L^2(\Omega)}. \tag{6.11}$$

The solutions to both problems exist and are unique, and $\sigma_s^A = \sigma_1 + \sigma_2$ by virtue of the orthogonal property (see Section 5.1.1). The solution to problem (6.11) is a polynomial of the space $P_\Gamma^\perp$ vanishing on Γ. Hence, the trace of the spline σ_s^A coincides with the trace σ_1 on Γ which is independent of the values $s_1, s_2, ..., s_{N-L}$. It is obvious that the solutions to problems (6.9) describe the entire set of solutions to (6.1), as they differ in the polynomial vanishing on Γ and annihilating the functional $\|D^m u\|_{L^2(\Omega)}$.

Thus, if the set A contains L-solvable set for the space P_Γ, then all the solutions to problems (2.6) coincide on Γ. Owing to Lemma 6.3, we obtain the statement of the theorem. □

6.1.4. Convergence Rates

For the traces of splines to converge, it is necessary for us to make use of the fact that $X = H^{m-1/2}(\Gamma)$ is compact embedded in $C(\Gamma)$. Let Φ be a unit sphere of the space $H^{m-1/2}(\Gamma)$. Then the set $G\Phi$ consisting of the functions $Gf \in W_2^m(\Omega)$ for $f \in \Phi$ is bounded by virtue of Lemma 6.1. Compactness of the embedding $W_2^m(\Omega) \subset C(\Omega)$ implies that the set $G\Phi$ is a precompact in $C(\Omega)$, i.e. a set of uniformly continuous functions. It is obvious that the set Φ also consists of uniformly continuous functions, i.e. Φ is a precompact in $C(\Gamma)$. The compactness of the embedding has thus been established.

Theorem 6.2. Let f belong to $H^{m-1/2}(\Gamma)$ and the sets $A_1, A_2, ...$ form a condensed h-net in Γ. Then the traces of the D^m-splines σ^{A_i} onto Γ converge to the function f in the norm of the space $H^{m-1/2}(\Gamma)$.

Proof. To use Theorem 3.2 on convergence of abstract T-splines, it is necessary to reduce the problem of calculating the trace of a D^m-spline on Γ to problem (1.1).

First, we do it for non-algebraic $(m-1)$-dimensional manifolds Γ. Assume that $X = H^{m-1/2}(\Gamma), \quad Y = H^{m-1/2}(\Gamma)/P_\Gamma$ is a factor space, $T : X \to Y$ is the operator of canonical embedding, which puts the function u in correspondence with a factor class $\dot{u} = u + P_\Gamma$. Henceforth, we define the norm in the space Y. The set of solutions to problems

$$\sigma_f^\Gamma = \arg \min_{u \in \Gamma^{-1}(f)} \|D^m u\|_{L^2(\Omega)} \tag{6.12}$$

for any functions $f \in H^{m-1/2}(\Gamma)$ will be denoted as $Sp(\Gamma)$. Since Γ is a non-algebraic manifold, then Γ contains L-solvable set for the space P_{m-1}. Then, we conclude that $Sp(\Gamma)$ is a Hilbert space, because the interpolating spline operator is a projector (see Section 1.4.2).

It is easy to establish that the factor space $W_2^m(\Omega)/P_{m-1}$ with norm

$$\| \dot{u} \|_m = \|u + P_{m-1}\| = \|D^m u\|_{L^2(\Omega)}$$

is a Hilbert space, and hence $Sp(\Gamma)/P_{m-1}$ is its Hilbert subspace with the norm $\| \cdot \|_m$. Prove that the factor space $Y = H^{m-1/2}(\Gamma)/P_\Gamma$ with the norm

$$\| \dot{u} \|_Y = \inf\{\|D^m f\|_{L^2(\Omega)} : f|_\Gamma = u\} \tag{6.13}$$

is a Hilbert space. Note that the definition of norm (6.13) is independent of the choice of a function u in the class $\dot{u} = u + P_{m-1}$. Introduce the mapping

$$t : H^{m-1/2}(\Gamma)/P_\Gamma \to Sp(\Gamma)/P_{m-1},$$

which puts the class $u + P_\Gamma \in Y$ in correspondence with the class $\sigma_u^\Gamma + P_{m-1}$, where σ_u^Γ is the solution to problem (6.12).

The orthogonal property enables us to conclude that the definition of the operator t is correct and independent of the choice of a representative in the class, and the operator t is a bijection. Since, the norms of the elements $u + P_\Gamma$ and $\sigma_u^\Gamma + P_{m-1}$ coincide and t is a bijection, the space $Y = H^{m-1/2}(\Gamma)/P_{m-1}$ is also a Hilbert space as in the case of $Sp(\Gamma)/P_{m-1}$.

The continuity of the operator T can be proved from the relations

$$\| \dot{u} \|_Y \le \|D^m Gu\|_{L^2(\Omega)} \le \|Gu\|_{W_2^m(\Omega)} \le k\|u\|_{H^{m-1/2}(\Gamma)}$$

using the operator G from Lemma 6.1. Hence, $\|T\| \le k$. So, the proof for non-algebraic manifolds is completed.

Now we indicate the modifications in the proof for the case of Γ being an algebraic $(m-1)$-dimensional manifold. In Section 6.1.3, we have proved that the problem of determining the spline trace onto the manifold Γ is equivalent to the solution of the problem

$$\sigma_1 = \arg \min_{u \in A^{-1}(f), u(R_i)=0} \|D^m u\|_{L^2(\Omega)}.$$

Hence, it is equivalent to the construction of D^m-spline on a closed subspace in $W_2^m(\Omega)$:

$$\{u \in W_2^m(\Omega) : u(R_i) = 0, \quad i = 1, ..., N - L\}.$$

The modifications in the proof of Theorem 3.1 for the case of an algebraic manifold consist in replacing the space $W_2^m(\Omega)$ with its closed subspace. □

Theorem 6.3. Assume that $f \in H^{m-1/2}(\Gamma)$, $m > n/2$ and the sets $A_1, A_2, \ldots$ form a condensable h-net on Γ. Then, for the traces of the splines σ^{A_i} onto the manifold Γ, the asymptotic estimates

$$\|\sigma^{A_i} - f\|_{H^s(\Gamma)} = o(h_i^{m-s-1/2})$$

hold for $i \to \infty$. The parameter s belongs to the interval $[0, m-1]$.

