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Numerical simulation of a special class of non-homogeneous Gaussian fields*

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We consider an algorithm for the stochastic simulation of the Gaussian three-dimensional fields with a discrete argument and with regard the dependence of horizontal correlation functions on the vertical coordinate. The area for uses for the algorithm in question for a specific class of correlation functions of the horizontal fields is investigated. Some examples of application of the algorithm for simulation of the three-dimensional air temperature in the atmosphere fields are given with regard for the real dependence of the horizontal correlations on altitude.

In this paper, some questions associated with the construction of algorithms for stochastic simulation of complexes of hydrometeorological fields are considered. These algorithms are intended for the solution of problems of the variational agreement of hydrothermodynamics and probability numerical models of atmospheric processes [1]. In the numerical simulation of the Gaussian fields with a discrete argument on finite difference grids, used for the construction of numerical hydrothermodynamic models, one of the main problem is the representation of a covariance matrix (or a covariance function) of the field. Since the dimension of a covariance matrix for such fields is extremely great, its complete representation on the basis of the available real information is practically impossible. Therefore, in the solution of practical problems of statistical meteorology [1, 2], one limits himself to the representation of a simplified covariance matrix. For example, for geopotential fields the representation of a correlation matrix as direct product of the vertical and the horizontal correlation matrices is acceptable (the covariance matrix turns, as a result of multiplication of values of the field, to the appropriate climatic standard deviations). Horizontally, the property of homogeneity and isotropy is valid for these fields with a sufficient degree of accuracy. In this case, the structure of a field depends on height weakly. Therefore, the algorithm for simulation of a field is reduced to one of the most well-known "algorithms on rows and columns" [1, 3]. Contrary to geopotential, the horizontal correlations of the temperature field essentially depend on height (or in the isobaric system of coordinates on the constant pressure level). In the present paper, a modification of this algorithm in-

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tended for the simulation of fields in which the horizontal correlations vary from level to level is considered.

1. For simplicity, let us consider the vertical section of a field $\{\xi_{ij}^p\}$ at fixed j and further use the notation $\{\xi_i^p\}$, $p = 1, \ldots, m$, $i = 1, \ldots, n$.

The method for the simulation of multidimensional fields [1, 3] applied in this paper is reduced to the following transformations. Let us have the Gaussian field $\{\xi_i^p\}$ with $M\xi_i^p = 0$, $M(\xi_i^p)^2 = 1$, and $M\xi_i^p\xi_j^q = 0$, if $p \neq q$, and $M\xi_i^p\xi_j^q = r_{ij}^{(p)}$ if $p = q$. Here $r_{ij}^{(p)}$ is an entry of a correlation matrix, or the correlation function $r_{ij}^{(p)} = r^{(p)}(|j-i|) = r^{(p)}(k)$, $k = 0, \ldots, n-1$.

Based on this field we build the field $\{\eta_i^p\}$ in which for any $i = 1, \ldots, n$ the following transformation is applied

$$\begin{pmatrix} \eta_i^1 \\ \eta_i^2 \\ \vdots \\ \eta_i^m \end{pmatrix} = A \begin{pmatrix} \xi_i^1 \\ \xi_i^2 \\ \vdots \\ \xi_i^m \end{pmatrix}, \tag{1}$$

where

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ a_{12} & a_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{1m} & a_{2m} & \ldots & a_{mm} \end{pmatrix}$$

is the lower triangular matrix of m dimension such that $AA^T = R_v$, where R_v is a correlation matrix of the vertical section of the field $\{\eta_i^p\}$. In this case, the correlation of the field $\{\eta_i^p\}$ at levels p $(p = 1, \ldots, m)$ is equal to

$$M\eta_i^p\eta_j^p = a_{p1}^2 r_{ij}^{(1)} + a_{p2}^2 r_{ij}^{(2)} + \ldots + a_{pp}^2 r_{ij}^{(p)} = \gamma_{ij}^{(p)}.$$

Thus, the correlations of the field $\{\xi_i^p\}$ are connected to correlations of the field $\{\eta_i^p\}$ by the relations

$$\begin{aligned} \gamma_{ij}^{(1)} &= a_{11}^2 r_{ij}^{(1)} \\ \gamma_{ij}^{(2)} &= a_{21}^2 r_{ij}^{(1)} + a_{22}^2 r_{ij}^{(2)} \\ &\ldots\ldots \\ \gamma_{ij}^{(m)} &= a_{m1}^2 r_{ij}^{(1)} + a_{m2}^2 r_{ij}^{(2)} + \ldots + a_{mm}^2 r_{ij}^{(m)}. \end{aligned} \tag{2}$$

