9. Optimal Approximation of Linear Operators

In this chapter, we introduce the general problem of optimal approximation of a linear operator by the value of another operator. The spline interpolating method helps us to solve this problem and to obtain numerical formulas of optimal approximation for a wide variety of functional spaces and linear operators. More exactly, it is possible when the reproducing kernels or mappings are known and effectively calculated. Besides, in these cases the exact estimates of errors on classes and extremal elements can be obtained, too.

Further, we give various examples which include the Sobolev cubature formulas, cubature formulas on a unit sphere and on subspaces. All these methods we consider from the point of view of the indefinite coefficient method. The main application of the theory are the cubature formulas for integration of mesh functions. For example, consider the integral

$$\int_{\Omega} \omega(X)u(X)dX,$$

where Ω is a domain or a manifold, $X \in \Omega$, $\omega(X)$ - weight function, $u(X)$ is the function known on a mesh only.

The method of integration may consist in the preliminary construction of coefficients of cubature formula and its application to a concrete mesh function. The second method consists in the interpolation of the mesh function and subsequent explicit integration of analytic representation. The first method is more preferable if repeated calculations are produced with the help of the same cubature formula.

9.1. General Approach

Let X, Z be Hilbert spaces, V be a Banach space, $A : X \to Z$ and $G : X \to V$ be linear continuous operators. Let there be given the element Ax from Z, but we do not know x. The question arises how to find the value Gx on such an information.

It easy to see that if $V = X$ and G is equal to the identity operator, then the problem consists in inversion of the operator A. In general, the problem of finding Gx has not a unique solution, but we formulate a very simple and natural variational problem which leads to an approximate solution, defined with the help of the interpolating splines.

For the operator A, choose the linear operator $T : X \to Y$ in such a way that (T, A) would be a spline-pair (see Chapter 1). Denote by $S : Z \to X$ the operator of spline-interpolation. Then the following properties of the operator of spline-interpolation are valid.

1. S is a right inverse operator for the operator A, i.e. for every $z \in Z$ $ASz = z$.
2. S is a left inverse operator for the operator A in the subspace $N(T)$, i.e. for every $t \in N(T)$ $SAt = t$.
3. (Minimal energy property). For every $u \in X$ $\|TSAu\|_Y \le \|Tu\|_Y$.
4. (Orthogonal property). For every $u \in X$

$$\|T(u - SAu)\|_Y^2 = \|Tu\|_Y^2 - \|TSAu\|_Y^2.$$

5. If $u - SAu \in N(T)$, then $u = SAu$.

Proof. Properties 1-4 are well-known from Chapter 1. Prove Property 5. According to Property 1 $A(u - SAu) = Au - ASAu = 0$. Consequently, $u - SAu \in N(T) \cap N(A)$. Since (T, A) is a spline-pair, then $N(T) \cap N(A) = \{0\}$ and $u - SAu = 0$. Property 5 is proved. □

Now we are ready to pose the variational problem for optimal approximation and to solve it.

Definition 9.1. *Operator $E_{\text{opt}} : Z \to V$ is called the optimal restoration operator for calculation of Gx by the value Ax, if it minimizes the following functional:*

$$\Phi(E) = \sup_{\|Tu\|_Y = 1} \|Gu - EAu\|_V \tag{9.1}$$

on the set of linear continuous operators $E \in \mathcal{L}(Z, V)$.

Theorem 9.1. $E_{\text{opt}} = GS$.

Proof. Let us verify that $\Phi(GS)$ is bounded. Since G is a linear bounded operator, then

$$\|Gu - GSAu\|_V \le \|G\| \cdot \|u - SAu\|_X. \tag{9.2}$$

The spline-pair defines the norm

$$\sqrt{\|Tu\|_Y^2 + \|Au\|_Z^2},$$

which is equivalent to the main norm. Thus, we have

$$\|u - SAu\|_X \le c\sqrt{\|T(u - SAu)\|_Y^2 + \|A(u - SAu)\|_Z^2}.$$

From Property 1 it follows $A(u - SAu) = 0$ and with the help of Property 4 we obtain

$$\|u - SAu\|_X \le c\|Tu\|_Y. \tag{9.3}$$

So from (9.1-9.3) we have $\Phi(GS) \le c\|G\|$. To prove that GS is the optimal operator, it is necessary to establish $\Phi(GS) \le \Phi(E)$ for any $E \in \mathcal{L}(Z, V)$ or, which is the same,

$$\sup_{\|Tu\|_Y=1} \|Gu - GSAu\|_V \le \sup_{\|Tv\|_Y=1} \|Gv - EAv\|_V.$$

It is sufficient for any $u \in X$ ($\|Tu\|_Y = 1$) to find $v \in X$ ($\|Tv\|_Y = 1$) with condition

$$\|Gu - GSAu\|_V \le \|Gv - EAv\|_V. \tag{9.4}$$

Let $u \in X$ ($\|Tu\|_Y = 1$) be an arbitrary element. Consider two opposite cases: the element u is an interpolating spline and it is not. In the first case, $SAu = u$ and inequality (9.4) is valid for all v.