Proof. Lemma 6.2 implies the statement of the theorem, because the difference $\|\sigma^{A_i} - f\|_{H^{m-1/2}(\Gamma)}$ tends to zero by virtue of Theorem 6.2. □

Remark 6.1. A similar technique can be used to establish asymptotic estimates of the convergence in the space of continuous functions $C(\Gamma)$ of the form

$$\|\sigma^{A_i} - f\|_{C(\Gamma)} = o(h_i^{m-n/2}) .$$

6.2. Traces of D^m-Splines in $\mathbb{R}^n$ Onto a Manifold

We refer the reader for the definition of D^m-splines $\mathbb{R}^n$ and attendant notations to Section 5.3. It is not difficult to extend the results obtained in Section 6.1 about existence, convergence and convergence rates to this type of D^m-splines. We do not make this, because this could be similar and is done in (Bezhaev 1984). We only give definition of traces of interpolating smoothing splines and construct an algorithm for computing the trace of an interpolating smoothing spline. Also, we give three practical examples.

6.2.1. Interpolating Smoothing Splines on Manifolds

We refer the reader for the definition of interpolating smoothing spline to Section 5.3.2. Let functionals $k_1, \ldots, k_s$ be located on the manifold Γ. To be more precise, if we denote by $P_\Gamma^\perp$ the space of polynomials from P_{m-1}, vanishing on Γ, then it is sufficient for us to suppose that the functionals k_i are annihilated on functions of the space $P_\Gamma^\perp$. These conditions are satisfied, for example, by the functionals of point evaluations $k_i(u) = u(P_i)$, $\quad P_i \in \Gamma$.

The obvious condition $K_0(\bar{0}) \cap P_{m-1} \supset P_\Gamma^\perp$ implies that condition (5.91) for uniqueness of the interpolating smoothing function is violated for the non-zero space $P_\Gamma^\perp$. Nevertheless, the functionals $k_1, \ldots, k_s$ can satisfy a condition for the uniqueness of the trace of spline (5.88) on the manifold Γ.

Theorem 6.4. If the functionals $k_1, \ldots, k_s$ contain the L-solvable set for the space P_Γ, all the solutions to problem (5.88) coincide on the manifold Γ.

The proof of the theorem does not differ much from that of Theorem 6.1. Note only that the trace of a spline on the manifold Γ is determined by the trace of the unique solution to problem

$$\bar{\sigma}_{\rho}^{s1} = \arg \min_{u \in K_{s1}(\bar{r}),\, u(R_i)=0} |u|_m + \sum_{i=1}^{s1} \rho_i (k_i(u) - r_i)^2 \tag{6.14}$$

where $R_1, ..., R_{N-L}$ is L-solvable set for the space $P_\Gamma^\perp$.

6.2.2. An Algorithm for Computing the Trace of D^m-Spline on a Manifold

Let $e_1, ..., e_L$ be the basis of the space P_Γ, and $e_{L+1}, ..., e_N$ be the basis of the space $P_\Gamma^\perp$. Formulae (5.89)-(5.90) imply representation

$$\begin{aligned} \bar{\sigma}_{\rho}^{s1} &= \sum_{i=1}^{s} \lambda_i k_i (G_m(P - X)) + \sum_{i=1}^{N-L} a_i G_m(P - R_i) \\ &+ \sum_{i=1}^{L} \nu_i e_i(P) + \sum_{i=L+1}^{N} b_i e_i(P) \end{aligned} \tag{6.15}$$

where the expansion coefficients are found from the system of equations which schematically can be written down in the form

$$\begin{bmatrix} K + I_\rho & & & \\ KR & R & & \\ KE_1 & RE_1 & 0 & \\ KE_2 & RE_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\lambda} \\ \bar{a} \\ \bar{\nu} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} \bar{r} \\ \bar{0} \\ \bar{0} \\ \bar{0} \end{bmatrix}. \tag{6.16}$$

The matrix of the system is symmetric and non-singular. The block K has the elements $k_i k_j (G_m(P - X))$; the block R - the elements $GG_m(R_i - R_j)$; the block KR - the elements $k_j(G_m(P - R_i))$; the blocks KE_1, RE_1, KE_2, RE_2 have the common elements $k_j(e_i(P))$, $e_i(R_j), i = 1, ..., L,\quad k_j(e_i(P))$, $e_i(R_j), i = L + 1, ..., N$, respectively.

Consider a group of equations defined by the four lower blocks: 4×4 and 4×3 are zero, KE_2 is also zero by definition of the functionals k_i. Hence, the unknowns $a_1, ..., a_{N-L}$ are determined independent of the system of equations $RE_2 \bar{a} = \bar{0}$. By construction, the points $R_1, ..., R_{N-L}$ constitute the L-solvable set for the space $P_\Gamma^\perp$, therefore the square matrix RE_2 is non-singular, and the coefficients a_i are equal to zero. This fact makes it possible to reduce system of equations (6.16) to a system of a smaller dimension

$$\begin{bmatrix} K + I_\rho & KE_1^T & 0 \\ KE_1 & 0 & 0 \\ KR & RE_1 & RE_2 \end{bmatrix} \begin{bmatrix} \bar{\lambda} \\ \bar{\nu} \\ \bar{b} \end{bmatrix} = \begin{bmatrix} \bar{r} \\ \bar{0} \\ \bar{0} \end{bmatrix}. \tag{6.17}$$

Here, we have taken into account the fact that $KE_2 = 0$, and the second and the third groups of equations are rearranged. In view of the non-singularity of the matrix RE_2, we obtain the following system of equations for finding the coefficients $\bar{\lambda}$ and $\bar{\nu}$:

$$\begin{bmatrix} K + I_\rho & KE_1^T \\ KE_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\lambda} \\ \bar{\nu} \end{bmatrix} = \begin{bmatrix} \bar{r} \\ \bar{0} \end{bmatrix}. \tag{6.18}$$

The coefficients $b_{L+1}, ..., b_N$ can be found from the remaining equations of system (6.16), but we are not interested in these coefficients as the terms of expansion (6.15) corresponding to them vanish on the sphere. Thus, we have proved the following

Theorem 6.5. Let the linearly independent continuous linear functionals $k_1, ..., k_s$ contain L-solvable set for the space P_Γ and vanish on the functions of the space $P_\Gamma^\perp$. Then the trace of the solution to problem (6.14) onto the manifold Γ can be presented in the form

$$\sigma(P) = \sum_{i=1}^{s} \lambda_i k_i(G_m(P - X)) + \sum_{i=1}^{L} \nu_i e_i(P) \tag{6.19}$$

where the expansion coefficients can be found from system of linear equations (6.17).

