Bull. Nov. Comp. Center, Num. Anal., 9 (2000), 85–89

Dissipativity of generalized Maxwell–Leontovich operator

M.V. Urev

This paper completes the analysis of the Maxwell operator with impedance boundary conditions for arbitrary time dependence.

Let Ω be a finite, simply connected domain of the three-dimensional Euclidean space R^3, whose boundary S is sufficiently smooth. Let $T > 0$ and $Q_T = \Omega \times (0,T)$, $S_T = S \times (0,T)$; $\boldsymbol{n}$ is the inward unit normal to S.

We define $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^3$ as the space of square integrable fields over Ω and

$$\boldsymbol{H}^1(\Omega) = (H^1(\Omega))^3 = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega) : \nabla \boldsymbol{u} \in \boldsymbol{L}^2(\Omega)\}.$$

Let $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}^1(\Omega)$ be the usual Hilbert spaces equipped with the natural scalar products $(\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$ and $(\cdot,\cdot)_{\boldsymbol{H}^1(\Omega)}$, respectively.

We define the following subspaces of $\boldsymbol{L}^2(\Omega)$:

- $\boldsymbol{J}(\Omega)$ is the closure of the set $\tilde{\boldsymbol{J}}(\Omega)$ of smooth solenoidal vector-functions in $\boldsymbol{L}^2(\Omega)$;
- $\boldsymbol{J}_0(\Omega)$ is the closure in $\boldsymbol{L}^2(\Omega)$ of a set of smooth solenoidal vector-functions satisfying the condition $u_n|_S = 0$;
- $\mathcal{G}(\Omega)$ is a closure in $\boldsymbol{L}^2(\Omega)$ of the lineal $\widetilde{\boldsymbol{G}}(\Omega)$ of gradients of continuously differentiable in Ω harmonic function.

Introduce the following functional spaces:

$$\mathcal{H}(\Omega) = \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega), \qquad \mathcal{H}(Q_T) = L^2((0,T); \mathcal{H}(\Omega)),$$
$$\mathcal{H}(Q_R) = L^2(R; \mathcal{H}(\Omega)).$$

Let us also introduce the following subspaces of $\mathcal{H}(Q_T)$:

$$\mathcal{J}(Q_T) = \boldsymbol{J}(Q_T) \times \boldsymbol{J}(Q_T), \qquad \mathcal{J}_0(Q_T) = \boldsymbol{J}_0(Q_T) \times \boldsymbol{J}_0(Q_T),$$
$$\mathcal{G}(Q_T) = \boldsymbol{G}(Q_T) \times \boldsymbol{G}(Q_T),$$

where $\boldsymbol{J}(Q_T)$, $\boldsymbol{J}_0(Q_T)$, and $\boldsymbol{G}(Q_T)$ are the subspaces of $\boldsymbol{L}^2(Q_T)$ which elements for almost all $t \in (0,T)$ are in the subspaces $\boldsymbol{J}(\Omega)$, $\boldsymbol{J}_0(\Omega)$, and $\boldsymbol{G}(\Omega)$ respectively.

For the space $\mathcal{J}(Q_T)$, the following Weyl's decomposition into direct sum orthogonal to $\mathcal{H}(Q_T)$ holds:

$$\mathcal{J}(Q_T) = \mathcal{J}_0(Q_T) \oplus \mathcal{G}(Q_T).$$

We define Maxwell's operator $\mathcal{A} : \mathcal{J}(Q_T) \to \mathcal{J}(Q_T)$ with a new impedance boundary condition. The operator $\mathcal{A}$ is determined on the set $D(\mathcal{A})$: $\boldsymbol{u} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} \in \tilde{J}(Q_T) \times \tilde{J}(Q_T)$, where $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ satisfy the following boundary condition:

$$(\boldsymbol{u}_1 - C D^{\alpha}_{0+} \boldsymbol{u}_2 \times \boldsymbol{n}) \times \boldsymbol{n} = 0 \quad \text{on } S_T. \tag{1}$$