In the second case take $\omega = u - SAu$. The element $T\omega$ is not equal to zero, because if $T\omega = 0$, then according to Property 5 $u = SAu$, and, consequently, u is an interpolating spline (case 1). So, we can consider the element

$$v = \frac{u - SAu}{\|T\omega\|_Y}.$$

It is easy to see that $\|Tv\|_Y = 1$. We have

$$\|Gv - EAv\|_V = \frac{1}{\|T\omega\|_Y}\|Gu - GSAu - E(Au - ASAu)\|_V.$$

Since $Au = ASAu$ and $E(0) = 0$, then

$$\|Gv - GSAv\|_V = \frac{1}{\|T\omega\|_Y}\|Gu - GSAu\|_V. \tag{9.5}$$

Using Property 4 we can easily obtain $\|T\omega\|_Y \le 1$:

$$\|T\omega\|_Y^2 = \|T(u - SAu)\|_Y^2 = \|Tu\|_Y^2 - \|TSAu\|_Y^2 \le \|Tu\|_Y^2 = 1.$$

Now from equality (9.5) follows (9.4) and the Theorem is proved. □

Remark 9.1. We need only the property $E(0) = 0$, and we do not fully use the linearity. Thus, $E_{opt} = GS$ is valid for a wide class of the non-linear operators.

9.2. Optimal Approximation of Linear Functionals

Let X be a semi-Hilbert space with the semi-norm $|\cdot|_P$ and the kernel of the semi-norm $P \subset X$. Connect with a linearly independent set of functionals $k_1, ..., k_N$ from X^* and the semi-norm $|\cdot|_P$ the space of splines, which are the solutions to problems

$$\begin{aligned} & k_i(u) = r_i, \quad i = 1, ..., N, \quad u \in X, \\ & \sigma = \arg\min |u|_P. \end{aligned} \tag{9.6}$$

Definition 9.2. *Let L be a linear continuous functional from X^*. The formula of approximation of the functional by the linear combination*

$$L \approx \sum_{i=1}^{N} a_i k_i \tag{9.7}$$

is said to be exact on the space of splines if any spline (9.6) obeys the equality

$$L(\sigma) = \sum_{i=1}^{N} a_i k_i(\sigma). \tag{9.8}$$

Theorem 9.2. The coefficients of formulae (9.7)-(9.8) are determined from the system of linear algebraic equations

$$\begin{bmatrix} G & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} L_\pi \\ L_P \end{bmatrix} \tag{9.9}$$

with the matrix which is a transpose of that of system (2.48). The vectors L_π and L_P have the components $L(\pi_P(k_i))$, $i = 1, ..., N$, and $L(p_i)$, $i = 1, ..., S$.

Proof. Write down systems of equations (2.48) and (9.9) in the abbreviated forms $H\boldsymbol{\Lambda} = \boldsymbol{R}$ and $H^T\boldsymbol{A} = \boldsymbol{L}$, respectively. Making use of the parentheses $(\cdot,\cdot)$ for denoting the scalar product of vectors, from representation (2.44) we obtain

$$\begin{aligned} L(\sigma) &= (\boldsymbol{\lambda}, L_\pi) + (\boldsymbol{c}, L_P) = (\boldsymbol{\Lambda}, \boldsymbol{L}) = (H^{-1}\boldsymbol{R}, \boldsymbol{L}) \\ &= (\boldsymbol{R}, (H^T)^{-1}\boldsymbol{L}) = (\boldsymbol{R}, \boldsymbol{A}) = (\boldsymbol{z}, \boldsymbol{a}) = \sum_{i=1}^{N} a_i L_i(\sigma) \end{aligned}$$

and this completes the proof of the Theorem. □

Definition 9.3. *Approximation formula (9.7) is said to be an interpolation one, if it is exact on the elements of the space P. The interpolation formula is called an optimal one, if its coefficients are chosen from the condition for the minimization of the following functional:*

$$\Phi(\boldsymbol{a}) = \sup_{u\in X,\ |u|_P=1} \left| L(u) - \sum_{i=1}^{N} a_i k_i(u) \right| . \tag{9.10}$$

Theorem 9.3. The optimal approximation formula is exact on the space of interpolating splines.

Proof. The interpolation condition for formula (9.1) can be written in the form $B^T\boldsymbol{a} = L_p$, thus coinciding with the second group of equations of system (9.9). Taking into account main property (2.7) of the reproducing mapping π_p, we can rewrite functional (9.10) in the form of scalar product

$$\Phi(a) = \sup_{u\in X,\ |u|_P=1} \left| (\pi_P(L - \sum_{i=1}^{N} a_i k_i), u)_P \right| . \tag{9.11}$$

It is obvious that the supremum is attained on the function

$$\pi_P(L - \sum_{i=1}^{N} a_i k_i) / \left| \pi_P(L - \sum_{i=1}^{N} a_i k_i) \right|_P$$

and is equal to

$$\Phi(\boldsymbol{a}) = \left| \pi_P(L - \sum_{i=1}^{N} a_i k_i) \right|_P . \tag{9.12}$$

Then, making use of the properties of the norm, we can conclude that the coefficients of the optimal formula can be determined as the solution to the following problem of the constrained optimization of quadratic functional:

$$\begin{aligned} &B^T\boldsymbol{a} = L_P, \\ &\Phi^2(\boldsymbol{a}) = (\pi_P(L - \sum_{i=1}^{N} a_i k_i), \pi_P(L - \sum_{i=1}^{N} a_i k_i))_P \to \min . \end{aligned} \tag{9.13}$$

