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Coloured cause-effect structures

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We present an extension of the class of cause-effect structures by coloured tokens. As an example of coloured c-e structure we use the well-known problem of dining philosophers. Relationships between the classes of coloured c-e structures and coloured Petri Nets introduced by K. Jensen are investigated.

Introduction

In order to describe concurrent systems, L.Czaja in [1] has introduced cause-effect structures (CESs) inspired by condition/event Petri nets (PNs). CES can be defined as a triple (X, C, E) where X is the set of nodes, C and E are the cause and effect functions from X to the set of formal polynomials over X such that $x \in X$ occurs in $C(y)$ iff y occurs in $E(x)$. Each polynomial $C(x)$ ($E(x)$) denotes a family of cause (effect) subsets of the node x. The operator $*$ combines nodes into subsets, and the operator $+$ combines subsets into families.

Section 1 contains the basic definitions of the structure and behaviour of CESs.

There is an interesting question about the relationships between CESs and PNs. In [5] Raczunas states that every CES has strongly equivalent PN, i.e., two bijections exist: between the so-called firing components of CES and the transitions of PN, and between the nodes of CES and the places of PN; moreover, the bijections must preserve pre- and postsets of firing components and transitions. Raczunas investigated the converse mapping from PNs to CESs. He remarked that a strong equivalence is not the case for converse mapping. So he defines an equivalence relation between the places in PNs and, similarly, in CESs, as a coincidence of the presets of equivalent places as well as their postsets. Then he introduced an (ordinary) equivalence between PNs and CESs by means of weakening the requirement of bijective correspondence between places down to bijective correspondence between equivalent classes of places. But the issue remained open: to find a subclass of PNs or an extension of the class of CESs such that the strong equivalence will take place.

We have decided this problem in [7] by introducing an extension of cause-effect structures — two-level CESs (TCESs). TCESs are a convenient intermediate class between PNs and CESs, because they are strongly equivalent to the class of PNs and we can transform any TCES into structurally equivalent CES with the help of folding-transformation. On the other hand, each CES has a strongly equivalent TCES.

Unfortunately, an expressive power of CESs is not sufficient for using them in real-life applications. Some supplementary constructions are necessary. Then the problem is to extend the obtained results to CESs with time and/or coloured nodes or to other high-level semantics of CESs. Note that the extension with time for CESs has been received in [2], and the relationship between time CESs and time PNs has been investigated in [6].

In ordinary CESs, a token or an active state of a node represents the presence of control and/or some resource in it. But it does not signify a qualitative difference between resources functioning in the CES. Moreover, each node should not have more than one token-resource at the same time. In some cases it is important to distinguish the resources qualitatively. This difference is represented by colours of tokens. Moreover, each node can have more than one differently coloured token.

This work is devoted to construction of the class of coloured CESs (CCESs). There are different ways of introducing the semantics of coloured tokens in CESs. We choose one of them that strictly correlates with the semantics of coloured tokens in Petri nets ([3]).

Then we give an algorithm of mapping of CCESs into CPNs. The correctness of this mapping is established. An example of running the algorithm to solve the problem of n dining philosophers is considered. The problem of the converse mapping from coloured PNs into CCESs is decided with the help of the class of two-level CESs.

1. Cause-effect structures

Cause-effect structures are represented as directed graphs with an additional structure imposed on the set of nodes. These graphs with operations $+$ and $*$, corresponding to nondeterministic choice and parallelism, constitute a near-semi-ring where "near" means that distributivity of $*$ over $+$ holds conditionally. Fig.1 presents a simple example of CES (with arrows annotated by some executable statements and conditions) representing a parallel program that computes a sum of natural numbers x and y by adding and subtracting 1 and testing for 0.

A CES is completely represented by the set of annotated nodes: each node x is subscribed by a formal polynomial $E(x)$ built of (the names of) its successors and superscribed by a formal polynomial $C(x)$ built of its predecessors; it may be either in an active or passive state. If a CES, like in the example, represents a program, then the active state of a node means the presence of control in it and its readiness for execution of statements assigned

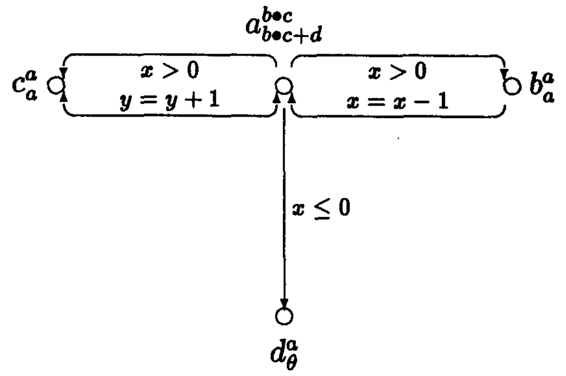

Figure 1

to outgoing arcs. We should note that the statements and conditions on arcs are used only with illustrative goals and they are absent in "real" CESs.