6.2.3. Approximation of Surfaces with Known Normals at the Points

Let a surface Φ star-like with respect to the origin in $\mathbb{R}^3$ be given by the prescribed points $(x_i, y_i, z_i), i = 1, ..., s$. Define the points on the sphere $Q_1, ..., Q_s$ and the values $r_1, ..., r_s$ by the following relations:

$$Q_i = (x_i, y_i, z_i)/r_i, \quad r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}. \tag{6.20}$$

It is natural to assume that the prescribed points do not coincide with the origin. If $u(x, y, z)$ satisfies the conditions $u(Q_i) = r_i, \quad i = 1, ..., s$, the trace of the function u onto the sphere $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ defines the surface described by the radius-vector

$$\mathbf{r} = r(\varphi, \eta)(\cos\varphi \sin\eta, \sin\varphi \sin\eta, \cos\eta)$$

where

$$r(\varphi, \eta) = (\cos\varphi \sin\eta, \sin\varphi \sin\eta, \cos\eta) \tag{6.21}$$

and this surface goes through these prescribed points.

Note that the standard technique is to construct a function $r(\varphi, \eta)$ satisfying the above-mentioned requirements $r(Q_i) = r_i$. However, the approach suggested makes it possible to obtain a more flexible rule.

Thus, the problem of surface fitting by given points has been reduced to the problem of approximating a function on the sphere Γ, i.e. on a manifold in

$\mathbb{R}^3$. To make use of Theorem 6.5, it is necessary to specify the definitions of the spaces P_Γ and $P_\Gamma^\perp$.

Lemma 6.4. Monomials of the form

$$P^\alpha = x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}, \quad |\alpha| = m-2, m-1 \tag{6.22}$$

do not become singular on Γ and form the basis in P_Γ. Polynomials of the form

$$(x^2+y^2+z^2-1)x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}, \quad |\alpha| \le m-3 \tag{6.23}$$

form the basis in the space $P_\Gamma^\perp$.

The lemma is directly implied by the fact that the polynomial vanishing on the sphere can be evenly divided by $(x^2+y^2+z^2-1)$, and a non-zero linear combination of monomials (6.22) cannot be annihilated of Γ (cannot be evenly divided by $x^2+y^2+z^2-1$):

According to Theorem 6.2, to construct a trace of a D^m- spline, it is sufficient to know the basis P_Γ. Note that the basis in P_Γ can be also given in terms of spherical harmonics of degree $m-1$ in the Cartesian coordinates, but basis (6.22) is much simpler.

By prescribing the linear functionals $k_{Q_i}(u) = u(Q_i)$ in $D^{-m}L^2$ for $m \ge 2$, we can thus solve the problem of surface fitting by given points or construct an approximation to the surface by using the trace of an interpolating smoothing spline. In Fig. 6.1 one can see a 3-dimensional surface, interpolated by 14 points, and then scaled on 14×14 grid. Fig. 6.2 illustrates the possibilities of D^m-splines for a function given on torus.

Now, we prove that the prescribed normal to the surface at the prescribed point defines a pair of linear functionals vanishing on the functions of the space $P_\Gamma^\perp$. Indeed, if $(\bar{x},\bar{y},\bar{z}) \in \Gamma$, then the normal to the surface, defined by the trace of the function u onto the sphere Γ, satisfies the following relations (Bezhaev 1987):

$$\begin{cases} \lambda n_1 = \dfrac{\partial u}{\partial x}(\bar{x}^2-1) + \dfrac{\partial u}{\partial y}\bar{x}\bar{y} + \dfrac{\partial u}{\partial z}\bar{x}\bar{z} + u\bar{x} \\ \lambda n_2 = \dfrac{\partial u}{\partial x}\bar{x}\bar{y} + \dfrac{\partial u}{\partial y}(\bar{y}^2-1) + \dfrac{\partial u}{\partial z}\bar{y}\bar{z} + u\bar{y} \\ \lambda n_3 = \dfrac{\partial u}{\partial x}\bar{x}\bar{z} + \dfrac{\partial u}{\partial y}\bar{y}\bar{z} + \dfrac{\partial u}{\partial z}(\bar{z}^2-1) + u\bar{z} \end{cases} \tag{6.24}$$

where λ is the normalizing multiplier.

This multiplier can be eliminated from equations (6.24) leading them to the following two equations (in the general form) relating the value of the function and those of its derivatives:

$$k_1(u) = 0, \quad k_2(u) = 0, \tag{6.25}$$

where the functionals k_1 and k_2 are continuous in $D^{-m}L^2$ for $m \ge 3$, which vanish on the functions of the sphere $P_\Gamma^\perp$. If a prescribed point of the surface Φ is

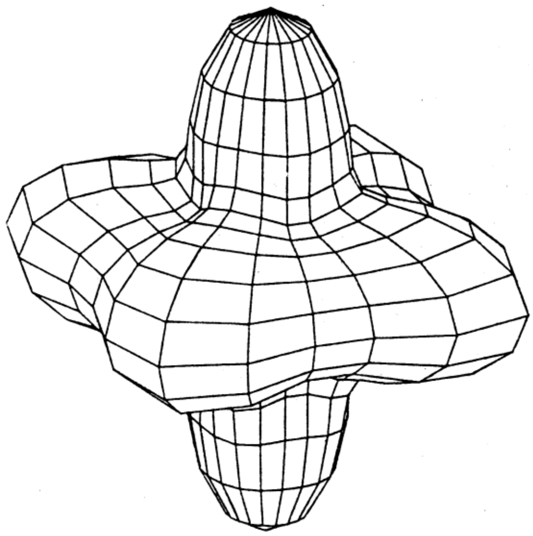

Fig. 6.1. 3-dimensional surface, interpolated by 14 points, and then scaled on 14 × 14 grid.