For problem (9.13) make up the Lagrange function

$$\begin{aligned} D(\boldsymbol{a},\boldsymbol{d}) &= \frac{1}{2}\Phi^2(\boldsymbol{a}) + (\boldsymbol{d}, B^T\boldsymbol{a} - L_P) \\ &= \frac{1}{2}\Bigg[(\pi_P(L), \pi_P(L))_P - 2\sum_{i=1}^{N} a_i(\pi_P(L), \pi_P(k_i))_P \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j (\pi_P(L_i), \pi_P(k_j))_P \Bigg] + (d, B^T\boldsymbol{a} - L_P). \end{aligned}$$

Putting the partial derivative in the vector $\boldsymbol{d}$ equal to zero yields the second group of equations of system (9.9). Put the partial derivatives in the vector $\boldsymbol{a}$ equal to zero. We have

$$\frac{\partial D(\boldsymbol{a},\boldsymbol{d})}{\partial a_m} = -(\pi_P(L),\pi_P(k_m))_P + \sum_{i=1}^{N} a_i(\pi_P(k_i),\pi_P(k_m))_P + (P_l d)_m = 0,$$

$m = 1, ..., N.$

Making some manipulations with the expression

$$\frac{\partial D(\boldsymbol{a},\boldsymbol{d})}{\partial a_m} = -(\pi_P(L - \sum_{i=1}^{N} a_i k_i),\pi_P(k_m))_P + (P_l d)_m$$
$$= (\sum_{i=1}^{N} a_i k_i - L)(\pi_P(k_m)) + (P_l d)_m = 0,$$
$$m = 1, ..., N$$

we have

$$\sum_{i=1}^{N} a_i L_i \pi_P(k_m) + (P_l d)_m = L\pi_P(k_m), \quad m = 1, ..., N. \tag{9.14}$$

Equations (9.14) imply that the vectors $\boldsymbol{a}$ and $\boldsymbol{d}$ satisfy the first group of equations of (9.9) as well. This completes the proof of the Theorem. □

Definition 9.4. *Approximation formula (9.7) is called an optimal one on the subspace $E \subset X$, if it is an interpolation one and its coefficients are chosen from the condition for minimization of functional*

$$\Phi_E(a) = \sup_{u \in E,\ |u|_P = 1} |L(u) - \sum_{i=1}^{N} a_i k_i|. \tag{9.15}$$

Theorem 9.4. The coefficients of the optimal formula of approximation on the subspace E are determined from the following system of linear algebraic equations:

$$\begin{bmatrix} T & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} L(\pi) \\ \boldsymbol{0} \end{bmatrix} \tag{9.16}$$

with the matrix coincident with that of system of equations (7.22). The vector $L(\varphi)$ consists of the components $L(\varphi_1), ..., L(\varphi_M)$, where $\varphi_1, ..., \varphi_M$ is the basis of the space E.

Proof. By Theorem 9.3, the optimal formula is exact on the interpolating splines; in this case, it is exact on the splines on subspaces (see (7.19), where instead of the functionals L_i it is necessary to consider the functionals k_i). Pursue the proof of this Theorem in the way similar to that of Theorem 9.3. Write down systems (7.22) and (9.16) in the form $H\boldsymbol{\Lambda} = \boldsymbol{R}$ and $H\boldsymbol{A} = \boldsymbol{L}$, respectively. Making use of representation (7.21), we obtain

$$L(\sigma) = (\boldsymbol{d}, L(\varphi)) = (\boldsymbol{\Lambda}, \boldsymbol{L}) = (H^{-1}\boldsymbol{R}, \boldsymbol{L}) = (\boldsymbol{R}, H^{-1}\boldsymbol{L})$$
$$= (\boldsymbol{R}, \boldsymbol{\Lambda}) = (\boldsymbol{r}, \boldsymbol{a}) = \sum a_i k_i(\sigma).$$

This completes the proof of the Theorem. □

9.3. Prolongation of Mesh Functions and Cubature Formulas Based on Indefinite Coefficient Method

Let Ω be a compact in $\mathbb{R}^n$. The totality of chaotically located points $\{P_1, ..., P_N\}$ from Ω is said to be a scattered mesh. The function f is said to be a mesh function if it is defined only on the introduced scattered mesh, i.e. only the values $f(P_1), ..., f(P_N)$ are known.

Definition 9.5. *Prolongation method I for mesh functions is said to be given if the linear operator $I\colon \mathbb{R}^N \to \mathbb{R}^\Omega$ is defined which to any vector $(f(P_1), ..., f(P_N))$ puts into correspondence the function $\sigma : \Omega \to \mathbb{R}$. Prolongation method I is said to be interpolating if*

$$\sigma(P_i) = f(P_i), \quad i = 1, ..., N. \tag{9.17}$$

The remaining methods are said to be smoothing.

Most linear prolongation methods are constructed by using the indefinite coefficient method. Let $\omega_1(X), ..., \omega_M(X)$ be functions given in Ω. Denote by $r = (f(P_1), ..., f(P_N))^T$ the column vector of mesh values and by $\alpha = (\alpha_1, ..., \alpha_M)^T$ the column vector of real numbers.

