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A new Monte Carlo method for calculation of covariance function of solution of biharmonic equation*

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The article is devoted to the new Monte Carlo method for the calculation of covariance function of the solution of biharmonic equation when its right-hand side is a random field. The comparison of this method with the randomization algorithm of the Monte Carlo method (see [1]) is presented. The numerical results of the solution of the concrete problem are given.

Consider the problem of the oscillation of the plate in the bounded domain $G \subset R^2$ under the influence of the random field loading $\sigma = \sigma(\bar{x})$. The solution of the boundary value problem

$$\Delta^2 u(\bar{x}) = \sigma(\bar{x}), \quad u|_{\partial G} = \varphi(\bar{x}), \quad \Delta u|_{\partial G} = \psi(\bar{x}) \tag{1}$$

is also the random field. It needs to define its covariance function $v = E\{u(\bar{x})u(\bar{y})\}$ at the points x_0 and y_0. In the methods existed earlier the product of solutions in two given points for different realizations of the field $\sigma(\bar{x})$ was averaged (so called randomization algorithms of the Monte Carlo method (see [1])). The method given below enables us to use the covariance function $K(\bar{x}, \bar{y})$ of the field σ directly.

Multiply the initial equation by the same equation depending on the other variable y ($\Delta^2 u(\bar{y}) = \sigma(\bar{y})$). We obtain the expression

$$\Delta_x^2 u(\bar{x}) \Delta_y^2 u(\bar{y}) = \sigma(\bar{x})\sigma(\bar{y}),$$

where Δ_x is the Laplace operator, operating in variable $\bar{x}$.

Once the mathematical expectation from the both parts of the expression has been taken, we obtaine the equation in the unknown covariance function

$$\Delta_x^2 \Delta_y^2 v(\bar{x}, \bar{y}) = K(\bar{x}, \bar{y}).$$

Let us now rewrite it as a system of two equations

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$$\Delta_x^2 w(\bar{x},\bar{y}) = K(\bar{x},\bar{y}), \quad \Delta_y^2 v(\bar{x},\bar{y}) = w(\bar{x},\bar{y}). \tag{2}$$

The boundary conditions for this system are obtained in the following way:

$$\begin{aligned}
&w(\bar{x},\bar{y})\big|_{x\in\partial G} = \Delta_y^2 E\{u(\bar{x})(\bar{y})\}\big|_{x\in\partial G} = \varphi(\bar{x})E\{\Delta_y^2 u(\bar{y})\} = \varphi(\bar{x})E\{\sigma(\bar{y})\},\\
&\Delta_x w(\bar{x},\bar{y})\big|_{x\in\partial G} = \Delta_x\Delta_y^2 E\{u(\bar{x})(\bar{y})\}\big|_{x\in\partial G} = \psi(\bar{x})E\{\Delta_y^2 u(\bar{y})\} = \psi(\bar{x})E\{\sigma(\bar{y})\},\\
&v(\bar{x},\bar{y})\big|_{y\in\partial G} = E\{u(\bar{x})(\bar{y})\}\big|_{y\in\partial G} = \varphi(\bar{y})E\{u(\bar{x})\},\\
&\Delta_y v(\bar{x},\bar{y})\big|_{y\in\partial G} = \Delta_y E\{u(\bar{x})(\bar{y})\}\big|_{y\in\partial G} = \psi(\bar{y})E\{u(\bar{x})\}.
\end{aligned} \tag{3}$$

To define the latter two conditions we are to solve the equations

$$\Delta^2 E\{u(\bar{x})\} = E\{\sigma(\bar{x})\}, \quad E\{u\}\big|_{\partial G} = \varphi(\bar{x}), \quad \Delta E\{u\}\big|_{\partial G} = \psi(\bar{x}).$$

We solve each of the equations of system (2) using "walk on spheres" algorithm of the Monte Carlo method (see [1]).