If a node is active, then we try to move control from it simultaneously to all its successors which form a product in its lower (subscript) polynomial — if they are passive. Symmetrically, if a node is passive, then we try to move control to it simultaneously from all its predecessors which form a product of its upper (superscript) polynomial — if they are active (if no predecessors or successors exist, then the upper or lower polynomial is Θ, sometimes omitted). This rule generally gives complex interdependences between nodes in the aspect of control flow: a group of nodes must "negotiate" the possibility of changing their state with each other. Such groups of nodes will play a role analogous to that of transitions in PNs. They are called firing components. The set of all firing components of the CES U is denoted by $FC[U]$.

Let us introduce these notions more formally.

Definition 1.1. Let X be a set called a space of nodes and let θ be a symbol called neutral. The least set Y such that

$\theta \in Y$,

$X \subseteq Y$,

if $K \in Y$ and $L \in Y$, then $(K + L) \in Y$ and $(K + L) \in Y$,

is a set of polynomials over X denoted by $F[X]$.

Definition 1.2. We say that the algebraic system $A = (F[X], +, *, \theta)$ is a near-semi-ring of polynomials over X if the following axioms hold for all $K \in F[X], L \in F[X], M \in F[X], x \in X$:

(+)	$\theta + K = K + \theta = K$	(*)	$\theta * K = K * \theta = K$
(++)	$K + K = K$	(**)	$x * x = x$
(+++)	$K + L = L + K$	(* * *)	$K * L = L * K$

$(++++)\ K+(L+M)=(K+L)+M \quad (*\,*\,**)\ \ K*(L*M)=(K*L)*M$
$(+*) \qquad K*(L+M)=K*L+K*M$
provided that either $L=M=\theta$ or $L\neq\theta$ and $M\neq\theta$

Definition 1.3. Let X be a space of nodes and $(F[X],+,*,\theta)$ be a near-semi-ring of polynomials. A CES over X is a pair (C,E) of functions:
$C: X \longrightarrow F[X]$ (cause function)
$E: X \longrightarrow F[X]$ (effect function)
such that x occurs in the polynomial $C(y)$ iff y occurs in $E(x)$ (then x is a cause of y and y is an effect of x). The set of all CESs over X is denoted by $CE[X]$.
The CES is completely represented by the set of annotated nodes x.

Definition 1.4. Let us define addition and multiplication of functions by $(C1+C2)(x) = C1(x)+C2(x), (C1*C2)(x) = C1(x)*C2(x)$, then an algebra of CES is obtained as follows. Let $\theta : X \longrightarrow F[X]$ be a constant function $\theta(x)=\theta$, let, for brevity, the CES (θ,θ) be denoted by θ, and let $+$ and $*$ on CES be defined by

$(C1,E1)+(C2,E2)=(C1+C2,E1+E2)$
$(C1,E1)*(C2,E2)=(C1*C2,E1*E2)$

Obviously, if $Ui=(Ci,Ei)\in CE[X](i=1,2)$, then $U1+U2\in CE[X]$ and $U1*U2\in CE[X]$, that is, in the resulting structure, x is a cause of y iff y is an effect of x.

Definition 1.5. A CES U is decomposable iff there are CESs V and W such that
$\theta\neq V\neq U, \theta\neq W\neq U$ and either $U=V+W$ or $U=V*W$.

Definition 1.6. Let U and V be CESs. V is a substructure of U iff $V+U=U$. Then we write $V\leq U$. $SUB[U]=\{V : V\leq U\}$. Easy checking ensures that $\leq$ is a partial order. The set of all minimal ($wrt \leq$) and not equal to θ elements of $SUB[U]$ is denoted by $MIN[U]$.

Definition 1.7. For a CES U, let $Q=(C_Q,E_Q)$ be a minimal substructure of U such that for every node x in Q:
(i) polynomials $C_Q(x)$ and $E_Q(x)$ do not comprise '+',
(ii) exactly one polynomial, either $C_Q(x)$ or $E_Q(x)$, is θ.

Then, Q is called a firing component of U. A set of firing components is denoted by $FC[U]=\{Q\in MIN[U] : (i),(ii) hold\}$. We denote by $FC[U]^*$ a set of all finite strings over $FC[U]$, an empty string including. We denote by ${}^\bullet Q$ the set of nodes x in Q with $C_Q(x)=\theta$, and we denote by $Q^\bullet$ the set of nodes x in Q with $E_Q(x)=\theta$.

Definition 1.8. A state s is a subset of the space of nodes X. A node x is active in the state s iff $x \in s$, and it is passive, otherwise.