Definition 9.6. *Let K be a non-singular $M \times M$ matrix and U be a rectangular $M \times N$ matrix. The prolongation by the indefinite coefficient method is said to be a function*

$$\sigma(X) = \sum_{i=1}^{M} \alpha_i \omega_i(X) \tag{9.18}$$

whose expansion coefficients α_i, $i = 1, ..., M$, are determined from the following linear algebraic system:

$$K\alpha = Ur. \tag{9.19}$$

Definition 9.7. *Let L be a linear functional on the space $\mathbb{R}^\Omega$. The approximation formula*

$$L(f) \simeq \sum_{i=1}^{N} a_i f(P_i) \tag{9.20}$$

is said to be exact for prolongation method I, if the coefficients $a_1, ..., a_N$ *are chosen from condition*

$$L(\sigma) = \sum_{i=1}^{N} a_i f(P_i) \tag{9.21}$$

for any $(f(P_1), ..., f(P_N)) \in \mathbb{R}^N$, $\sigma = I(f(P_1), ..., f(P_N))$.

Henceforth, expression (9.20) is said to be a cubature formula by analogy with the standard term when the functional to be approximated is an integral

$$L(f) = \int_V f(X)dX$$

in the bounded measurable subset V from Ω.

Theorem 9.5. The coefficients of the cubature formula exact for the indefinite coefficient method can be written in the form

$$\boldsymbol{a} = (a_1, ..., a_N)^T = U^T(K^T)^{-1}L_\omega, \tag{9.22}$$

where the matrices U^T and K^T are transposed to U and K, and the vector L_ω has the components $L(\omega_1), ..., L(\omega_M)$.

Proof. Denote by $(\ , \ )_M$ the scalar product in an M-dimensional space of vectors and by $\boldsymbol{\omega}$ the vector function $(\omega_1(X), ..., \omega_M(X))$. Then the statement of the Theorem is implied by the obvious equalities

$$L(\sigma) = L((\boldsymbol{\alpha}, \boldsymbol{\omega})_M) = (\boldsymbol{\alpha}, L_\omega)_M = (K^{-1}U\boldsymbol{r}, L_\omega)_M$$
$$= (\boldsymbol{r}, U^T(K^T)^{-1}L_\omega)_N = \sum_{i=1}^{N} a_i f(P_i).$$

□

Remark 9.2. If the prolongation method of the indefinite coefficient type is defined, the calculation of the corresponding cubature formula raises no difficulties. We must only know how to calculate components of the vector L_ω.

9.4. Cubature Formulas Based on Interpolating and Smoothing Prolongation Methods

9.4.1. Lagrange Method

Let Ω be a segment $[0,1]$ and the set $\{x_1 < \ldots < x_N\}$ be an arbitrary mesh of nodes on the segment [0, 1]. Set $M = N$ and choose for the functions $\omega_i(x)$ the basis monomials x^{i-1}, $i = 1, \ldots, N$.

It is known that for any values of $f(x_1), \ldots, f(x_N)$ there exists an interpolating Lagrange polynomial

$$\sigma(x) = \sum_{i=1}^{N} \alpha_i x^{i-1}$$

satisfying the conditions $\sigma(x_i) = f(x_i)$, $i = 1, \ldots, N$, whose coefficients can be determined from the system of equations

$$\begin{bmatrix} 1 & x_1 & \ldots & x_1^{N-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & x_N & \ldots & x_N^{N-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_N \end{bmatrix} = \begin{bmatrix} f(x_1) \\ \cdot \\ \cdot \\ \cdot \\ f(x_N) \end{bmatrix}.$$

Theorem 9.5 implies that the coefficients of the cubature formula

$$L(f) \simeq \sum_{i=1}^{N} a_i f(x_i)$$

corresponding to the Lagrange method are determined from the adjoint linear algebraic system

$$\begin{bmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_N \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ x_1^{N-1} & \ldots & x_N^{N-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_N \end{bmatrix} = \begin{bmatrix} L(1) \\ L(x) \\ \cdot \\ \cdot \\ \cdot \\ L(x^{N-1}) \end{bmatrix}. \qquad (9.23)$$

For the following three functionals $L(f) = \int_0^1 f(x)dx$, $L(f) = f(0)$, $L(1) = f'(0)$ the right-hand sides of system (9.23) are of the form

$$(1, 1/2, \ldots, 1/N)^T, \quad (1, 0, \ldots, 0)^T, \quad (0, 1, 0, \ldots, 0)^T.$$

9.4.2. Interpolation and Smoothing by D^m-Splines

Let $m > 0$ be an integer. Prescribe a class P_{m-1} of polynomials of total degree not exceeding $m-1$ of n variables with the basis functions

$$X^\gamma = x_1^{\gamma_1}...x_n^{\gamma_n}, \quad |\gamma| = \gamma_1 + ... + \gamma_n \le m-1 \tag{9.24}$$

whose dimension is equal to

$$Q = \binom{m-1+n}{n} = \frac{(m-1+n)!}{(m-1)!n!}.$$

Assume that the points $P_1, ..., P_N$ from $\mathbb{R}^n$ do not lie on the $(m-1)$-th order algebraic surface. Suppose that $m > n/2$.

