Bull. Nov. Comp. Center, Num. Anal., 1 (1993)

Quasi-polynomial finite elements in elliptic boundary value problems with small parameter

V.A. Vasilenko

The uniform error estimates with respect to a small parameter are obtained here for the finite element approximation of the elliptic boundary value problem with a small parameter. The space of trial functions is the space of special L-splines with the basis of local functions.

1. General formulation

Let H be the Hilbert space and $f : H \to R^1$ be linear bounded functional over H. Let us consider the family of the symmetric positive definite bilinear forms $a_\alpha : H \times H \to R^1$ depending on the real parameter α, $0 < \alpha \leq 1$,

$$\forall\, u, v \in H \quad a_\alpha(u,v) = a_\alpha(v,u),$$

$$a_\alpha(u,v) \geq \gamma(\alpha)(u,u)_H, \tag{1}$$

where $\gamma(\alpha) > 0$ and is independent of $u \in H$. We need to find the elements $u_\alpha \in H$ which provide the condition

$$\forall\, v \in H \quad a_\alpha(u_\alpha, v) = f(v). \tag{2}$$

In accordance with the Rietz algorithm we consider in H the family of N-dimensional subspaces H_α^N, $0 < \alpha \leq 1$. The approximate solutions u_α^N of problems (2) can be found from the following condition

$$\forall\, v_\alpha^N \in H_\alpha^N \quad a_\alpha(u_\alpha^N, v_\alpha^N) = f(v_\alpha^N).$$

Our aim is the construction of the family of subspaces H_α^N to provide the uniform (with respect to α) convergence of the Rietz algorithm when N tends to infinity.

2. a_α-splines

Let us consider in H the finite set of the linear bounded functionals $k_i H \to R^1$ which are linear independent. We call $\sigma_\alpha^N \in H$ the a_α-splines if for some real values $r_1, r_2, \ldots, r_N$ the element σ_α^N is the solution of the following variational problem

$$\begin{aligned} \sigma_\alpha^N &= \arg \min_{v \in K_r^N} \quad a_\alpha(v,v), \\ K_r^N &= \{v \in H : k_i(v) = r_i, \quad i = 1,2,\ldots,N\}. \end{aligned} \tag{3}$$

Since the bilinear form a_α is coercitive (see (1)), problem (3) is always uniquely solvable [1, 2]. If the element $u \in H$ exists such that $r_i = k_i(u)$, then we say spline σ_α^N interpolates $u \in H$. In this case we denote the resolvent operator for problem (3) by S_α^N, i.e. $\sigma_\alpha^N = S_\alpha^N u$. Then the space of a_α-splines H_α^N is $S_\alpha^N H$.

What is the structure of a_α-splines like? To clarify it we consider the auxiliary problems which look like (2) and can be formulated in the following form: find the elements $k_i^\alpha \in H$ from the condition

$$\forall\, v \in H \quad a_\alpha(k_i^\alpha, v) = k_i(v),$$

for $i = 1,2,\ldots,N$. Then the set K_r^N can be described by the following formula

$$K_r^N = \{v \in H : a_\alpha(k_i^\alpha, v) = r_i, \quad i = 1,2,\ldots,N\}.$$

Taking into account the fact that $a_\alpha(v,v)^{1/2}$ is the norm in H which is reproduced by the scalar product $a_\alpha(u,v)$, we obtain that the solution σ_α^N of problem (3) is the normal spline [ ] and can be always represented in the form

$$\sigma_\alpha^N = \sum_{i=1}^{N} \lambda_i^\alpha k_i^\alpha,$$

where λ_i^α are real coefficients which satisfy the linear algebraic system

$$\sum_{i=1}^{N} a_\alpha(k_i^\alpha, k_j^\alpha)\lambda_i^\alpha = r_j, \quad j = 1,2,\ldots,N. \tag{4}$$