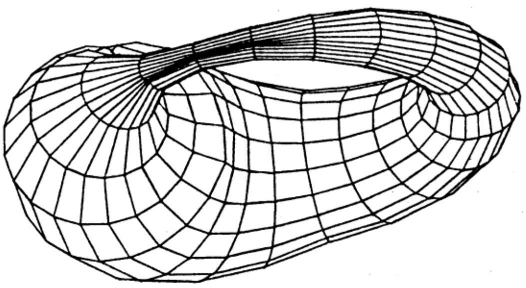

Fig. 6.2. Interpolation on torus

an interpolation one, the value of the function u is defined at the point $(\bar{x}, \bar{y}, \bar{z})$, and when eliminating λ, the terms containing u in representation (6.24) can be taken to the right-hand side of equalities (6.25).

Fig. 6.3 illustrates the sphere, interpolated with the help of prescribed points and normals. One normal is not correct, it is shifted by 45 degrees, therefore the sphere is deformed.

Remark 6.2. The geometric arguments imply that the vector $\mathbf{n} = (n_1, n_2, n_3)$ of the normal cannot be orthogonal to the vector $(\bar{x}, \bar{y}, \bar{z})$. Equation (6.24) and the orthogonality condition $n_1\bar{x} + n_2\bar{y} + n_3\bar{z} = 0$ imply $u(\bar{x}, \bar{y}, \bar{z}) = 0$, i.e. the

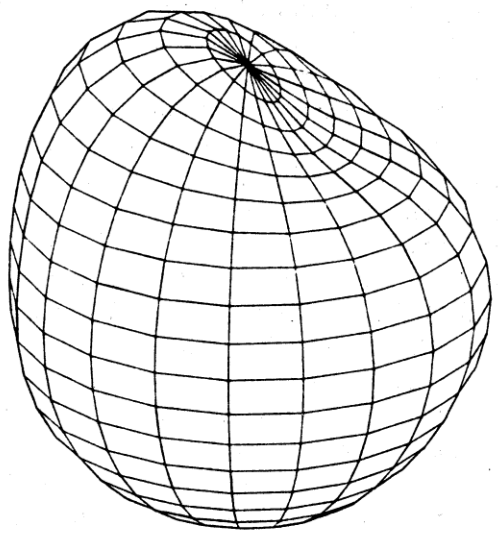

Fig. 6.3. Interpolation with prescribed normals

prescribed point coincides with the origin, which is impossible and confirms the arguments.

Remark 6.3. Linear functionals used to construct splines are often given in local coordinates defined on a manifold. It is obvious that they can be reduced to linear functionals for the functions defined on $\mathbb{R}^n$. To transform the functionals given in the spherical coordinates, we can use equation (6.24). This formula is used in a particularly simple manner to express the functionals of the partial derivatives of the function r in terms of the functionals of a combination of the partial derivatives of the function u.

Remark 6.4. It would not be difficult to give a rigorous justification for the method of spline traces on a manifold in terms of the reproducing kernels of the semi-Hilbert space $H^{m-1/2}(\Gamma)$ with the special semi-norms $|f|_\Gamma = \inf\{|u|_m : u \in D^{-m}L^2, u|_\Gamma = f\}$. The reproducing kernel in such a space is the trace of the Green function (see Section 5.3.1) on the manifold Γ. This fact gives us another possibility to prove characterization Theorem 6.5.

6.2.4. Discussion

Other methods of solving spline approximation problems on the sphere were investigated in (Wahba 1981, Freeden 1984, Dierckx 1984, Bezhaev 1987). An algorithm based on B-splines was constructed in (Dierckx 1984), with the problem of poles specially solved. The energy functional was chosen as a sum of squares of defect of spline's derivatives at the nodes of B-splines. The problem of poles was more successfully solved in (Bezhaev 1987) on the basis of trigonometric T-splines. Here, the energy functional can be chosen in the manner similar to that in (Wahba 1981).

In (Freeden 1984, Wahba 1981), energy functionals are introduced in the explicit forms by means of the Laplace-Beltrami operators on the sphere. This is, to a certain extent, a generalization of Atteia's splines for the case of the sphere as very simple functionals approximate the energy of a thin plate described by the function on the sphere. But unfortunately, the solution to the problem of interpolation and smoothing can be obtained only in the form of a series. Therefore, (Wahba 1981) deduced an energy functional equivalent to the original one. The generalization of similar splines to non-spherical manifolds seems difficult to achieve.

6.3. Spline-Approximations in Thin Layer

In Sections 6.1, 6.2 the traces of analytical splines on smooth manifolds were considered. To find the trace on the manifold $\Gamma \subset R^{n-1}$ we need to solve an analytical or a finite element problem for the variational spline interpolation in any domain $\Omega \subset R^n$ which includes Γ. The number of arithmetic operations to find a spline and trace extremely depends on the size of the domain Ω, especially for the finite element case. Since the interpolation points form any h-net lying only in Γ, it is natural to require that Ω be any "thin layer" near Γ of the size h. But how to provide in this case the error estimates, which we have already obtained? The consideration of this question is the goal of this paragraph.

6.3.1. Analytical Approach

Let $\Gamma \subset R^{n-1}$ be a smooth bounded manifold and $\Omega_* \subset R^n$ be a bounded domain of the fixed size which includes Γ. Let us consider the family of the subdomains $\Omega \subset \Omega_*$, each including the manifold Γ. We suppose that Ω_* and cvery Ω have sufficiently smooth boundaries.