Theorem 9.6. Let $0 < s < m$ and functions ω_i, $i = 1, ..., N$, be given of the following form

$$\omega_i(X) = (-1)^{[s]+1} \begin{cases} \|X - P_i\|^{2s}, & \text{s is not an integer} \\ \|X - P_i\|^{2s} \ln \|X - P_i\|, & \text{s is an integer} \end{cases} \tag{9.25}$$

where $[s]$ means the integer part of the number s. Let us order polynomials (9.24) in an arbitrary way $e_i(X)$, $\quad i = 1, ..., Q$. Then the matrix

$$K = \begin{bmatrix} G & E \\ E^T & 0 \end{bmatrix} = \begin{bmatrix} \omega_1(P_1) & ... & \omega_N(P_1) & e_1(P_1) & ... & e_Q(P_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \omega_1(P_N) & ... & \omega_N(P_N) & e_1(P_N) & ... & e_Q(P_N) \\ e_1(P_1) & ... & e_1(P_N) & 0 & ... & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e_Q(P_1) & ... & e_Q(P_N) & 0 & ... & 0 \end{bmatrix}$$

is non-singular, and function

$$\sigma(x) = \sum_{i=1}^{N} \alpha_i \omega_i(X) + \sum_{i=1}^{Q} c_i e_i(X) \tag{9.26}$$

with the coefficients $\alpha = (\alpha_1, ..., \alpha_N)^T$, $c = (c_1, ..., c_Q)^T$ satisfying system

$$\begin{bmatrix} G & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{0} \end{bmatrix} \tag{9.27}$$

determines the interpolating method of prolongation of the function f by the values $\boldsymbol{r} = (f(P_1), ..., f(P_N))^T$.

Theorem 9.7. Let I_N be the identity $N \times N$ matrix. For any $\lambda > 0$ matrix

$$K_\lambda = \begin{bmatrix} G + \lambda I_N & E \\ E^T & 0 \end{bmatrix} \tag{9.28}$$

is non-singular, and function (9.26) with the coefficients satisfying system

$$\begin{bmatrix} G + \lambda I_N & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ c \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{9.29}$$

defines the smoothing prolongation method.

These theorems summarize the results of (Duchon 1977). For the particular case $s = m - n/2$ they are contained in Section 5.3. The functions $\sigma(X)$ are the solutions to some variational problems and are called D^m-splines. Theorems 9.6-9.7 readily imply

Theorem 9.8. Depending on the prolongation method the coefficients of cubature formula (9.20), which exact either for interpolation or smoothing by D^m-splines are determined, correspondingly, from the following system:

$$\begin{bmatrix} G & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} L_\omega \\ L_e \end{bmatrix}, \quad \begin{bmatrix} G + \lambda I_N & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} L_\omega \\ L_e \end{bmatrix} \tag{9.30}$$

where $L_\omega = (L(\omega_1), ..., L(\omega_N))^T$, $L_e = (L(e_1), ..., L(e_Q))^T$, $\boldsymbol{d}$ is an auxiliary vector. The matrices K and K_λ are not transposed as they are symmetric.

Let $\Omega \subset \mathbb{R}^n$ be a bounded simply-connected domain with the Lipschitz boundary. Set $m > n/2$ and $s = m - n/2$, and let the set $\{P_1, ..., P_N\}$ form an h-net in the domain Ω. Then the estimates of approximation of the function $f \in W_2^m(\Omega)$ by the interpolating D^m-spline σ are of the form

$$\|D^k(\sigma - f)\|_{L^p(\Omega)} \leq ch^{m-k-n/2+n/p}\|D^m f\|_{L^2(\Omega)}. \tag{9.31}$$

Here, $k \geq 0$ and $p \geq 2$, are such that $k - n/p \leq m - n/2$ excluding the case of $p = \infty$ and $k = m - n/2$. The constant c is independent of the location of the points $P_1, ..., P_N$ and the parameter h.

Theorem 9.9. The cubature formula

$$L(u) = \int_\Omega u(X)dx \simeq \sum_{i=1}^N c_i u(P_i) \tag{9.32}$$

exact for interpolation by D^m-splines obeys the following error estimates:

$$\Delta L(f) = |\int_\Omega f(X)dX - \sum_{i=1}^N c_i f(P_i)| \leq ch^m\|D^m f\|_{L^2(\Omega)} \tag{9.33}$$

where $f \in W_2^m(\Omega)$.

Proof. From formula (9.21) we have

$$\Delta L(f) = |\int_{\Omega}(f(X) - \sigma(X))dX|. \tag{9.34}$$

Making use of the Schwarz inequality and interpolation estimates (9.31) we obtain

$$\Delta L(f) \leq (\text{mes} \quad \Omega)^{1/2}\|f - \sigma\|_{L^2(\Omega)} \leq (\text{mes} \quad \Omega)^{1/2} c h^m \|D^m f\|_{L^2(\Omega)}.$$

This completes the proof of the Theorem. □

Remark 9.3. The latter cubature formulae, whose coefficients are defined from equations (9.30) may be named the Sobolev cubature formulas because it was S.Sobolev who first invented them.

9.4.3. Approximation by Traces of D^m-Splines on the Sphere

Let us introduce the unit sphere

$$S_{n-1} = \{(x_1, ..., x_n) \in R^n : \quad x_1^2 + ... + x_n^2 = 1\}$$

of the space $\mathbb{R}^n$ and points $P_1, ..., P_N$ on the sphere S_{n-1}.