If $r_j = k_j(u)$, $j = 1,2,\ldots,N$, $u \in H$, then the operator $S_\alpha^N : H \to H$ is the orthogonal projector of H on to the linear span of the elements $k_1^\alpha, k_2^\alpha, \ldots, k_N^\alpha$ is the scalar product $a_\alpha(u,v)$, because system (4) is the condition of minimum for the quadratic functional

$$a_\alpha\left(u - \sum_{i=1}^{N} \lambda_i^\alpha k_i^\alpha, \quad u - \sum_{i=1}^{N} \lambda_i^\alpha k_i^\alpha\right)$$

with respect to the variables $\lambda_1^\alpha, \lambda_2^\alpha, \ldots, \lambda_N^\alpha$.

Let us introduce an operator $R_\alpha : H \to H$ by the equality

$$\forall\, v \in H \quad a_\alpha(R_\alpha, u, v) = (u, v)_H. \tag{5}$$

Then we have

$$k_i^\alpha = R_\alpha k_i, \quad i = 1, 2, \ldots, N$$

and for the solution u_α of the initial problem (2) we obtain $u_\alpha = R_\alpha f$. If K^N is the linear span of the elements $k_1, k_2, \ldots, k_N$ and $K_N : H \to E_N$ is the operator defined by formula

$$K_N u = [k_1(u), k_2(u), \ldots, k_N(u)],$$

then the adjoint operator $K_N^* : E_N \to H$ acts according to the rule

$$K_N^* \lambda = \sum_{i=1}^{N} \lambda_i k_i, \quad \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]$$

and the operator S_α^N of the spline interpolation with a_α-splines can be written in the following form

$$S_\alpha^N = R_\alpha K_N^* (K_N R_\alpha K_N^*)^{-1} K_N.$$

In accordance to the Rietz algorithm we find the approximate solution u_α^N of problem (2) in the space of a_α-splines:

$$\forall\, v \in H \quad a_\alpha(u_\alpha^N, S_\alpha^N v) = f(S_\alpha^N v).$$

It is clear that the resolvent operator R_α^N of this problem can be written in the following form

$$u_\alpha^N \equiv R_\alpha^N f = R_\alpha K_N^* (K_N R_\alpha K_N^*)^{-1} K_N R_\alpha f,$$

and $R_\alpha^N = S_\alpha^N R_\alpha$. It means that the Rietz solution u_α^N is a_α-spline which interpolates the exact solution u_α. Hence

$$\begin{aligned} &\alpha(R_\alpha f - R_\alpha^N f, R_\alpha f - R_\alpha^N f) \\ &\quad = a_\alpha(R_\alpha f - S_\alpha^N R_\alpha f, R_\alpha f - S_\alpha^N R_\alpha f) \\ &\quad = a_1(R_1 f, R_\alpha f - S_\alpha^N R_\alpha f) - a_\alpha(S_\alpha^N R_\alpha f, R_\alpha f - S_\alpha^N R_\alpha f). \end{aligned}$$

It is evident that

$$\begin{aligned} a_\alpha(S_\alpha^N R_\alpha f, R_\alpha f - S_\alpha^N R_\alpha f) &= a_\alpha(\sum_{i=1}^N \lambda_i^\alpha k_i^\alpha, R_\alpha f - S_\alpha^N R_\alpha f) \\ &= \sum_{i=1}^N \lambda_i^\alpha (k_i, R_\alpha f - S_\alpha^N R_\alpha f)_H = 0, \end{aligned}$$

because $S_\alpha^N R_\alpha f$ interpolates $R_\alpha f$. In the particular case $\alpha = 1$ we have

$$a_1(S_1^N R_1 f, R_\alpha f - S_\alpha^N R_\alpha f) = 0.$$

Thus,

$$a_\alpha(R_\alpha f - R_\alpha^N f, R_\alpha f - R_\alpha^N f) = a_1(R_1 f - R_1^N f, R_\alpha f - R_\alpha^N f).$$