Consider the second equation of the system. It is obvious that its solution satisfies the following integral relations

$$\begin{aligned}
v(\bar{x},\bar{y}_0) = &\int\limits_{S(\bar{y}_0,r)} v(\bar{x},\bar{s})ds - \frac{r^2}{4}\int\limits_{S(\bar{y}_0,r)} \Delta_s v(\bar{x},\bar{s})ds + \\
&\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^r z\frac{r^2 - z^2(\ln(r/z)+1)}{4} w(\bar{x},\bar{y}_0+(z\cos\varphi, z\sin\varphi))d\varphi dz,\\
\Delta_y v(\bar{x},\bar{y}_0) = &\int\limits_{S(\bar{y}_0,r)} \Delta_s v(\bar{x},\bar{s})ds - \\
&\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^r z\ln(r/z) w(\bar{x},\bar{y}_0+(z\cos\varphi, z\sin\varphi))d\varphi dz,
\end{aligned}$$

where $S(\bar{y}_0, r)$ is arbitrary circle with the centre at the point $\bar{y}_0$ entirely lying in G.

We obtain the system of integral equations

$$V = KV + F.$$

The random estimate of the solution is

$$\bar{\xi} = \sum_{n=0}^{N-1} Q_n F(\bar{y}_n) + Q_N F(\bar{y}_N), \tag{4}$$

where $\{\bar{y}_n\}_{n=0}^N$ is the "walk on spheres" Markov chain which terminates at the random point $\bar{y}_n$ in ε-neighbourhood of the boundary, Q_n are matrix weights

$$Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_n = Q_{n-1} \begin{bmatrix} 1 & -r^2/4 \\ 0 & 1 \end{bmatrix},$$

where r is the radius of the maximum inscribed circle with the centre at the point $\bar{y}_{n-1}$, $F(\bar{y}_N) = F(\bar{y}_*)$, where $\bar{y}_*$ is the boundary point nearest to the point $\bar{y}_N$. This estimate is ε-biased.

The integrals $F_i(\bar{y}_n)$ are estimated by the Monte Carlo method using one random point. As a density in the segment $[0, r]$ we choose the function $f = 4\rho \ln(r/\rho)/r^2$, therefore the random estimate is

$$\begin{aligned} F_1(\bar{y}_n) &= \frac{r^2 - \rho^2(\ln(r/\rho)+1)}{4} \cdot \frac{r^2}{4\ln(r/\rho)} w(\bar{x}, \bar{y}_n + \rho\bar{\omega}), \\ F_2(\bar{y}_n) &= -\frac{r^2}{4} w(\bar{x}, \bar{y}_n + \rho\bar{\omega}), \end{aligned} \tag{5}$$

where ρ is the random variable, distributed with the density f, $\bar{\omega}$ is the unit isotropic vector.

Thus, to find the solution of the problem $v(\bar{x}_0, \bar{y}_0)$, we need the values of function $w(\bar{x}_0, \bar{y}_n + \rho\bar{\omega})$, $n = 0, \ldots, N-1$. They are found from the first equation of system (2) by means of the method under consideration, using only one Markov chain $\{\bar{x}_m\}_{m=0}^M$ for every w.

For this the "walk on spheres" Markov chain is simulated starting from the point $\bar{x}_0$: $\{\bar{x}_m\}_{m=0}^M$. The random estimates for $w(\bar{x}_0, \bar{y}_n + \rho_n\bar{\omega}_n)$ by analogy with estimate (4) are

$$\bar{\zeta}_n = \sum_{m=0}^{M-1} Q_m F(\bar{x}_m) + Q_M F(\bar{x}_M). \tag{6}$$