In Chapter 6, an algorithm has been proposed for prolongation of the mesh spherical function $f(P_1), ..., f(P_N)$ by using the spline traces on the sphere S_{n-1}. We formulate two following theorems, which are particular cases of more general theorems of Chapter 6.

Theorem 9.10. Assume that the points $P_1, ..., P_N$ are not the roots of the generalized polynomial of the system of functions

$$X^\gamma = x_1^{\gamma_1} ... x_n^{\gamma_n}, \quad m - 2 \leq |\gamma| \leq m - 1. \tag{9.35}$$

Then the traces of the interpolating and smoothing D^m-splines on the surface S_{n-1} are uniquely determined.

Remark 9.4. The matrices K and K_λ for $m > 2$ with the interpolating points on the sphere are singular. Nevertheless, systems (9.27) and (9.29) will be compatible. Hence, there will exist an affine space of interpolating D^m-splines for the mesh spherical function $f(P_1), ..., f(P_N)$. Note that any two D^m-splines will differ in the polynomial

$$(x_1^2 + ... + x_n^2 - 1)P(x)$$

where $P \in P_{m-3}$, which is annihilated on the sphere S_{n-1}. The trace is thus uniquely determined. The same is true for smoothing D^m-splines as well.

Remark 9.5. The singularity of the matrices K and K_λ does not exclude the possibility of using Theorem 9.5 for constructing coefficients of cubature formulas. We can show that the cubature formulae exact for approximation by using the traces of D^m-splines are determined from system (9.30). The solutions to these systems will be unique with respect to the vector $\boldsymbol{a}$.

Remark 9.6. It has been shown in Chapter 6 that for obtaining the non-singular matrices K and K_λ it is necessary to remove linearly dependent functions on the sphere S_{n-1} from linear expansion (9.26). It can be achieved by replacing bases (9.24) with (9.35).

Theorem 9.11. Let $m > n/2$, $s = m - n/2$ and the set $\{P_1, ..., P_N\}$ form an h-net on the sphere S_{n-1}. Then the estimates of approximation of the function $f \in H^{m-1/2}(S_{n-1})$ by the trace of the interpolating D^m-spline σ are of the form

$$\|\sigma - f\|_{H^k(S_{n-1})} \leq ch^{m-k-1/2}\|f\|_{H^{m-1/2}(S_{n-1})} \tag{9.36}$$

where $0 \leq k \leq m - 1/2$, $H^k(S_{n-1})$ is a Sobolev space of the fractional index on the sphere S_{n-1}.

Theorem 9.12. The cubature formula

$$L(u) = \int\limits_{S_{n-1}} u(X)dS_X \simeq \sum_{i=1}^{N} c_i u(P_i) \tag{9.37}$$

exact for interpolation by traces of D^m-splines obeys the error estimates

$$\Delta L(f) \leq ch^{m-1/2}\|f\|_{H^{m-1/2}(S_{n-1})}. \tag{9.38}$$

Proof. The proof is carried out in the same way as for Theorem 9.9:

$$\begin{aligned}\Delta L(f) &\leq \int\limits_{S_{n-1}} |\sigma(X) - f(X)|dS_X \leq (\text{mes } S_{n-1})^{1/2}\|\sigma - f\|_{H^0(S_{n-1})} \\ &\leq (\text{mes } S_{n-1})^{1/2} ch^{m-1/2}\|f\|_{H^{m-1/2}(S_{n-1})}.\end{aligned}$$

This completes the proof of the Theorem. □

9.4.4. Finite Element Approximation

Let $P_1, ..., P_N$ be points in Ω and $\omega_1, ..., \omega_M$ be a basis of the M-dimensional subspace of the functions $\mathbb{R}^\Omega$. Assume that the matrix $T = \{t_{ij}\}$, $i, j = 1, ..., M$ is symmetric positive semidefinite and the matrix $A = \{\omega_j(P_i)\}$, $i = 1, ..., N$, $j = 1, ..., M$, is of the rank N.

Lemma 9.1. If $\ker T \cap \ker A = \{0\}$, the matrices

$$K = \begin{bmatrix} T & A^* \\ A & 0 \end{bmatrix}, \quad K_\alpha = \alpha T + A^* A, \quad \alpha > 0$$

are non-singular.

Definition 9.8. *The function*

$$\sigma(x) = \sum_{i=1}^{M} \sigma_i \omega_i(X) \tag{9.39}$$

is said to be a finite element interpolating spline if its coefficients $\boldsymbol{\sigma} = (\sigma_1, ..., \sigma_M)^T$ *can be determined from system*

$$\begin{bmatrix} T & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{r} \end{bmatrix} \tag{9.40}$$

and a finite element smoothing spline, if its coefficients can be determined from the system

$$(\alpha T + A^* A)\boldsymbol{\sigma} = A^* \boldsymbol{r}. \tag{9.41}$$

Lemma 9.2. If the rank $A \leq N$, $\ker T \cap \ker A = \{0\}$, system

$$\begin{bmatrix} T & A^*A \\ A^*A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{A} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ A^*\boldsymbol{r} \end{bmatrix} = \begin{bmatrix} 0 \\ A^* \end{bmatrix} \boldsymbol{r} \tag{9.42}$$

has a unique solution with respect to the vector σ.