Definition 1.9. For $Q \in FC[U]$, let $[[Q]]$ denote a binary relation in the set of all states: $(s,t) \in [[Q]]$ iff ${}^\bullet Q \subseteq s, Q^\bullet \cap s = \emptyset, t = (s - {}^\bullet Q) \bigcup Q^\bullet$. The semantics $[[U]]$ of a CES U is a union of relations:

$$[[U]] = \bigcup_{Q \in FC[U]} [[Q]]$$

The semantics $[[U]]^*$ of U is a transitive extension of $[[U]]$.

Definition 1.10. A CES U is a structural deadlock iff $FC[U] = \emptyset$, i.e., U cannot change its state, regardless of the state.

2. Cause-effect structures with coloured tokens

In this section we suggest an extension of the set of CESs by semantics of coloured tokens. Each node of a coloured CES (briefly CCES) may have more than one token simultaneously iff all colours of its tokens are different.

A CCES, as well as CES, is completely represented by the set of annotated nodes. But its formal polynomials range over the extended set of pairs — <node, colour>. Moreover, each its polynomial is 'coloured', that is, each its monomial is associated with some colour-label. For instance, the record
$\{x^{1<a,2>*<b,1>+2<a,3>}, a_{2<x,1>+3<x,2>}, b_{1<x,1>}\}$ means that:

– the node x can receive a token of colour 1 simultaneously from the nodes a and b (if a has a token of colour 2, b has that of colour 1) or a token of colour 2 from the node c (if it has a token of colour 1);

– symmetrically, the node a can send a token of colour 2 or colour 3 to the node x, but x will receive this token repaint in colours 1 or 2, respectively (the first case demands synchronization with the node b iff it has a token of colour 1).

Thus, cause and effect functions in CCESs reflect the space of nodes X not in the set of polynomials $F[X]$, as it is in CESs, but in the set of polynomials coloured by a special function $Cl \times F[X_C]$. According to these remarks, all formal definitions of CCESs are the same as those for CESs.

Moreover, a CCES can be seen as a CES over an extended space of nodes (of pairs <node, colour>). It allows us to introduce such high-level semantics as coloured tokens in a very simple manner. Let us introduce these notions more formally.

Definition 2.1. Let X be a set called a space of nodes and let $Cl = [1..n]$ be a set of colours. Then an extended set $X_C = \{< x, i > | x \in X, i \in Cl\}$ is called a space of coloured nodes.

Definition 2.2. Let X_C be a space of coloured nodes and let θ be a symbol called neutral. The least set Y which satisfies the following conditions:
$\theta \in Y, X_C \subseteq Y$,
if $K \in Y$ and $L \in Y$, then $(K + L) \in Y$ and $(K + L) \in Y$,
is called a set of polynomials over X_C and denoted by $F[X_C]$.

Definition 2.3. We say that the algebraic system $A = (F[X_C], +, *, \theta)$ is a near-semi-ring of polynomials over X_C if the axioms from Definition 1.2 hold for all $K \in F[X_C], L \in F[X_C], M \in F[X_C], x \in X_C$.

Definition 2.4. Let $P \in F[X_C]$ and its canonical form be $\sum M_i$, where M_i is a monomial. The product of two sets $Cl \times F[X_C]$ is a set of 'coloured' polynomials $\{\sum l_i M_i\}$, where l_i is an arbitrary colour, some of which may be identical. Moreover, each monomial M_i may be 'coloured', that is, $M_i = M_i^1 * l_i^2 M_i^2 * .. * l_i^n M_i^{-n}$, where monomials M_i^j are arbitrary components of M_i ('arbitrary', because decomposition of M_i into components can be made in different ways).

Definition 2.5. Let X_C be a space of coloured nodes and $(F[X_C], +, *, \theta)$ be a near-semi-ring of polynomials. A coloured CES (CCES) over X is a pair (C, E) of functions:
$C : X \to Cl \times F[X_C]$ (cause function),
$E : X \to Cl \times F[X_C]$ (effect function),
such that $< x, i >$ occurs in the monomial $jC^j(y)$ from $C(y) = \sum jC^j(y)$ iff $< y, j >$ occurs in $iE^i(x)$ from $E(x)$ (then $< x, i >$ is a cause of $< y, j >$ and $< y, j >$ is an effect of $< x, i >$). The set of all CCESs over X is denoted by $CCE[X]$. The CES is completely represented by the set of annotated nodes.

Definition 2.6. Addition and multiplication of cause functions are defined by:
$(C_1 + C_2)(x) = C_1(x) + C_2(x)$;
$(C_1 * C_2)(x) = C_1(x) * C_2(x)$.
They are the same for effect functions. Then an algebra of CCESs is obtained as follows. Let $\Theta : X \to F[X_C]$ be a constant function $\Theta(x) = \Theta$, for brevity, let the CES (Θ, Θ) be denoted by Θ, and let $+$ and $*$ on CCES be defined by:
$(C_1, E_1) + (C_2, E_2) = (C_1 + C_2, E_1 + E_2)$,
$(C_1, E_1) * (C_2, E_2) = (C_1 * C_2, E_1 * E_2)$.
Obviously, if $U_i = (C_i, E_i) \in CCE[X](i = 1, 2)$, then $U_1 + U_2 \in CCE[X]$ and $U_1 * U_2 \in CCE[X]$.