Let us introduce at Γ the Hilbert functional space $x(\Gamma)$ with the scalar product $(u, \vartheta)_{x(\Gamma)}$ and the norm $\|u\|_{x(\Gamma)} = (u, u)^{1/2}_{x(\Gamma)}$. We assume that the space $x(\Gamma)$ is continiously embedded to the space $C(\Gamma)$ of the continious functions in Γ,

$$\forall u \in x(\Gamma) \quad \|u\|_{C(\Gamma)} \leq K\|u\|_{x(\Gamma)}, \quad K = \text{const}. \tag{6.26}$$

Let ω_h be a set of points in Γ which forms h-net in Γ in the sense of the distance inside Γ, and $\varphi_* \in x(\Gamma)$ be the fixed function. The normal spline which interpolates this function on the mesh ω_h is the solution σ_h of problem

$$\begin{cases} a_h\sigma_h \overset{\mathrm{df}}{=} \sigma_h|_{\omega_h} = \varphi_*|_{\omega_h} = a_h\varphi_*, \\ \|\sigma_h\|^2_{x(\Gamma)} = \min. \end{cases} \tag{6.27}$$

Here a_h is the trace operator to the mesh ω_h. Then by the general theory (see Chapter 1) the resolvent spline-projector $S_h : x(\Gamma) \to x(\Gamma)$ can be written in the form

$$S_h = a_h^*(a_h a_h^*)^{-1} a_h \tag{6.28}$$

and error estimates for the normal spline $\sigma_h = S_h\varphi_*$ can be obtained in the usual way (see Chapter 3), but the structure of the spline σ_h may be too complicated for calculations.

Let $X(\Omega)$ be some Hilbert functional space in the domain Ω, which is also embedded to the space $C(\Omega)$. Denote by $P_\Omega : x(\Gamma) \to X(\Omega)$ the prolongation operator with the minimal $X(\Omega)$-norm; exactly $\forall u \in x(\Gamma)$ $P_\Omega u$ is the solution of the following variational problem: find $U_\Omega = P_\Omega u$ from the conditions: U_Ω in Γ is u and $\|U_\Omega\|_{X(\Omega)}$ is minimal. We assume that the operator P_Ω is bounded. If $p_\Omega : X(\Omega) \to x(\Gamma)$ is a trace operator (also bounded), then we have a spline interpolation problem: find $P_\Omega u$ from conditions

$$\begin{cases} p_\Omega(P_\Omega u) = u, \\ \|P_\Omega u\|^2_{X(\Omega)} = \min. \end{cases} \tag{6.29}$$

Hence, according to the general results from Chapter 1, the resolvent operator of this spline-problem is

$$P_\Omega = p_\Omega^*(p_\Omega p_\Omega^*)^{-1}. \tag{6.30}$$

Consider now the spline-interpolation problem: find $\Sigma_h^\Omega \in X(\Omega)$ from the conditions

$$\begin{cases} a_h\left(P_\Omega \Sigma_h^\Omega\right) = a_h\varphi_*, \\ \|\Sigma_h^\Omega\|^2_{X(\Omega)} = \min. \end{cases} \tag{6.31}$$

The trace $\sigma_h^\Omega = p_\Omega \Sigma_h^\Omega$ can be represented in the following form:

$$\sigma_h^\Omega = B_\Omega a_h^*(a_h B_\Omega a_h^*)^{-1} a_h\varphi_*, \tag{6.32}$$

where $B_\Omega = p_\Omega p_\Omega^* : x(\Gamma) \to x(\Gamma)$. Let

$$m_h^\Omega = a_h^*(a_h B_\Omega a_h^*)^{-1} a_h,$$

then

$$\begin{aligned} m_h^\Omega - S_h &= a_h^*[(a_h B_\Omega a_h^*)^{-1} - (a_h a_h^*)^{-1}]a_h \\ &= a_h^*(a_h a_h^*)^{-1} a_h(I - B_\Omega)a_h^*(a_h B_\Omega a_h^*)^{-1} a_h = S_h(I - B_\Omega)m_h^\Omega, \end{aligned}$$

therefore,

$$m_h^{\Omega}\varphi_* = S_h\varphi_* + S_h(I - B_{\Omega})m_h^{\Omega}\varphi_*.$$

Using $\|S_h\| \le 1$, we have

$$\|m_h^{\Omega}\varphi_*\|_{x(\Gamma)} \le \|\varphi_*\|_{x(\Gamma)} + \|I - B_{\Omega}\|_{x(\Gamma)\to x(\Gamma)} \cdot \|m_h^{\Omega}\varphi_*\|_{x(\Gamma)}.$$

Under the assumption $\|I - B_{\Omega}\|_{x(\Gamma)\to x(\Gamma)} < 1$ we obtain

$$\|\sigma_h^{\Omega}\|_{x(\Gamma)} \le \frac{\|B_{\Omega}\|_{x(\Gamma)\to x(\Gamma)}}{1 - \|I - B_{\Omega}\|_{x(\Gamma)\to x(\Gamma)}} \cdot \|\varphi_*\|_{x(\Gamma)}. \qquad (6.33)$$

The constant in this inequality is independent of h, but possibly dependent on Ω.

Denote the constants $C_1(\Omega)$ and $C_2(\Omega)$ by the formulae

$$\begin{aligned} C_2(\Omega) &= \|P_{\Omega}\|_{x(\Gamma)\to X(\Omega)}, \\ C_1(\Omega) &= 1/\|p_{\Omega}\|_{X(\Omega)\to x(\Gamma)}. \end{aligned} \qquad (6.34)$$

Then we have

$$\forall u \in x(\Gamma) \quad C_1(\Omega)\|u\|_{x(\Gamma)} \le \|P_{\Omega}u\|_{X(\Omega)} \le C_2(\Omega)\|u\|_{x(\Gamma)}. \qquad (6.35)$$

By formula (6.30) we obtain

$$\|P_{\Omega}u\|^2_{X(\Omega)} = (p^*_{\Omega}(p_{\Omega}p^*_{\Omega})^{-1}u, p^*_{\Omega}(p_{\Omega}p^*_{\Omega})^{-1}u)_{X(\Omega)} = (B_{\Omega}^{-1}u, u)_{x(\Gamma)}. \quad (6.36)$$