Definition 9.9. *Function (9.39) is said to be a finite element quasi-interpolating spline if its coefficients can be determined from system (9.42). If the rank* $A = N$, *the quasi-interpolating spline coincides with the interpolating one.*

The proof of Lemmas 9.1 and 9.2 is trivial. The general scheme of construction of interpolating, smoothing and quasi-interpolating finite element splines is done in Chapter 4.

Theorem 9.13. Depending on the prolongation method the coefficients of cubature formula (9.20) exact for interpolation, smoothing or quasi-interpolation by finite element splines can be determined, correspondingly, in the following way:

$$\begin{bmatrix} T & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{a} \end{bmatrix} = \begin{bmatrix} L_\omega \\ \mathbf{0} \end{bmatrix}, \quad \boldsymbol{a} = A(\alpha T + A^* A)^{-1} L_\omega,$$

$$\boldsymbol{a} = [0 \quad A] \begin{bmatrix} T & A^*A \\ A^*A & 0 \end{bmatrix}^{-1} \begin{bmatrix} L_\omega \\ \mathbf{0} \end{bmatrix}.$$

Proof. The proof is carried out similarly to that of Theorem 9.5. □

9.5. Exact Integration of Certain Special Functions

9.5.1. Exact Integration of Radial Functions $\|X - P_i\|^{2s}$ and $\|X - P_i\|^{2s} \ln \|X - P_i\|$ on the Unit Sphere S_{n-1}, $n \geq 3$

To construct cubature formulas on the surface of the sphere

$$L(u) = \int_{S_{n-1}} u(X)dS \simeq \sum_{i=1}^{N} a_i u(P_i) \tag{9.43}$$

by using the traces of D^m-splines (see Subsection 9.4.3), it is necessary to know the value of the vector L_ω, i.e. it is necessary to find integrals of radial functions. Before passing on to calculations we recall the Catalan integration formula for $n \geq 3$ (Fikhtengolts 1969):

$$\int_{S_{n-1}} f((P,X))dS = 2\frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} f(\|P\|u)(1-u^2)^{(n-3)/2}du \tag{9.44}$$

and the basic properties of B- and Γ-functions (Gradshtein, Ryzhik 1971):

$$\int_{-1}^{1}(1-x)^\alpha(1-x)^\beta dx = 2^{\alpha+\beta+1}B(\beta+1,\ \alpha+1),\ \alpha > -1,\ \beta > -1 \tag{9.45}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x) \tag{9.46}$$

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(n) = (n-1)! \tag{9.47}$$

$$\Gamma(\frac{1}{2}) = \pi, \quad \Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}(2n-1)!!}{2^n}. \tag{9.48}$$

First, calculate

$$L(\omega_i) = \int_{S_{n-1}} \|X - P_i\|^{2s}dS.$$

We have

$$L(\omega_i) = \int_{S_{n-1}} (\|X\|^2 - 2(X,P_i) + \|P_i\|^2)^s dS$$

and as the points X and P_i lie on the unit sphere S_{n-1}, the integral can be reduced to a one-dimensional one by using the Catalan formula in the following way:

$$L(\omega_i) = \int\limits_{S_{n-1}} (2 - 2(X, P_i))^s dS$$

$$= 2^{s+1} \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int\limits_{-1}^{1} (1-u)^{s+(n-3)/2}(1+u)^{(n-3)/2} du.$$

Making use of (9.45) and (9.46) we reduce the last expression to the following one:

$$L(\omega_i) = 2^{s+1} \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} 2^{s+n-2} B(s + \frac{n+1}{2}, \frac{n-1}{2})$$

$$= 2^{2s+n-1} \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \frac{\Gamma(s+\frac{n-1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(s+n-1)}$$

$$= 2^{2s+n-1} \pi^{(n-1)/2} \frac{\Gamma(s+\frac{n-1}{2})}{\Gamma(s+n-1)}.$$

Thus, we have the general formula

$$\int\limits_{S_{n-1}} \|X - P_i\|^{2s} dS = 2^{2s+n-1} \pi^{(n-1)/2} \frac{\Gamma(s+\frac{n-1}{2})}{\Gamma(s+n-1)} \tag{9.49}$$

which for odd $n = 2k+1$ can be simplified as:

$$\int\limits_{S_{2k}} \|X - P_i\|^{2s} dS = \frac{2^{2(s+k)}\pi^k}{(s+k)(s+k+1)...(s+2k-1)}. \tag{9.50}$$

To find the integral

$$\int\limits_{S_{n-1}} \|X - P_i\|^{2s} \ln \|X - P_i\| dS \tag{9.51}$$

differentiate both sides of equality (9.49) with respect to the parameter s. We have

$$\frac{\partial}{\partial s} \left(\int\limits_{S_{n-1}} \|X - P_i\|^{2s} dS \right) = \int\limits_{S_{n-1}} \|X - P_i\|^{2s} \ln \|X - P_i\| dS.$$

Thus,

$$\int\limits_{S_{n-1}} \|X - P_i\|^{2s} \ln \|X - P_i\| dS$$
$$= \frac{\partial}{\partial s} \left(2^{2s+n-1} \pi^{(n-1)/2} \frac{\Gamma(s+\frac{n-1}{2})}{\Gamma(s+n-1)} \right). \tag{9.52}$$

The exact representation of the derivative can be easily found in case of $n = 2k+1$ where the explicit differentiation of expression (9.50) is possible.