The boundary condition (1) can be written in the equivalent form

$$\Big(\boldsymbol{u}_2 + \frac{1}{C} I^{\alpha}_{0+} \boldsymbol{u}_1 \times \boldsymbol{n}\Big) \times \boldsymbol{n} = 0 \quad \text{on } S_T, \tag{2}$$

where we use the following notations: for $0 < \alpha < 1$

$$(D^{\alpha}_{0+} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\xi)\, d\xi}{(t-\xi)^{\alpha}}, \qquad (I^{\alpha}_{0+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\xi)\, d\xi}{(t-\xi)^{1-\alpha}}.$$

In [1], an impedance boundary condition for the Maxwell equations is constructed for a domain of vacuum bounded by a highly conductive medium. The mentioned condition is a partial case of the boundary condition (1) when $\alpha = 0.5$ and $C = \sqrt{\mu/(4\pi\sigma)}$, where μ is the magnetic permeability of the boundary S; σ is the conductivity of S. This boundary condition provides an approximation of the same order of accuracy as the classical Leontovich condition and extends the latter to the case of arbitrary time dependence.

For $\boldsymbol{u} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} \in D(\mathcal{A})$, define the operator $\mathcal{A}(\boldsymbol{u}) = \{\operatorname{rot} \boldsymbol{u}_2, -\operatorname{rot} \boldsymbol{u}_1\}$.

It is possible to prove (see [2]) that $D(\mathcal{A})$ is a dense set in $\mathcal{J}(Q_T)$. In [2], it was shown that the conjugate operator $\mathcal{A}^*$ is determined on the set $D(\mathcal{A}^*)$: $\boldsymbol{v} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} \in \tilde{J}(Q_T) \times \tilde{J}(Q_T)$, where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ satisfy the boundary condition

$$\Big(\boldsymbol{v}_2 - \frac{1}{C} I^{\alpha}_{T-} \boldsymbol{v}_1 \times \boldsymbol{n}\Big) \times \boldsymbol{n} = 0 \quad \text{on } S_T$$

and, for $\boldsymbol{v} = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} \in D(\mathcal{A}^*)$, $\mathcal{A}^* \boldsymbol{v} = -\{\operatorname{rot} \boldsymbol{v}_2, -\operatorname{rot} \boldsymbol{v}_1\}$. Here

$$(I^{\alpha}_{T-} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(\xi)\, d\xi}{(\xi - t)^{1-\alpha}}.$$

Note that $D(\mathcal{A}^*)$ is a dense set in $\mathcal{J}(Q_T)$. Then we can close the operator $\mathcal{A}$ in $\mathcal{J}(Q_T)$.

Definition. Denote the closure of the operator A in the space $\mathcal{J}(Q_T)$ by $\bar{\mathcal{A}}$ and call this operator as the generalized Maxwell–Leontovich operator.

Lemma. *For any function $f \in L_2(0,T)$ the following inequalities take place:*

$$\int_0^T (I_{0+}^{\alpha} f)(t) f(t)\, dt \geq 0, \tag{3}$$

$$\int_0^T (I_{T-}^{\alpha} f)(t) f(t)\, dt \geq 0. \tag{4}$$

Proof. The operator I_{0+}^{α} is bounded as operator from $L_2(0,T)$ to $L_2(0,T)$. The set $C_0^{\infty}(0,T)$ is dense in $L_2(0,T)$. Thus, it is sufficient to prove inequality (3) for the functions belonging to $C_0^{\infty}(0,T)$.