In other words,

$$\forall u \in x(\Gamma) \quad C_1^2(\Omega)\|u\|^2_{x(\Gamma)} \le (B_{\Omega}^{-1}u, u)_{x(\Gamma)} \le C_2^2(\Omega)\|u\|^2_{x(\Gamma)}, \qquad (6.37)$$

$$\forall u \in x(\Gamma) \frac{1}{C_2^2(\Omega)}\|u\|^2_{x(\Gamma)} \le (B_{\Omega}u, u)_{x(\Gamma)} \le \frac{1}{C_1^2(\Omega)}\|u\|^2_{x(\Gamma)}, \qquad (6.38)$$

because the operator B_{Ω} is self-adjoint. It means that

$$\|B_{\Omega}\|_{x(\Gamma)\to x(\Gamma)} = 1/C_1^2(\Omega),$$

$$\begin{aligned} \|I - B_{\Omega}\|_{x(\Gamma)\to x(\Gamma)} &= \sup_{u\ne 0} \frac{((I - B_{\Omega})u, u)_{x(\Gamma)}}{(u,u)_{x(\Gamma)}} \le 1 - \inf_{u\ne 0} \frac{(B_{\Omega}u, u)_{x(\Gamma)}}{(u,u)_{x(\Gamma)}} \\ &\le 1 - 1/C_2^2(\Omega). \end{aligned}$$

The last value is always less then one. Finally, inequality (6.33) can be transformed to

$$\|\sigma_h^{\Omega}\|_{x(\Gamma)} \le \frac{\|\varphi_*\|_{x(\Gamma)}}{C_1^2(\Omega)(1 - 1 + 1/C_2^2(\Omega))} = \left[\frac{C_2(\Omega)}{C_1(\Omega)}\right]^2 \times \|\varphi_*\|_{x(\Gamma)} \qquad (6.39)$$

or, in the other form,

$$\|\sigma_h^{\Omega}\|_{x(\Gamma)} \le \|P_{\Omega}\|^2_{x(\Gamma)\to X(\Omega)}\|p_{\Omega}\|^2_{X(\Omega)\to x(\Gamma)}\|\varphi_*\|_{x(\Gamma)}. \qquad (6.40)$$

The main question is: in what situation the constants in inequalities (6.37), (6.38) are independent of Ω? This problem can be solved by the special choice of norms in the spaces $X(\Omega)$.

Let ε be a real parameter, $0 < \varepsilon \leq 1$, and $\{\Omega_\varepsilon\}_{0<\varepsilon\leq 1}$ be some parametric family of domains with the sufficiently smooth boundaries such that $\Omega_1 = \Omega_*$, $\Omega_{\varepsilon_1} \subset \Omega_{\varepsilon_2}$, when $\varepsilon_1 < \varepsilon_2$, each domain Ω_ε includes the manifold Γ and mes $(\Omega_\varepsilon) \to 0$, when $\varepsilon \to 0$. Let us introduce a bijective and sufficiently smooth mapping $\varphi_\varepsilon : \Omega_1 \to \Omega_\varepsilon$ and define the norm in the Hilbert space $X(\Omega_\varepsilon)$ by the formula

$$\|U_\varepsilon(x_\varepsilon)\|_{X(\Omega_\varepsilon)} = \|U_\varepsilon(\varphi_\varepsilon(x))\|_{X(\Omega_1)}. \tag{6.41}$$

Then the element $\sigma_\varepsilon = P_{\Omega_\varepsilon} u, \quad u \in x(\Gamma)$ is the solution the prolongation problem

$$\begin{cases} p_{\Omega_\varepsilon}\sigma_\varepsilon = u \\ \|\sigma_\varepsilon\|_{X(\Omega_\varepsilon)} = \min \end{cases} \tag{6.42}$$

or, in other words,

$$\begin{cases} p_{\Omega_1}(\sigma_\varepsilon \circ \varphi_\varepsilon) = u \\ \|\sigma_\varepsilon \circ \varphi_\varepsilon\|_{X(\Omega_1)} = \min . \end{cases} \tag{6.43}$$

The symbol $\circ$ means the superposition of the functions. Thus, $\sigma_\varepsilon \circ \varphi_\varepsilon = P_{\Omega_1} u$ and $\sigma_\varepsilon = P_{\Omega_\varepsilon} u = (P_{\Omega_1} u) \circ \varphi_\varepsilon^{-1}$. Therefore

$$\begin{aligned} \|P_{\Omega_\varepsilon}\|_{x(\Gamma)\to X(\Omega_\varepsilon)} &= \sup_{\|u\|_{x(\Gamma)}=1} \|P_{\Omega_\varepsilon} u\|_{X(\Omega_\varepsilon)} = \sup_{\|u\|_{x(\Gamma)}=1} \|P_{\Omega_\varepsilon} u \circ \varphi_\varepsilon\|_{X(\Omega_1)} \\ &= \sup_{\|u\|_{x(\Gamma)}=1} \|P_{\Omega_1} u\|_{X(\Omega_1)} = \|P_{\Omega_1}\|_{x(\Gamma)\to X(\Omega_1)}. \end{aligned}$$

In the same way:

$$\begin{aligned} \|p_{\Omega_\varepsilon}\|_{X(\Omega_\varepsilon)\to x(\Gamma)} &= \sup_{\|U_\varepsilon\|_{X(\Omega_\varepsilon)}=1} \|p_{\Omega_\varepsilon} U_\varepsilon\|_{x(\Gamma)} = \sup_{U_\varepsilon \circ \varphi_\varepsilon \neq 0} \frac{\|p_{\Omega_1}(U_\varepsilon \circ \varphi_\varepsilon)\|_{x(\Gamma)}}{\|U_\varepsilon \circ \varphi_\varepsilon\|_{X(\Omega_1)}} \\ &= \|p_{\Omega_1}\|_{X(\Omega_1)\to x(\Gamma)}. \end{aligned}$$

Finally, the constant $[C_2(\Omega_\varepsilon)/C_1(\Omega_\varepsilon)]^2$ is independent of ε.