Definition 2.7. A CCES U is decomposable iff there are CCESs V and W such that $\Theta \neq V \neq U, \Theta \neq W \neq U$ and either $U = V + W$ or $U = V * W$.

Definition 2.8. Let U and V be CCESs. V is a substructure of U iff $V + U = U$. Then we write $V \leq U$. $SUB[U] = \{V | V \leq U\}$. Easy checking ensures that $\leq$ is a partial order. The set of all minimal (*wrt* $\leq$) and $\neq \Theta$ elements of $SUB[U]$ is denoted by $MIN[U]$.

Definition 2.9. For a CCES U, let $Q = (C_Q, E_Q)$ be a minimal substructure of U such that for every node $x \in Q$:
(i) polynomials $C_Q(x), E_Q(x)$ do not comprise $+$,
(ii) exactly one polynomial, either $C_Q(x)$ or $E_Q(x)$, is Θ.

Then Q is called a firing component of U. A set of firing components is denoted by $FC[U] = \{Q \in MIN[U] | (i)$ and (ii) hold$\}$. We denote by $^\bullet Q$ the set of pairs $\{< x, i > | x^{\Theta}_{iE_Q(x)} \in Q\}$, and we denote by $Q^\bullet$ the set of pairs $\{< y, j > | y_{\Theta}^{jC_Q(y)} \in Q\}$.

Remark 1. The set of firing components of a CCES may have groups of firing components whose sets of input and output nodes are identical but have tokens of different colours. Each such a group is analogous to some transition in a coloured PN, and each firing component of such a group is analogous to a binding from the set of all bindings of this transition. Thus, the set of firing components breaks down into groups of equivalent components.

Definition 2.10. A state s is a subset of the space of coloured nodes X_C. A node x is active in the state s iff $\exists i \in Cl :< x, i > \in s$, and it is passive otherwise.

$s(x) = \{i | < x, i > \in s\}$.

Definition 2.11. For $Q \in FC[U]$, let $[Q]_C$ denote a binary relation in the set of all states:
$(s, t) \in [Q]_C$ iff $^\bullet Q \subseteq s, Q^\bullet \cap s = \emptyset, t = (s - {^\bullet Q}) \bigcup Q^\bullet$.

The semantics $[U]_C$ of a CCES U is a union of relations:
$[U]_C = \bigcup_{Q \in FC[U]} [Q]_C$.

Example 1. As an example of a CCES, we use the well-known dining philosophers' system. The philosophers' structure is given in Fig.2. The nodes of the structure are
$X = \{I(idle), L(hasLeft), E(eating), R(hasRight), F(freeForks)\}$.
There are n colours according to the number of philosophers. There are four groups of equivalent firing components:
$Q_i^1 = \{< I, i >, < F, i >\} \bigcup \{< L, i >\}$ — the i-th philosopher takes his left fork iff he is idle and a necessary fork is free;
$Q_i^2 = \{< L, i >, < F, [i+1] >\} \bigcup \{< E, i >\}$ — the i-th philosopher takes his right fork and begins to eat;

$Q_i^3 = \{< E, i >\} \bigcup \{< R, i >, < F, i >\}$ — the i-th philosopher finishes eating and returns his left fork to freeForks;
$Q_i^4 = \{< R, i >\} \bigcup \{< I, i >, < F, [i+1] >\}$ — the i-th philosopher returns his right fork and stays idle.
The initial state of this CCES is $s_0 = \{< I, 1 >, .., < I, n >, < F, 1 >, .., < F, n >\}$.

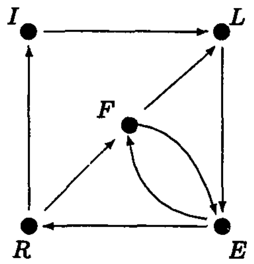

Figure 2

Here $C(I) = \sum i < R, i >, E(I) = \sum i < L, i >$;
$C(L) = \sum i < I, i > * < F, i >, E(L) = \sum i < E, i >$;
$C(E) = \sum i < L, i > * < F, [i+1] >, E(E) = \sum i < R, i > * < F, i >$;
$C(R) = \sum i < E, i >, E(R) = \sum i < I, i > * < F, [i+1] >$;
$C(F) = \sum i(< E, i > + < R, [i-1] >)$,
$E(F) = \sum i(< L, i > + < E, [i-1] >)$,
where $i = 1, .., n$.