Here the following random estimates for the components of the vector $F_i(\bar{x}_m)$

$$\begin{aligned} F_1(\bar{x}_m, \bar{y}_n + \rho_n\bar{\omega}_n) &= \frac{r^2 - \rho^2(\ln(r/\rho)+1)}{4} \cdot \frac{r^2}{4\ln(r/\rho)} \times \\ &\qquad K(\bar{x}_m + \rho\bar{\omega}, \bar{y}_n + \rho_n\bar{\omega}_n), \\ F_2(\bar{x}_m, \bar{y}_n + \rho_n\bar{\omega}_n) &= -\frac{r^2}{4} K(\bar{x}_m + \rho\bar{\omega}, \bar{y}_n + \rho_n\bar{\omega}_n) \end{aligned} \tag{7}$$

are used, where ρ is the random variable distributed with the density $f = 4\rho\ln(r/\rho)/r^2$, $\bar{\omega}$ is the unit isotropic vector. Furthermore, we see from (6), (7) that every $\bar{\zeta}_n$ can be calculated using only the Markov chain $\{\bar{x}_m\}_{m=0}^M$.

This algorithm is possible to realize in the following way:

1. The Markov chain $\{\bar{y}_n\}_{n=0}^N$ and the random values ρ_n, distributed with the density f are simulated. Thus, points $\bar{y}_n + \rho_n\bar{\omega}_n$ are fixed.
2. The Markov chain $\{\bar{x}_m\}_{m=0}^M$ and values ρ_m, distributed with the density f are simulated.
3. The integrals $F_i(\bar{x}_m)$ are calculated according to (7).
4. The random estimates $\bar{\zeta}_n$ are found using (6).
5. The integrals $F_i(\bar{y}_n)$ are calculated according to (5), where $w(\bar{x}_0, \bar{y}_n + \rho_n\bar{\omega}_n)$ defines the non-biased estimate $\bar{\zeta}_n$.
6. Solution of the initial problem is found from (4).

The efficiency of the given algorithm (A) was examined in comparison with the algorithm B (see [1]) for numerical solutions of the problem of the oscillation of the plate $G = \{x_1, x_2 : 0 \le x_1, x_2 \le 1\}$ simply supported at the boundary and the random field σ, having the spectral density $p(\lambda) = \frac{\alpha^2}{2\pi}e^{-\alpha|\lambda|}$.

Here, taking into consideration the piecewise linear boundaries, the boundary value problem is as follows:

$$\Delta^2 u(\bar{x}) = \sigma(\bar{x}), \quad u\big|_{\partial G} = \Delta u\big|_{\partial G} = 0.$$

System (2), (3) becomes

$$\Delta_x^2 w(\bar{x}, \bar{y}) = K(\bar{x}, \bar{y}), \quad \Delta_y^2 v(\bar{x}, \bar{y}) = w(\bar{x}, \bar{y}),$$
$$w(\bar{x}, \bar{y})\big|_{x\in\partial G} = \Delta_x w(\bar{x}, \bar{y})\big|_{x\in\partial G} = v(\bar{x}, \bar{y})\big|_{y\in\partial G} = \Delta_y v(\bar{x}, \bar{y})\big|_{y\in\partial G} = 0.$$

The randomised spectral model (see [2])

$$\sigma(\bar{x}) = \sum_{k=1}^{M} p_k^{1/2}[\xi_k \sin(\lambda_k\bar{x}) + \eta_k \cos(\lambda_k\bar{x})] \tag{8}$$

was used in the algorithm B for simulating σ, where ξ_k, η_k are independent standard Gaussian variables, $p_k = \int_{\Lambda_k} p(\lambda)d\lambda$, where $\Lambda_1, \ldots, \Lambda_M$ is some arbitrary partition of the plane R^2, and $\lambda_1, \ldots, \lambda_M$ are distributed in these parts in accordance with the densities $p_k(\bar{x}) = p(\lambda)/p_k$.