If $r_{ij}^{(1)} = r_{ij}^{(2)} = \ldots = r_{ij}^{(m)}$, we have a standard method "on rows and columns" [1, 3]. Note that similar relations take place for correlation functions. Also, let us write down (2) in the matrix transformation form:

$$\vec{\gamma}_{ij} = B\vec{r}_{ij},$$

$$\vec{\gamma}_{ij} = \begin{pmatrix} \gamma_{ij}^{(1)} \\ \vdots \\ \gamma_{ij}^{(m)} \end{pmatrix}, \quad \vec{r}_{ij} = \begin{pmatrix} r_{ij}^{(1)} \\ \vdots \\ r_{ij}^{(m)} \end{pmatrix}, \quad B = \begin{pmatrix} a_{11}^2 & 0 & \dots & 0 \\ a_{12}^2 & a_{22}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{1m}^2 & a_{2m}^2 & \dots & a_{mm}^2 \end{pmatrix}.$$

The matrix B has the following properties:

(i1) $\sum_{i=1}^{p} a_{pi}^2 = 1$, i.e., the matrix B is a stochastic one;

(i2) $\lambda_1(B) = a_{11}^2 = 1$, $B\vec{e} = \lambda_1(B)\vec{e}$, $\lambda_i(B) = a_{ii}^2 \le 1$, $i = 1, \dots, m$, where $\lambda(B)$ is an eigenvalue of the matrix B, and $\vec{e} = (1, \dots, 1)^T$ is the eigenvector corresponding to $\lambda_1(B)$.

Thus, if we transform the field $\{\xi_i^p\}$ with per-row correlation matrices $R^{(p)} = (r_{ij}^{(p)})$ to the field $\{\eta_i^p\}$ with the help of (1), then the correlation matrices of its rows $\Gamma^{(p)} = (\gamma_{ij}^{(p)})$ are defined by relations (2).

When modeling the stochastic fields with a given correlation structure, the inverse problem is solved: let $\Gamma^{(1)}$, $\Gamma^{(2)}$, ..., $\Gamma^{(m)}$ are given correlation matrices of rows of the field $\{\eta_i^p\}$. What should be the correlation matrices of rows of the field $\{\xi_i^p\}$, that after transformation (1) will help to obtain a field $\{\eta_i^p\}$ with these required correlation matrices?

Let $\vec{\gamma}_{ij} = \vec{\gamma}_{ij}^* = (\gamma_{ij}^{*(1)}, \dots, \gamma_{ij}^{*(m)})^T$ be given correlations. Then the corresponding correlations $\vec{r}_{ij}^*$ are determined by the equation

$$\vec{r}_{ij}^* = C\vec{\gamma}_{ij}^*, \qquad C = (c_{ij}) = B^{-1}, \tag{3}$$

With regard for (i1), (i2), and (3), we write out the basic properties of the matrix C:

(j1) since $\lambda_i(C) = \lambda_i(B^{-1}) = 1/\lambda_i(B)$, $\lambda_{\max}(B) = \lambda_1(B) = 1$ and the corresponding eigenvector consists of units, then $\lambda_{\min}(C) = \lambda_1(C) = 1$ and from $C\vec{e} = \vec{e}$ it follows that $c_{p1} + c_{p2} + \ldots + c_{pp} = 1$;

(j2) since $c_{pp} = \lambda_p(C) \ge 1$, other c_{ij} may be of different signs, hence the matrix C is not stochastic, and not in all the cases the quantities r_{ij} make sense of the correlation coefficients forming a non-negatively determined correlation matrix.

2. In the given paragraph, a special class of the correlation matrices $\gamma^{(p)} = (\gamma_{ij}^{(p)})$, $p = 1, \dots, m$, will be considered as well as the conditions, at which the above-considered method has the solution.

Let

$$\gamma^{(p)} = (1-\varepsilon_p)\gamma + \varepsilon_p \gamma_0, \qquad p = 1,\dots .m, \tag{4}$$

where γ and γ_0 are any fixed correlation matrices. Then with regard for (3), the corresponding $r^{(i)}$ are determined by the equations

$$\begin{aligned} r^{(1)} &= c_{11}(1-\varepsilon_1)\gamma + c_{11}\varepsilon_1\gamma_0, \\ r^{(2)} &= [c_{21}(1-\varepsilon_1) + c_{22}(1-\varepsilon_2)]\gamma + (c_{21}\varepsilon_1 + c_{22}\varepsilon_2)\gamma_0, \\ &\dots\dots \\ r^{(m)} &= \left[\sum_{i=1}^{m} c_{mi}(1-\varepsilon_i)\right]\gamma + \left(\sum_{i=1}^{m} c_{mi}\varepsilon_i\right)\gamma_0. \end{aligned}$$

For the matrices $r^{(i)}$ have sense of correlation matrices, it is necessary that $r^{(i)}$ be positive definite, i.e., $\lambda(r^{(i)}) \geq 0$.