One can note that an expressive power of this semantics is greater than Example 1 demonstrates. The semantics of coloured tokens allows each node to send or to receive more than one token simultaneously. This fact is denoted by presenting more than one colour-labels in some monomial of effect or, respectively, cause polynomial. For instance, this example can be simplified as follows:

$$\{I^{\sum i<E,i>}_{\sum i<L,i>}, L^{\sum i<I,i>*<F,i>}_{\sum i<E,i>}, E^{\sum i<L,i>*<F,[i+1]>}_{\sum i[<I,i>*<F,i>*<F,[i+1]>]}, F^{\sum i<E,i>*[i+1]<E,i>}_{\sum i<L,i>+i<E,[i-1]>}\}$$

where $i = 1, .., n$.

That is, a node R is reduced and two groups of equivalent firing components Q^3 and Q^4 are replaced by the one:
$Q^3 = \{< E, i >\} \bigcup \{< I, i >, < F, i >, < F, [i+1] >\}$ – the i-th philosopher finishes eating and returns simultaneously both his forks to freeForks.

3. Relationships between CCESs and CPNs

3.1. Coloured Petri nets

CPNs were defined in [3]. In order to avoid unnecessary technical difficulties, we present here a definition of a CPN which is slightly different from that given in [3]. The differences are not essential but they help us to make this presentation simpler.

A CPN consists of the following components:
* P, a finite set of places.
* T, a finite set of transitions such that $P \bigcap T = \emptyset$.
* $A \subseteq P \times T \bigcup T \times P$, the set of arcs. As for P/T-nets, we define ${}^\bullet t = \{p \in P|(p,t) \in A\}, t^\bullet = \{p \in P|(t,p) \in A\}, {}^\bullet p = \{t \in T|(t,p) \in A\}$ and $p^\bullet = \{t \in T|(p,t) \in A\}$.
* Each $p \in P$ has an associated non-empty set $C(p)$ of token colours.
* A marking M is a function which assigns a value $M(p,c)$ which is equal to 0 or 1 to each $p \in P$ and each $c \in C(p)$. That is, the set $M(p)$ denotes colours of tokens in the place p.
* The initial marking M_0 is a distinguished marking. It represents the initial state of the net.
* Each $t \in T$ has an associated non-empty set $B(t)$ of bindings of t. Each binding is some variant of firing of the transition t.
* Each $a \in A$ such that $a = (p,t)$ or $a = (t,p)$, where $p \in P$ and $t \in T$, has an associated arc expression $W(a)$. The arc expression is a function from $B(t)$ to $\{0,1\}^{C(p)}$. That is, given a binding b, the arc expression produces integer $W(a)(b)(c) \in \{0,1\}$ for each $c \in C(p)$. The arc expression denotes what subset of coloured tokens is taken from or produced for p when t occurs with the binding b.

The dynamics of CPN is defined as follows:
* Let $t \in T$ and $b \in B(t)$. Thus, t is enabled in a marking M for a binding b denoted by $M[(t,b) >$ if and only if
$\forall p \in {}^\bullet t : \forall c \in C(p) : M(p,c) \geq W(p,t)(b)(c)$,
That is, every input place p of t contains at least those coloured tokens which are required by the corresponding arc expression $E(a)$ for the binding b in question.
* If $M[(t,b) >$, then t may occur with the binding b producing a new marking M' such that
$\forall p \in P : M'(p) = (M(p) - W(p,t)(b)) \bigcup W(t,p)(b)$.

3.2. Algorithm of mapping CCESs into CPNs

Let us consider CCES

$$U = \{x^{\sum iC^i(x)}_{\sum iE^i(x)} | x \in X\} \tag{1}$$

with an initial marking $S_0 \subseteq X_C$ and with a set of colours Cl. Let the set $FC[U]$ break down into groups of equivalent firing components

$$FC[U] = \{Q^i | i = 1, .., n\} \tag{2}$$

Since all firing components of such a group have identical sets of input and output nodes, we can define the sets of input and output nodes for each group Q^i as follows:
${}^{\bullet}Q^i = \{x | < x, j > \in {}^{\bullet}Q, Q \in Q^i\}$;
$Q^{i\bullet} = \{y | < y, j > \in Q^{\bullet}, Q \in Q^i\}$,
where $Q \in Q^i$ is arbitrary.