Example 1. Let us consider the square

$$\Omega_1 = [-0.5, 0.5] \times [-0.5, 0.5]$$

in the (x, y)-plane and Γ is the interval $[-0.5, 0.5]$ in the x-axis. Let $m > 1$ and $x(\Gamma) = W_2^{m-1/2}(-0.5, 0.5)$. The function from this space can be prolonged to the space $X(\Omega_1) = W_2^m(\Omega_1)$. Denote by

$$\Omega_\varepsilon = [-0.5, 0.5] \times [-\varepsilon/2, \varepsilon/2], \quad 0 < \varepsilon < 1.$$

The simple mapping $\varphi_\varepsilon : \Omega_1 \to \Omega_\varepsilon$ is as follows:

$$x_\varepsilon = x, \quad y_\varepsilon = \varepsilon y$$

and the special norm in the space $X(\Omega_\varepsilon) = W_2^m(\Omega_\varepsilon)$ is

$$\|U_\varepsilon(x_\varepsilon, y_\varepsilon)\|_{X(\Omega_\varepsilon)} = \|U_\varepsilon(x, \varepsilon y)\|_{W_2^m(\Omega_1)}$$

$$= \left[\int_{-0.5}^{0.5}\int_{-0.5}^{0.5}\left(U_\varepsilon^2(x,\varepsilon y) + \sum_{k=0}^{m}(\varepsilon^k \frac{\partial^m U_\varepsilon}{\partial x^{m-k}\partial y^k}(x,\varepsilon y))\right)^2 dx\, dy\right]^{1/2}$$

$$= \left[\int_{-0.5}^{0.5}\int_{-\varepsilon/2}^{\varepsilon/2}\left(\varepsilon^{-1}U_\varepsilon^2(x_\varepsilon, y_\varepsilon) + \sum_{k=0}^{m}\varepsilon^{2k-1}(\frac{\partial^m U_\varepsilon}{\partial x_\varepsilon^{m-k}\partial y_\varepsilon^k}(x_\varepsilon, y_\varepsilon))\right)^2 dx_\varepsilon dy_\varepsilon\right]^{1/2}.$$

Since the difference $\sigma_h^{\Omega_\varepsilon} - \varphi_*$ is equal to zero on h-net ω_h, we have the following error estimates for the trace of a spline in Γ

$$\begin{aligned}\|\sigma_h^{\Omega_\varepsilon} - \varphi_*\|_{W_p^\alpha(\Gamma)} &\le C h^{m-1/2-\alpha-1/2+1/p}\|\sigma_h^{\Omega_\varepsilon} - \varphi_*\|_{W_2^{m-1/2}(\Gamma)} \\ &\le C(1 + \|P_{\Omega_1}\|^2 \cdot \|p_{\Omega_1}\|^2)h^{m-1+1/p-\alpha} \cdot \|\varphi_*\|_{W_2^{m-1/2}(\Gamma)}.\end{aligned} \tag{6.44}$$

Here $m > 1$, $2 \le p \le \infty$, $\alpha - 1/p \le m - 1$ (except for $p = \infty$ & $\alpha = m - 1$). Finally, the constant in the last error estimate is independent of h and ε.

6.3.2. Finite Element Case

Let $X_\tau(\Omega)$ be a finite-dimensional subspace in the Hilbert space $X(\Omega)$ (for example, $X_\tau(\Omega)$ is the finite element space with the element size of the order τ) and B_τ^Ω be the corresponding orthoprojector from $X(\Omega)$ to $X_\tau(\Omega)$, and the natural condition takes place

$$\forall U \in X(\Omega) \quad \|U - B_\tau^\Omega U\|_{X(\Omega)} \to 0, \quad \tau \to 0. \tag{6.45}$$

We preserve the notations of 6.3.1 and assume that the interpolation condition $a_h u = a_h \varphi_*$ is not contradictory in the space $X_\tau(\Omega)$, i.e. the function $U_\Omega^\tau \in X_\tau(\Omega)$ exists such that

$$p_\Omega U_\Omega^\tau|_{\omega_h} = \varphi_*|_{\omega_h}, \quad \varphi_* \in x(\Gamma), \tag{6.46}$$

where p_Ω is the trace operator from $X(\Omega)$ to the manifold $\Gamma \subset \Omega$. Let us formulate the following problem: find the normal spline $\Sigma_{h,\tau}^\Omega \in X_\tau(\Omega)$ from conditions

$$\begin{cases} a_h(p_\Omega \Sigma_{h,\tau}^\Omega) = a_h\varphi_*, \\ \|\Sigma_{h,\tau}^\Omega\|_{X(\Omega)} = \min\limits_{X_\tau(\Omega)}. \end{cases} \tag{6.47}$$

By the general results of Chapter 4 we have

$$\Sigma_{h,\tau}^{\Omega} = B_\tau^\Omega p_\Omega^* a_h^* (a_h p_\Omega B_\tau^\Omega p_\Omega^* a_h^*)^{-1} a_h \varphi_* \tag{6.48}$$

and the trace of this spline on Γ is

$$\sigma_{h,\tau}^{\Omega} = p_\Omega \Sigma_{h,\tau}^{\Omega} = C_\tau^\Omega a_h^* (a_h C_\tau^\Omega a_h^*)^{-1} a_h \varphi_*, \tag{6.49}$$

where $C_\tau^\Omega = p_\Omega B_\tau^\Omega p_\Omega^*$. Let

$$m_{h,\tau}^{\Omega} = a_h^* (a_h C_\tau^\Omega a_h^*)^{-1} a_h.$$

Then

$$m_{h,\tau}^{\Omega} - S_h = a_h^*[(a_h C_\tau^\Omega a_h^*)^{-1} - (a_h a_h^*)^{-1}] a_h = S_h (I - C_\tau^\Omega) m_{h,\tau}^{\Omega}. \tag{6.50}$$

Therefore

$$m_{h,\tau}^{\Omega} \varphi_* = S_h \varphi_* + S_h (I - C_\tau^\Omega) m_{h,\tau}^{\Omega} \varphi_*. \tag{6.51}$$

It is clear that $m_{h,\tau}^{\Omega}\varphi_*$ belongs to the space of the interpolating normal splines $Sp(h,\Gamma)$ on the mesh $\omega_h \subset \Gamma$. Taking into account $\|S_h\|_{x(\Gamma)\to x(\Gamma)} \le 1$, we obtain

$$\begin{aligned} \|\sigma_{h,\tau(h)}^{\Omega}\|_{x(\Gamma)} &\le \frac{\|C_{\tau(h)}^{\Omega}\|_{Sp(h,\Gamma)\to x(\Gamma)}}{1 - \|I - C_{\tau(h)}^{\Omega}\|_{Sp(h,\Gamma)\to x(\Gamma)}} \cdot \|\varphi_*\|_{x(\Gamma)} \\ &\le (1-\Theta_0)^{-1} \|C_{\tau(h)}^{\Omega}\|_{x(\Gamma)\to x(\Gamma)} \cdot \|\varphi_*\|_{x(\Gamma)} \\ &\le (1-\Theta_0)^{-1} \|p_\Omega\|_{X(\Omega)\to x(\Gamma)}^2 \cdot \|\varphi_*\|_{x(\Gamma)}. \end{aligned} \tag{6.52}$$