From [4, p. 143] it follows that

$$\lambda(r^{(i)}) \geq (1-x_i)\lambda(\gamma) + \lambda_{\min}(x_i\gamma_0), \qquad i = 1,\dots,m, \tag{5}$$

where $x_i = c_{i1}\varepsilon_1 + \ldots + c_{ii}\varepsilon_i$, $i = 1,\dots,m$, or, in the matrix form, $\vec{x} = C\vec{\varepsilon}$. Let us consider two cases:

I. Let $x_i \geq 0$. Then, with regard for (5), to fulfil the condition $\lambda(r^{(i)}) \geq 0$, it is sufficient that the following inequality hold:

$$x_i(\lambda(\gamma) - \lambda_{\min}(\gamma_0)) \leq \lambda(\gamma).$$

Depending on a sign of $\lambda(\gamma) - \lambda_{\min}(\gamma_0)$, we have:

- if $\lambda(\gamma) > \lambda_{\min}(\gamma_0)$, then

$$x_i \leq \frac{\lambda(\gamma)}{\lambda(\gamma) - \lambda_{\min}(\gamma_0)}. \tag{6}$$

- if $\lambda(\gamma) = \lambda_{\min}(\gamma_0)$, then the inequality is always fulfilled.
- if $\lambda(\gamma) < \lambda_{\min}(\gamma_0)$, then

$$x_i \geq \frac{\lambda(\gamma)}{\lambda(\gamma) - \lambda_{\min}(\gamma_0)}.$$

  In this case $\frac{\lambda(\gamma)}{\lambda(\gamma) - \lambda_{\min}(\gamma_0)} < 0$, but we assume that $x_i \geq 0$, hence the lower bound of x_i is equal to zero.

II. Let $x_i < 0$. Then with regard for (5) for performance of the condition $\lambda(r^{(i)}) \geq 0$, it is sufficient that the following inequality be valid:

$$x_i(\lambda(\gamma) - \lambda_{\max}(\gamma_0)) \leq \lambda(\gamma).$$

- if $\lambda(\gamma) > \lambda_{\min}(\gamma_0)$, then

$$x_i \le \frac{\lambda(\gamma)}{\lambda(\gamma) - \lambda_{\max}(\gamma_0)}.$$

In this case $\frac{\lambda(\gamma)}{\lambda(\gamma)-\lambda_{\max}(\gamma_0)} > 0$, but we assume that $x_i < 0$, hence the upper bound x_i is equal to zero.

- if $\lambda(\gamma) = \lambda_{\max}(\gamma_0)$, then the inequality is always fulfilled.
- if $\lambda(\gamma) < \lambda_{\max}(\gamma_0)$, then

$$x_i \ge \frac{\lambda(\gamma)}{\lambda(\gamma) - \lambda_{\max}(\gamma_0)}. \tag{7}$$

Since the function $f(x) = \frac{x}{x-a}$ (for $a > 0$) decreases, in inequalities (6), (7) it is necessary to replace the quantities $\lambda(\gamma)$ by $\lambda_{\max}(\gamma)$ and $\lambda_{\min}(\gamma)$, respectively.

Combining cases I and II, we obtain the final inequalities for $x_i = c_{i1}\varepsilon_1 + \ldots + c_{ii}\varepsilon_i$, $i = 1, \ldots, m$:

$$\frac{\lambda_{\min}(\gamma)}{\lambda_{\min}(\gamma) - \lambda_{\max}(\gamma_0)} \le c_{i1}\varepsilon_1 + \ldots + c_{ii}\varepsilon_i \le \frac{\lambda_{\max}(\gamma)}{\lambda_{\max}(\gamma) - \lambda_{\min}(\gamma_0)}. \tag{8}$$

Thus, since $\lambda_{\max}(\gamma) \ge 1$, $\lambda_{\min}(\gamma_0) \le 1$, then

$$\frac{\lambda_{\max}(\gamma)}{\lambda_{\max}(\gamma) - \lambda_{\min}(\gamma_0)} > 1, \qquad \frac{\lambda_{\min}(\gamma)}{\lambda_{\min}(\gamma) - \lambda_{\max}(\gamma_0)} \le 0.$$

Hence, choosing $x_i \in [0,1]$ we provide the condition $\varepsilon_i \in [0,1]$. It follows from $\vec{\varepsilon} = C^{-1}\vec{x} = B\vec{x}$ and from the fact that the matrix B is stochastic. If we choose $x_i < 0$ or $x_i > 1$, then it is possible to obtain the conditions $\varepsilon_i < 0$ or $\varepsilon_i > 1$. But in this case it is necessary in addition to verify the condition of the diagonal predominance of the matrix $r^{(i)}$.