Let us take $f \in C_0^{\infty}(0,T)$. Using the equality

$$\int_0^t \frac{f(\xi)\, d\xi}{(t-\xi)^{1-\alpha}} = \frac{1}{\alpha} \int_0^t (t-\xi)^{\alpha} f'(\xi)\, d\xi$$

and the following expansion

$$(t-\xi)^{\alpha} = -t^{\alpha} \sum_{k=0}^{\infty} a_k t^{-k} \xi^k, \quad 0 \leq \xi \leq t,$$

where $a_0 = -1$, $a_1 = \alpha$, $a_k = \alpha(1-\alpha) \times \ldots \times (k-1-\alpha)/k!$, $k \geq 2$, we obtain

$$\begin{aligned}
(I_{0+}^{\alpha} f, f) &= \int_0^T (I_{0+}^{\alpha} f)(t) f(t)\, dt \\
&= C_1 \alpha \int_0^T \left[\int_0^t \frac{f(\xi)\, d\xi}{(t-\xi)^{1-\alpha}} \right] f(t)\, dt \\
&= -C_1 \sum_{k=0}^{\infty} a_k \int_0^T \left[\int_0^t \xi^k f'(\xi)\, d\xi \right] t^{\alpha-k} f(t)\, dt \\
&= -C_1 \int_0^T t^{\alpha} f^2(t)\, dt \sum_{k=0}^{\infty} a_k + \\
&\quad C_1 \sum_{k=1}^{\infty} k a_k \int_0^T \left[\int_0^t \xi^{k-1} f(\xi)\, d\xi \right] t^{\alpha-k} f(t)\, dt,
\end{aligned}$$

where $C_1 = (\alpha\Gamma(\alpha))^{-1}$.

The first term in the right-hand side is equal to zero because

$$\sum_{k=0}^{\infty} a_k = (1-1)^{\alpha} = 0.$$

The second term can be written in the form

$$C_1 \sum_{k=1}^{\infty} k a_k I_k. \tag{5}$$

Let us show that $I_k \geq 0$ for $k \geq 1$. Really,

$$I_k = \int_0^T \left[\int_0^t \xi^{k-1} f(\xi)\, d\xi\right] t^{\alpha-k} f(t)\, dt = \int_0^T \psi_k(t)\psi_k'(t) t^{\alpha+1-2k}\, dt$$
$$= \frac{1}{2} T^{\alpha+1-2k} \psi_k^2(T) - \frac{\alpha+1-2k}{2} \int_0^T \psi_k^2(t) t^{\alpha-2k}\, dt \geq 0.$$

Here $\psi_k(t) = \int_0^t \xi^{k-1} f(\xi)\, d\xi$, and the inequality holds due to the fact that $\psi_k(t)$ is equal to zero in a neighbourhood of zero. Further, the following inequality takes place

$$ka_k|I_k| \leq \frac{C_1 T^{\alpha+1}}{\alpha+1} a_k.$$

By virtue of this inequality, series (5) converges. Inequality (4) can be proved in a similar way. □

Remark. Inequalities (3) and (4) are valid for the function $f(t)$ taking values in a Hilbert space X. It means that we can change $L_2(0,T)$ for $L_2((0,T);X)$ in the lemma.

Theorem. *The generalise Maxwell-Leontovich operator $\bar{\mathcal{A}}$ is a dissipative operator in the space $\mathcal{J}(Q_T)$.*

Proof. To prove the theorem, we must obtain the following inequalities:

$$\int_0^T \langle \bar{\mathcal{A}} \boldsymbol{u}, \boldsymbol{u} \rangle_0(t)\, dt \leq 0 \quad \forall \boldsymbol{u} \in D(\bar{\mathcal{A}}), \tag{6}$$

$$\int_0^T \langle \bar{\mathcal{A}}^* \boldsymbol{v}, \boldsymbol{v} \rangle_0(t)\, dt \leq 0 \quad \forall \boldsymbol{v} \in D(\bar{\mathcal{A}}^*), \tag{7}$$

where $\langle \cdot, \cdot \rangle_0$ is the scalar product in $\mathcal{H}(\Omega)$.