Then the strongly equivalent CPN N for U will be constructed as follows:

$*$ $P = \{x | x \in car[U] \subset X\}$ is a set of places;

$*$ $T = \{t_i\}_{i=1}^{n}$ is a set of transitions, where n is equal to the number of groups of equivalent firing components;

$*$ $A \subseteq P \times T \bigcup T \times P : A = \bigcup_{i=1}^{n} (\bigcup_{x_j \in {}^{\bullet}Q^i} \{(x_j, t_i)\} \bigcup_{y_k \in Q^{i\bullet}} \{(t_i, y_k)\}$ is a set of arcs;

$*$ $\forall p \in P : C(p) = Cl$;

$*$ the initial marking: $\forall x \in P : M_0(x) = S_0(x)$;

$*$ $\forall t_i \in T$: the number of bindings in $B(t_i)$ is equal to the number of equivalent firing components in Q^i, i.e. $\forall b \in B(t_i) : \exists! Q_b \in Q^i$;

$*$ $\forall p \in t^{\bullet} : W(t,p)(b) = Cl^+(p, Q_b)$, $\forall p \in {}^{\bullet}t : W(p,t)(b) = Cl^-(p, Q_b)$,
where
$Cl^+(p, Q_b) = \{i | < p, i > \in Q_b^{\bullet}\}$
$Cl^-(p, Q_b) = \{i | < p, i > \in^{\bullet} Q_b\}$
We need to redefine the notion of a strong equivalence (see [4]) for the case of the coloured token semantics:

Definition 3.1. CCES $U = (X_C, FC[U])$ and CPN $N = (P, T, A, C(P), B(T), W(A, B(T)))$ are strongly equivalent iff there exist two bijections $f : FC[U] \longrightarrow T \times B(T)$ and $g : X_C \longrightarrow P \times C(P)$ such that $g({}^{\bullet}Q) = {}^{\bullet}f(Q)$ and $g(Q^{\bullet}) = f^{\bullet}(Q)$ for any $Q \in FC[U]$.

Theorem 1. *The algorithm constructed above builds a strongly equivalent CPN for any CCES.*

Proof. Let us consider CCES in the form (1), with a set of firing components in the form (2). The algorithm gives rise to two bijections f and g:
$\forall <x,c>\in X_C : g(<x,c>) = (x,c)$.
$\forall i = 1,..,n : f : Q^i \longrightarrow t_i \times B(t_i)$,
such that each firing component $Q \in Q^i$ has an associated unique index, a binding b from $B(t_i)$, i.e. $f(Q^i_b) = (t_i, b)$.
By construction of the bijections f and g, we have:

$g(^\bullet Q^i_b) = \{(x,c)| <x,c>\in {}^\bullet Q^i_b\} = \{(x,c)|x \in {}^\bullet t_i, c \in W(x,t_i)(b)\} = {}^\bullet(t_i,b) = {}^\bullet f(Q^i_b)$;

$g(Q^{i\bullet}_b) = \{(y,c)| <y,c>\in Q^{i\bullet}_b\} = \{(y,c)|y \in t^\bullet_i, c \in W(t_i,y)(b)\} = (t_i,b)^\bullet = f^\bullet(Q^i_b)$,

i.e. the conditions of the definition of a strong equivalence hold. □

Example 2. Let us consider the CCES from Example 1. The algorithm will construct the following CPN:

* $P = \{I, L, E, R, F\}$;
* $T = \{t_i\}^4_{i=1}$;
* the arcs, for instance, for the transition t_1 are as follows:

$$A(t_1) = \bigcup_{x_j \in {}^\bullet Q^1} \{(x_j, t_1)\} \bigcup_{y_k \in Q^{1\bullet}} \{(t_1, y_k)\} = \{(I,t_1),(F,t_1)\} \bigcup \{(t_1, L)\};$$

* $\forall p \in P : C(p) = \{1,..,n\}$;
* the initial marking is as follows: $M_0(I) = \{1,..,n\} = M_0(F)$, i.e. n coloured tokens-philosophers are in the place I and n coloured tokens-forks are in the place F;
* each $t_i \in T$ has n bindings in $B(t_i)$;
* for instance, $W(L,t_2)(b) = W(t_2,E)(b) = [W(F,t_2)(b) - 1]$ for any $b \in B(t_2)$, i.e. any philosopher b which has his left fork (with the same number b) may take his right fork (with the number b plus 1 over mod n) and move to the state 'Eating'.

We can see the resulting CPN in Fig.3.

4. A converse mapping from CPNs to CCESs

The problem is whether a converse mapping from CPNs to CCESs is possible in the sense of a strong equivalence. As Raczunas has shown in [5] for ordinary classes of PNs and CESs, the strong equivalence does not hold for converse mapping.