It is clear that (see 6.2.1)

$$\begin{aligned} &\|I - p_\Omega B_\tau^\Omega p_\Omega^*\|_{Sp(h,\Gamma)\to x(\Gamma)} \le \|I - p_\Omega p_\Omega^*\|_{x(\Gamma)\to x(\Gamma)} \\ &\quad + \|p_\Omega (I - B_\tau^\Omega) p_\Omega^*\|_{Sp(h,\Gamma)\to x(\Gamma)} \le 1 - 1/\|P_\Omega\|_{x(\Gamma)\to X(\Omega)}^2 \\ &\quad + \|p_\Omega\|_{X(\Omega)\to x(\Gamma)} \cdot \|(I - B_\tau^\Omega) p_\Omega^*\|_{Sp(h,\Gamma)\to X(\Omega)}. \end{aligned} \tag{6.53}$$

Usually, the interpolating spline $\sigma_h \in Sp(h,\Gamma)$ belongs to the space $x^\beta(\Gamma)$ of smoother functions. For example, if $x(\Gamma) = W_2^{m-1/2}(\Gamma)$, then the normal spline σ_h on the mesh $\omega_h \subset \Gamma$ can be represented in the form

$$\sigma_h = \sum_{Q\in\omega_h} \lambda_Q k_Q, \tag{6.54}$$

where k_Q has the property

$$\forall u \in W_2^{m-1/2}(\Gamma) \quad (k_Q, u)_{W_2^{m-1/2}(\Gamma)} = u(Q), \tag{6.55}$$

$m-1/2 > (n-1)/2$, and $k_Q \in W_2^{m-1/2+\beta}(\Gamma)$ for $0 \le \beta < m-n/2$. Furthermore, it is obvious that for every $Q \in \omega_h$ we have

$$\forall U_\Omega \in X(\Omega) \quad (k_Q, p_\Omega U_\Omega)_{x(\Gamma)} = (p_\Omega^* k_Q, U_\Omega)_{X(\Omega)} = U_\Omega(Q) \tag{6.56}$$

and every normal spline in $X(\Omega)$ on the mesh ω_h can be written in the form

$$\Sigma_h = \sum_{Q\in\omega_h} \lambda_Q p^*_\Omega k_Q. \tag{6.57}$$

For the same reasons Σ_h belongs to the space $W_2^{m+\beta}(\Omega)$, $0 \le \beta < m - n/2$, and it is natural to assume that the orthoprojector B_τ^Ω has the following approximating property

$$\forall U_\Omega \in W_2^{m+\beta} \|U_\Omega - B_\tau^\Omega U_\Omega\|_{W_2^m(\Omega)} \le C_1\tau^\beta \|U_\Omega\|_{W_2^{m+\beta}(\Omega)} \tag{6.58}$$

and in a particalur case

$$\|p^*_\Omega\sigma_h - B_\tau^\Omega p^*_\Omega\sigma_h\|_{W_2^m(\Omega)} \le C_1\tau^\beta \|p^*_\Omega\sigma_h\|_{W_2^{m+\beta}(\Omega)}. \tag{6.59}$$

Usually error estimate (6.59) takes place in the local sense and the global estimate is obtained by the summation of local estimates. Therefore, the constant C_1 is independent of the size of the domain Ω.

The second assumption is: inequality

$$\|p^*_\Omega\sigma_h\|_{W_2^{m+\beta}(\Omega)} \le \frac{C_2}{h^\beta_{\min}} \|\sigma_h\|_{W_2^{m-1/2}(\Gamma)} \cdot \|p_\Omega\| \tag{6.60}$$

takes place with the constant C_2 independent of Ω and h, here $h_{\min}$ is the minimal distance between mesh points in ω_h. In this situation we have

$$\|(I - B_\tau^\Omega)p^*_\Omega\|_{Sp(h,\Gamma)\to X(\Omega)} = \sup_{\sigma_h\neq 0} \frac{\|p^*_\Omega\sigma_h - B_\tau^\Omega p^*_\Omega\sigma_h\|_{X(\Omega)}}{\|\sigma_h\|_{x(\Gamma)}}$$

$$\le \sup_{\sigma_h\neq 0} \frac{C_1\tau^\beta\|p^*_\Omega\sigma_h\|_{W_2^{m+\beta}(\Omega)}}{\|\sigma_h\|_{W_2^{m-1/2}(\Gamma)}} \le C_1 \cdot C_2(\tau/h_{\min})^\beta \cdot \|p_\Omega\|.$$

Finally, to provide (6.53) we need

$$1 - 1/\|P_\Omega\|^2 + C_1C_2\|p_\Omega\|^2(\tau/h_{\min})^\beta \le \Theta_0 < 1,$$

and we have the following inequality for two mesh parameters τ (a finite element mesh) and h (a scattered mesh)

$$\left(\frac{\tau}{h_{\min}}\right)^\beta \le \frac{1-(1-\Theta_0)\|P_\Omega\|^2}{C_1C_2 \cdot \|p_\Omega\|^2 \cdot \|P_\Omega\|^2}. \tag{6.61}$$

If we introduce in the space $X(\Omega)$ the special norm (see Section 6.1), then the constant in the right-hand side of (6.58) becomes independent of Ω, τ and $h_{\min}$ are proportional, and it is possible to solve the problem for the finite element analogue of the D^m-spline only in a sufficiently small thin layer near the manifold Γ, and the trace to Γ of this finite element analogue has the same accuracy as the exact normal spline, i.e.

$$\forall\varphi_* \in W_2^{m-1/2}(\Gamma)\ \|\sigma_h^\Omega - \varphi_*\|_{W_p^\alpha(\Gamma)} \le Ch^{m-n/2-\alpha+(n-1)/p}\|\varphi_*\|_{W_2^{m-1/2}(\Gamma)}$$

where the constant C is independent of φ_*, h and $\Omega \supset \Gamma$.