If $\boldsymbol{u} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} \in D(\mathcal{A})$, then using the Green formula we obtain

$$\int_0^T \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u} \rangle_0(t)\, dt = \int_0^T \langle \{\operatorname{rot} \boldsymbol{u}_2, -\operatorname{rot} \boldsymbol{u}_1\}, \{\boldsymbol{u}_1, \boldsymbol{u}_2\} \rangle_0(t)\, dt$$
$$= \int_0^T \int_S \boldsymbol{n} \times \boldsymbol{u}_2 \cdot \boldsymbol{u}_1\, dS\, dt.$$

The boundary condition (2) can be written as

$$\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{u}_2(\boldsymbol{x}, t) = -C_1 \int_0^t \frac{\boldsymbol{u}_{1\tau}(\boldsymbol{x}, \xi)\, d\xi}{(t-\xi)^{1-\alpha}}, \quad (\boldsymbol{x}, t) \in S_T,$$

where $C_1 = (C\Gamma(\alpha))^{-1}$. Using this form of the boundary condition, we obtain

$$\int_0^T \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u}\rangle_0(t)\,dt = -C_1 \int_S\int_0^T \left[\int_0^t \frac{\boldsymbol{u}_{1\tau}(\boldsymbol{x},\xi)\,d\xi}{(t-\xi)^{1-\alpha}}\right]\cdot \boldsymbol{u}_1(\boldsymbol{x},t)\,dt\,dS. \quad (8)$$

It follows from the lemma that for any $\boldsymbol{x} \in S$

$$\int_0^T \left[\int_0^t \frac{\boldsymbol{u}_{1\tau}(\boldsymbol{x},\xi)d\xi}{(t-\xi)^{1-\alpha}}\right]\cdot \boldsymbol{u}_1(\boldsymbol{x},t)\,dt \geq 0. \quad (9)$$

From equation (8) and inequality (9) it follows that inequality (6) is valid for any $\boldsymbol{u} \in D(\mathcal{A})$. Let $\boldsymbol{u} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} \in D(\bar{\mathcal{A}})$. Then there exists a sequence $\{\boldsymbol{u}^n\} \subset D(\mathcal{A})$ such that $\boldsymbol{u}^n \to \boldsymbol{u}$ in $\mathcal{H}(Q_T)$ and $\mathcal{A}\boldsymbol{u}^n \to \bar{\mathcal{A}}\boldsymbol{u}$ in $\mathcal{H}(Q_T)$. Thus,

$$\int_0^T \langle \bar{\mathcal{A}}\boldsymbol{u}, \boldsymbol{u}\rangle_0(t)\,dt = \lim_{n\to\infty}\int_0^T \langle \mathcal{A}\boldsymbol{u}^n, \boldsymbol{u}^n\rangle_0(t)\,dt \leq 0.$$

Inequality (7) can be proved in a similar way. □

Dissipativity of the generalized Maxwell–Leontovich operator for $t \in (-\infty, +\infty)$, i.e., for $\mathcal{J}(Q_R)$, is proved in [3]. For the classical Maxwell–Leontovich operator the proof is given in [4].

References

[1] Urev M.V. Boundary conditions for Maxwell equations with arbitrary time dependence // Comp. Math. and Math. Phys. – 1997. – Vol. 37, №. 12. – P. 1444–1451.

[2] Urev M.V. On the generalized Maxwell–Leontovich operator // Vychisl. Mat. – Novosibirsk, 1996. – Vol. 5. – P. 115–127 (in Russian).

[3] Urev M.V. Dissipativity of generalized Maxwell–Leontovich operator for $t \in (-\infty, +\infty)$. – Novosibirsk, 1998. – (Preprint / RAN. Sib. Branch. of Comp. Math. and Math. Geoph.; 1120) (in Russian).

[4] Krein S.G., Kulikov I.M. On the Maxwell–Leontovich operator // Differ. Uravn. – 1969. – Vol. 5, №. 7. – P. 1275–1282 (in Russian).