Let us prove that the variance of the algorithm A is less than the variance of the algorithm B. In the second case the random estimate of the solution is the product of two estimates

$$\bar{\xi}_x = \sum_{n=0}^{N-1} Q_n I_n \sigma(\bar{x}_n + \rho_n \bar{\omega}_n), \quad \bar{\xi}_y = \sum_{m=0}^{M-1} Q_m I_m \sigma(\bar{y}_m + \rho_m \bar{\omega}_m),$$

where

$$I_n \sigma(\bar{x}_n + \rho_n \bar{\omega}_n) = F(\bar{x}_n), \quad I_m \sigma(\bar{y}_m + \rho_m \bar{\omega}_m) = F(\bar{y}_m).$$

Their multiplication gives

$$\bar{\zeta} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} Q_n Q_m I_n I_m \sigma(\bar{x}_n + \rho_n \bar{\omega}_n) \sigma(\bar{y}_m + \rho_m \bar{\omega}_m). \tag{9}$$

Let $M[\bar{\zeta} \mid z]$ be the conditional expectation of the random variable ζ under the condition that the points $\bar{x}_n + \rho_n \bar{\omega}_n$ and $\bar{y}_m + \rho_m \bar{\omega}_m$ are fixed. Therefore, we can use a well-known relation

$$D\bar{\zeta} = MD[\bar{\zeta} \mid z] + DM[\bar{\zeta} \mid z]. \tag{10}$$

For the algorithm **A** estimate (9) is as follows:

$$\bar{\zeta}_* = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} Q_n Q_m I_n I_m K(\bar{x}_n + \rho_n \bar{\omega}_n, \bar{y}_m + \rho_m \bar{\omega}_m).$$

It is evident that the first term of sum (10) becomes zero and therefore,

$$D\bar{\zeta}_* = DM[\bar{\zeta}_* \mid z] = DM[\bar{\zeta} \mid z] < D\bar{\zeta}.$$

Thus, the variance of the algorithm **A** is less than the variance of the algorithm **B**.

Table 1 gives the results of comparison of the algorithms **A** and **B**. The covariance function v for the points $x = (1/2, 1/2)$ and $y = (1/2 + d, 1/2)$ was estimated. The calculations were performed for $\varepsilon = 0.01$, parameter $\alpha = 1$, a number of samples $N = 3000$. When σ is simulated according to (8), the simplest case of $M = 1$ is used. Then $\Lambda = R^2$.

From this table it is seen that the computational cost of the algorithm **A** is $1.5 \div 3$ times less.

The method under consideration leads to the substantial efficiency of the calculations for those random field, whose simulation has the great computational cost.

As an example, the comparison of the results of the calculations of covariance functions at the points $x = (0.5, 0.5)$ and $y = (0.5, 0.4)$ with the help of the algorithms **A** and **B** in parameter $\alpha = 0.1$, $\varepsilon = 0.01$ and $N = 10000$ is presented (see Table 2).

Table 1

d	A			B		
	$v \cdot 10^{-5}$	ρ, %	s	$v \cdot 10^{-5}$	ρ, %	s
0.0	1.37	3.5	0.18	1.26	4.3	0.26
0.1	1.32	4.8	0.39	1.25	6.0	0.58
0.2	1.04	3.6	0.22	1.16	6.2	0.63
0.3	0.87	5.5	0.51	0.82	7.2	0.83
0.4	0.39	5.7	0.52	0.38	11.2	1.95

ρ is a relative standard deviation error, s is the computational cost.

Table 2

Algorithm	M	$v \cdot 10^{-6}$	ρ, %	s
B	1	0.89	21	6.6
	2	1.013	17	5.4
	4	1.09	13.3	4.3
	6	1.22	12.22	4.5
	8	1.055	11.18	4.5
A		1.05	2.4	0.086

M is a number of terms in formula (8).

References

[1] K.K. Sabelfeld, The Monte Carlo Method in Boundary Value Problems, Nauka, Moscow, 1989 (in Russian).

[2] G.A. Mikhailov, The Optimisation of the Weight Monte Carlo Methods, Nauka, Moscow, 1982 (in Russian).