In [7] a two-level extension of CESs (TCESs) has been introduced. Any CES is completely represented by the set of annotated nodes $\{x^{C(x)}_{E(x)}\}$, where $E(x)$ and $C(x)$ are polynomials with operations + and $*$. We propose to exclude the operation + from formal polynomials, and by resulted elementary CES (or unalternative CES – UCES) we mean a two-level CES of the first syntactical level. Elementary CESs are united by the operation $\oplus$ into the

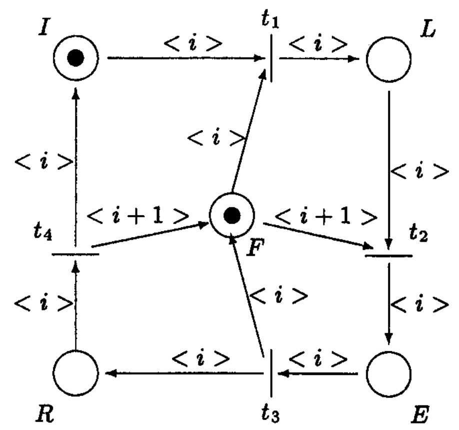

Figure 3

set called a two-level CES of the second syntactical level, or simply TCES. Thus, TCES is a set of the sets of annotated nodes.

So the operation $\oplus$ is a union of the sets of the upper level. It differs from the operation $+$ because it does not merge elementary CESs into a set of annotated nodes.

An operation $\otimes$ on the set of TCESs is the Cartesian product of the sets of the upper level. On the set of UCESs, the operation $\otimes$ is the same as the operation $*$ on the set of CESs.

A TCES in its canonical form is a sum of its firing components. Let us introduce the above notions more formally:

Definition 4.1. Let X be a set called a space of nodes and let θ be a symbol called neutral. The least set Y satisfying the following conditions:
$\theta \in Y, X \subseteq Y$;
if $K \in Y$ and $L \in Y$ then $(K * L) \in Y$,
is called a set of monomials over X and denoted by $M[X]$.

Definition 4.2. An algebraic system $A = (M[X], *, \theta)$ is a semi-group of monomials over X if the following axioms hold for all $K \in M[X]$, $L \in M[X]$, $M \in M[X]$, $x \in X$:

$(*)$ $\theta * K = K * \theta = K$
$(**)$ $x * x = x$
$(***)$ $K * L = L * K$
$(****)$ $K * (L * M) = (K * L) * M$

Definition 4.3. Let X be a space of nodes and $(M[X], *, \theta)$ be a semi-group of monomials. An 'unalternative' cause-effect structure (briefly UCES) over

X is a pair (C, E) of functions:
$C : X \longrightarrow M[X]$ (cause function)
$E : X \longrightarrow M[X]$ (effect function)
such that x occurs in the monomial $C(y)$ iff y occurs in $E(x)$ (then x is a cause of y and y is an effect of x). The set of all UCESs over X is denoted by $UCE[X]$. The UCES is completely represented by the set of annotated nodes x which is called its formula.

Definition 4.4. The sum of any (finite) number of UCESs is called a TCES, where the operation $\oplus$ satisfies the following axioms for any UCESs U, V and W:

(++)	$U \oplus U = U$
(+++)	$U \oplus V = V \oplus U$
(++++)	$U \oplus (V \oplus W) = (U \oplus V) \oplus W$

Another representation of a TCES U is the set of UCESs:
$\{Y_i \in UCE[X] | \sum Y_i = U\}$.
The set of all TCESs over X is denoted by $TCE[X]$.

The set of UCESs is a particular case of the set of CESs, when the operation $+$ is absent. So we may define a notion of a substructure on the sets of UCESs and TCESs.

Definition 4.5. Let U and V be UCESs. V is a substructure of U iff $V + U = U$, where the operation $+$ is an old one from the algebra of CESs. The TCES V is a substructure of the TCES U iff each UCES from V is a substructure of any UCES from U.

Definition 4.6. For a TCES U, let UCES $Q = (C_Q, E_Q)$ be a substructure of U such that for each node $x \in Q$ there exists only one monomial, either $C_Q(x)$ or $E_Q(x)$, being equal to θ. Then Q is called a firing component of U.

The set of all firing components is denoted by $FC[U]$. The set of all finite strings over $FC[U]$ including the empty one is denoted by $FC[U]^*$. The set of nodes $x \in Q$ with $C_Q(x) = \theta$, and the set of nodes $x \in Q$ with $E_Q(x) = \theta$ are denoted by ${}^\bullet Q$ and $Q^\bullet$, respectively.

Definition 4.7. A state s is a subset of the space of nodes X. A node x is active in the state s iff $x \in s$ and passive otherwise.

Definition 4.8. For $Q \in FC[U]$, let $[[Q]]$ denote a binary relation in the set of all the states: $(s, t) \in [[Q]]$ iff ${}^\bullet Q \subseteq s, Q^\bullet \cap s = \emptyset, t = (s - {}^\bullet Q) \bigcup Q^\bullet$. The semantics $[[U]]$ of a GCES U is a union of relations:

$$[[U]] = \bigcup_{Q \in FC[U]} [[Q]]$$

The semantics $[[U]]^*$ of U is a transitive extension of $[[U]]$.