9.5.2. Integration of Monomials X^α on the Unit Sphere S_{n-1}, $n \geq 3$

To construct cubature formulae (9.43) we additionally need the vector L_e (see (9.30)), i.e. we must know how to calculate integrals of the form

$$\int_{S_{n-1}} X^\alpha dS = \int_{S_{n-1}} x_1^{\alpha_1} ... x_n^{\alpha_n} dS \tag{9.53}$$

for any multi-index $\alpha = (\alpha_1, ..., \alpha_n)$.

Find these integrals again making use of the Catalan formula. We have

$$\int_{S_{n-1}} (X, P)^m dS = 2\frac{\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} \|P\|^m u^m (1 - u^2)^{(n-3)/2} du \tag{9.54}$$

but, on the other hand,

$$\int_{S_{n-1}} (X, P)^m dS = \sum_{|\alpha|=m} \frac{m!}{\alpha!} P^\alpha \int_{S_{n-1}} X^\alpha dS. \tag{9.55}$$

Let the point P 'run through' the space $\mathbb{R}^n$. Then expression (9.55) is a polynomial of n variables of degree m; in this case, monomials of less degree are not contained in the polynomial.

For odd $m = 2k + 1$ the integrand in (9.54) is odd in the variable u and, hence, the integral is equal to zero. Therefore, monomial (9.55) as well is identically equal to zero. It means that all its coefficients are zero, i.e. integrals (9.53) are equal to zero for odd $m = \alpha_1 + ... + \alpha_n$. Then prove that integrals (9.53) are equal to zero, even if at least one of the indices $\alpha_1, ..., \alpha_n$ is odd.

Consider the case of $m = 2k$. Then expression (9.54) is equal to

$$\frac{2\pi^{(n-1)/2}(p_1^2 + ... + p_n^2)^k}{\Gamma(\frac{n-1}{2})} \int_{-1}^{1} (u^2)^k (1 - u^2)^{(n-3)/2} du \tag{9.56}$$

i.e. it is a polynomial of degree $m = 2k$ containing only even degrees of the variables $p_1, ..., p_n$. Thus, if at least one of the indices $\alpha_1, ..., \alpha_n$ is odd, integrals (9.53) are equal to zero.

Taking into account the above-said, rewrite the right-hand side of (9.55):

$$\sum_{|\alpha|=k} \frac{(2k)!}{(2\alpha)!} P^{2\alpha} \int_{S_{n-1}} X^{2\alpha} dS. \tag{9.57}$$

Calculate the integral in (9.56) in the explicit form. We have (taking into account the change of variables $u^2 = (1 - v)/2$)

$$\int_{-1}^{1}(u^2)^k(1-u^2)^{(n-3)/2}du = 2\int_{0}^{1}(u^2)^k(1-u^2)^{(n-3)/2}du$$

$$= \frac{1}{2^{k+(n-2)/2}}\int_{-1}^{1}(1-v)^{k-1/2}(1+v)^{(n-3)/2}dv$$

$$= B(k+\frac{1}{2}, \frac{n-1}{2}) = \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(k+\frac{n}{2})}.$$

Expression (9.56) can be rewritten in the following way:

$$\frac{2\pi^{(n-1)/2}\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{n}{2})}\sum_{|\alpha|=k}\frac{k!}{\alpha!}P^{2\alpha}.$$

Comparing the last expression with (9.57) we have

$$\int_{S_{n-1}} X^{2\alpha}dS = 2\pi^{(n-1)/2}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{n}{2})}\frac{k!(2\alpha)!}{\alpha!(2k)!}$$

$$= 2\pi^{(n-1)/2}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+\frac{n}{2})}\frac{(2\alpha-1)!!}{(2k-1)!!}.$$

9.6. Discussion

Spline methods for constructing cubature formulas were studied by many authors. It bas been shown in (Laurent 1972) that the formulas exact for spline interpolation methods are optimal in the sense of Sard and Golomb-Weinberger. The spline methods are optimal for approximation of certain nonlinear functionals as well.

In the one-dimensional case of $n = 1$, there are results (Nikol'sky 1974; Mysovsky 1981) concerning the construction of cubature formulas with an optimal choice of coefficients $a_1, ..., a_n$ and nodes $P_1, ..., P_n$.

An alternative technique of construction of interpolating and smoothing spherical spline functions and also cubature formulas of integration on the sphere is the method contained in (Freeden 1981). However, the method given here seems to the authors to be simpler and more convenient. Both methods are likely to be optimal on the same spaces for different norms.

In addition to the considered functional-integral on the sphere S_{n-1}, in Section 9.5 we could consider a functional of the form

$$L(u) = \int_{S_{n-1}} P(X)u(X)dS,$$

where $P(X)$ is a polynomial. In this case, it is also possible to exactly define the vectors L_ω and L_e, i.e. exact integration of radial functions with a polynomial weight. There is, for example, a possibility of construction of cubature formulas for determining the Fourier coefficients of expansion of functions by using spherical harmonics.

Better possibilities for constructing cubature formulas arise in using finite element prolongation methods (see Subsection 9.4.4). In this case, the functions $\omega_1, ..., \omega_M$ have a simple polynomial form and the (exact or approximate) determination of the vector $L_\omega = (L(\omega_1), ..., L(\omega_M))$ does not raise great difficulties.