3. The practical realization of the algorithm is reduced to approximation of real correlation functions at isobaric levels by functions of the form (4), where γ and γ_0 are any correlation functions. In this case, the solution is found with an arbitrary matrix R_v. But ε_i are not arbitrary. The area of changes of these parameters is determined by the system of inequalities (8). Therefore another approximate method for the simulation of the temperature fields with real correlation functions presented in Figure 1 (9,10), borrowed from work [2], was applied. From the given correlation functions $\gamma^{*(1)}(\rho)$, $\gamma^{*(2)}(\rho)$, ..., $\gamma^{*(m)}(\rho)$ the functions $r^{*(1)}(\rho)$, $r^{*(2)}(\rho)$, ..., $r^{*(m)}(\rho)$ were calculated with the help of (3) (for calculation of the corresponding entries of the matrix C, as R_v is a real correlation matrix of the vertical profiles of a temperature field for the mid-latitudes from [2] was taken). It has turned out that any of these functions is not correlation. Therefore they were approximated by the corresponding correlation functions from the set

$$\begin{aligned} r(\rho) &= e^{-\alpha\rho^2}, \\ r(\rho) &= e^{-\alpha\rho^2} J_0(\beta\rho), \\ r(\rho) &= e^{-\alpha\rho^2}(A + BJ_0(\beta\rho)), \quad A + B = 1, \end{aligned} \tag{9}$$

and, in addition, for each of the functions $r^{*(k)}(\rho)$ a suitable correlation function from this set was selected, and the corresponding parameters were chosen so that this function in the best way (in the root-mean square sense) approximated $r^{*(k)}(\rho)$ at discrete values ρ of the given interval.

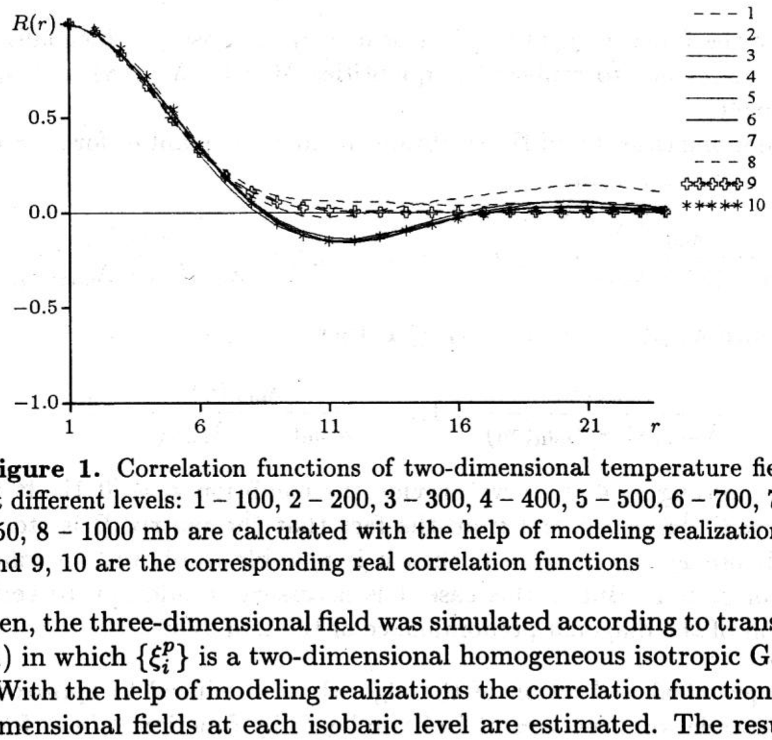

Figure 1. Correlation functions of two-dimensional temperature field at different levels: 1 – 100, 2 – 200, 3 – 300, 4 – 400, 5 – 500, 6 – 700, 7 – 850, 8 – 1000 mb are calculated with the help of modeling realizations, and 9, 10 are the corresponding real correlation functions

Then, the three-dimensional field was simulated according to transformation (1) in which $\{\xi_i^p\}$ is a two-dimensional homogeneous isotropic Gaussian field. With the help of modeling realizations the correlation functions of the two-dimensional fields at each isobaric level are estimated. The results are given in Figure 1 (functions 1–8). As it was noted above, in the same figure the real functions 9 and 10 from [2] are presented.