Definition 4.9. A TCES in the form $\{a_x, x^a\}$ is called a basic TCES.

Definition 4.10. Two operations are defined over the $TCE[X]$:
– an operation of nondeterministic choice $\oplus$:
$U \oplus V = \{Y \in UCE[X] | Y \in U or Y \in V\}$;
– an operation of combination $\otimes$: if U and V are UCESs, then $U \otimes V = \{x^{C_U(x)*C_V(x)}_{E_U(x)*E_V(x)} : x \in X\}$;
if $U = U_1 \oplus U_2$, then $U \otimes V = U_1 \otimes V \oplus U_2 \otimes V$ and also for V.

Definition 4.11. The algebraic system $< TCE[X], \oplus, \otimes, \theta >$ is a commutative semi-ring (with a unit θ) of TCESs if the following equality axioms hold for all $U, V, W \in TCE[X]$ and $Y \in UCE[X]$:

		(*)	$\theta \otimes U = U \otimes \theta = U$
(++)	$U \oplus U = U$	(**)	$Y \otimes Y = Y$
(+++)	$U \oplus V = V \oplus U$	(***)	$U \otimes V = V \otimes U$
(++++)	$U \oplus (V \oplus W) = (U \oplus V) \oplus W$	(****)	$U \otimes (V \otimes W) = (U \otimes V) \otimes W$
(+*)	$U \otimes (V \oplus W) = U \otimes V \oplus U \otimes W$		

The fulfillment of the equalities directly follows from Definitions 4.2 and 4.10.

This class preserves both the illustrative possibilities of PNs and the compact algebraic form of CESs. The problem of converse mapping is solved for the class of TCESs. But a TCES has only structural distinctions from a CES, their semantics are the same. So the semantics of coloured tokens is extended to the class of TCESs without essential changes. Thus, each CPN has a strongly equivalent coloured TCES.

Theorem 2. *Any CPN has a strongly equivalent coloured TCES.*

Proof. Let CPN $N = (P, T, A, C(P), B(T), W(A), M_0)$, then CTCES $U = (X_C, FC[U], S_0)$ will be strongly equivalent to N, where:
– the set of nodes X is equal to the set P of places of N;
– the set of coloured nodes $X_C = \{< x, i > | x \in P, i \in C(x)\}$;
– the initial state $S_0 = \{< x, i > | x \in P, i \in M_0(x)\}$;
– the set $FC[U]^g$ is $\{Q_t\}$ such that

$$\forall t \in T : Q_t = \{x_{E(x)} | (x, t) \in A\} \bigcup \{y^{C(y)} | (t, y) \in A\},$$

where $E(x) = \prod_{(t,y)\in A} y, C(y) = \prod_{(x,t)\in A} x$;
– we replace each firing component $Q_t \in FC[U]^g$ by a group of equivalent coloured firing components: $\forall b \in B(t)$:

$^\bullet Q_t^b = \{< x, i > | x_{E(x)} \in Q_t, i \in W(x,t)(b)\}$;

$Q_t^{b\bullet} = \{< y, j > | y^{C(y)} \in Q_t, j \in W(t,y)(b)\}$.

But a TCES in its canonical form is the sum of its firing components, so we complete our construction.
Finally, we check the condition of the Definition 3.1. Bijections g and f are as follows:

$g((x,i)) =< x, i >; f((t,b)) = Q_t^b$

and we have:

$g(^\bullet(t,b)) = g(\{(x,i)|(x,t) \in A, i \in W(x,t)(b)\}) = \{< x, i > | x \in {}^\bullet Q_t, i \in W(x,t)(b)\} = {}^\bullet Q_t^b = {}^\bullet f((t,b))$

$g((t,b)^\bullet) = g(\{(y,j)|(t,y) \in A, j \in W(t,y)(b)\}) = \{< y, j > | y \in Q_t^\bullet, j \in W(t,y)(b)\}) = Q_t^{b\bullet} = f^\bullet((t,b))$

i.e. the conditions of the definition of a strong equivalence are true. □

It is easy to ensure that, applying the algorithm from the proof of Theorem 2 to the CPN from Example 2, we obtain the CCES from Example 1 in the two-level form. That is, this CCES is presented by the sum of all its firing components without operation + in its cause and effect functions.

Conclusion

In the paper we present the extension of the class of cause-effect structures by the semantics of coloured tokens. This semantics correlates with that introduced by K.Jensen in [3] for Petri nets. But possibilities of CESs allow us to introduce it in a more simple and convenient way.

We investigate the relationships of this extended class with the class of coloured PNs. A simple algorithm is constructed that gives us a strongly equivalent CPN for each CCES. The problem is in the fact that not any CPN has a strongly equivalent CCES. We solve this problem by mapping CPNs into an extended class, TCESs, introduced in [7].

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