In conclusion, we consider an example, in which the correlation functions $\gamma^{(i)}$, $i = 1, \ldots, m$ $(m = 8)$, are given. Let us consider correlation functions of the two kinds:

$$\gamma^{(1)} = (1 - \varepsilon_1)\gamma + \varepsilon_1\gamma_0, \qquad \gamma^{(2)} = (1 - \varepsilon_2)\gamma + \varepsilon_2\gamma_0,$$

where γ and γ_0 are any correlation functions, and for $n \le m$ at levels i_1, i_2, $\ldots$, i_n they are equal to $\gamma^{(1)}$, and at other levels they are equal $\gamma^{(2)}$

(let in our concrete case, $\gamma^{(1)}$ correspond to levels 1, 2, 8, and $\gamma^{(2)}$ correspond to levels 3–7).

For simplicity, we solve the system of inequalities $0 \le x_i \le 1$ for $i = 1, \dots, m$, where $\vec{x} = (x_1, \dots, x_m)^T = C\vec{\varepsilon}$, and $\vec{\varepsilon} = (\varepsilon^{(1)}, \dots, \varepsilon^{(m)})^T$. Taking into account the fact that $\varepsilon^{(i_k)} = \varepsilon_1$ for $k = 1, \dots, n$ and $\varepsilon^{(j)} = \varepsilon_2$ (in our case $\varepsilon^{(1)} = \varepsilon^{(2)} = \varepsilon^{(8)} = \varepsilon_1$ and $\varepsilon^{(3)} = \dots = \varepsilon^{(7)} = \varepsilon_2$) and reducing similar components, we obtain the following system of inequalities

$$0 \le \Big(\sum_{i_k \le j} c_{ji_k}\Big)\varepsilon_1 + \Big(1 - \sum_{i_k \le j} c_{ji_k}\Big)\varepsilon_2 \le 1 \quad , \quad j = 1, \dots, m.$$

For an arbitrary matrix R_v this system of inequalities can be written down in the following form

$$a_j(\varepsilon_2) \le \varepsilon_1 \le b_j(\varepsilon_2), \qquad j = 1, \dots, m.$$

Thus, for any $\varepsilon_2 \in [0, 1]$ we obtain the inequality

$$\max_j a_j(\varepsilon_2) \le \varepsilon_1 \le \min_j b_j(\varepsilon_2),$$

which determines the area D of permissible values ε_1 and ε_2. In Figure 2, this area is given for a real correlation matrix R_v from work [2].

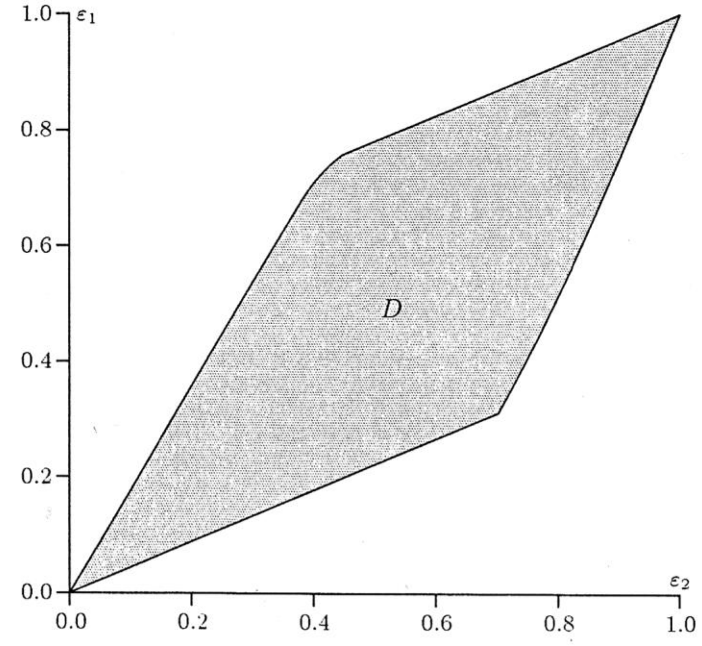

Figure 2. The area D of permissible values of the parameters ε_1 and ε_2

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