Denote $\psi_\alpha^N = R_\alpha f - R_\alpha^N f$. Then

$$\begin{aligned} a_\alpha(\psi_\alpha^N, \psi_\alpha^N) &= a_1(R_1 f, \psi_\alpha^N) - a_1(\sum_{i=1}^N \lambda_i^1 R_1 k_i, \psi_\alpha^N) \\ &= (f - \sum_{i=1}^N \lambda_i^1 k_i, \psi_\alpha^N)_H. \end{aligned} \tag{6}$$

Assume now that the Hilbert space H is continuously embedded to the Hilbert space H and the constant $C > 0$ exists (independent of α) such that

$$\forall\, u \in H \quad (u, u)_Y \le C a_\alpha(u, u). \tag{7}$$

Let us introduce the operator $L : H \to Y$ by the identity

$$\forall\, v \in H \quad (Lu, v)_Y = (u, v)_H.$$

Theorem. *If the elements f and k_i, $i = 1, 2, \ldots, N$ belong to the domain of definition of the operator L and condition (7) takes place, then for the Rietz approximation using a_α-splines as the trial functions the following uniform with respect to $\alpha \in (0,1]$ error estimate is valid*

$$a_\alpha(R_\alpha f - R_\alpha^N f, R_\alpha f - R_\alpha^N f) \le C \| L(f - \sum_{i=1}^N \lambda_i^1 k_i) \|_Y^2. \tag{8}$$

Proof. Using (6) we have

$$\begin{aligned} a_\alpha(\psi_\alpha^N, \psi_\alpha^N) &= \left(L(f - \sum_{i=1}^{N} \lambda_i^1 k_i), \psi^N\right)_Y \\ &\leq \left\|L(f - \sum_{i=1}^{N} \lambda_i^1 k_i)\right\|_Y \cdot \|\psi_\alpha^N\|_Y \\ &\leq \left\|L(f - \sum_{i=1}^{N} \lambda_i^1 k_i)\right\|_Y \cdot C^{1/2} a_\alpha(\psi_\alpha^N, \psi_\alpha^N)^{1/2}. \end{aligned}$$

Reducing the factor $a_\alpha(\psi_\alpha^N, \psi_\alpha^N)^{1/2}$ we obtain (8). □

3. a_α-splines in a simple problem with a small parameter

Let H be the Sobolev space $\overset{\circ}{W}{}_2^1(0,1)$ of the quadratic integrable functions and their 1-st derivatives with zero Dirichlet conditions at the end points of the interval $(0,1)$. We introduce the family of belinear form

$$a_\alpha(u,v) = \int_0^1 (\alpha u'v' + uv)dx, \quad \alpha \in (0,1].$$

Let us divide the interval $[0,1]$ by the uniform mesh of points

$$x_i = i \times h, \quad h = 1/N, \quad i = 0,1,\ldots,N,$$

and define linear bounded in H functionals k_i by the formula

$$k_i(u) = \int_{x_{i-1}}^{x_i} u(x)dx, \quad i = 1,2,\ldots,N.$$

In accordance with the previous consideration every a_α-spline which is the solution of the problem

$$\sigma_\alpha = \arg\min_{u \in K_r^h} a_\alpha(u,u),$$

$$K_r^h = \{u \in \overset{\circ}{W}{}_2^1(0,1) : k_i(u) = r_i, \quad i = 1,2,\ldots,N\},$$

is the linear combination of the elements $k_i^\alpha \in H$, where k_i^α is the solution of the projective problem

$$\forall\, v \in \overset{\circ}{W}{}_2^1(0,1) \quad a_\alpha(k_i^\alpha, v) = k_i(v).$$

In other words, k_i^α is the solution of the following boundary value problem

$$-\alpha^2(k_i^\alpha)'' + k_i^\alpha = \varkappa_i, \quad k_i^\alpha(0) = k_i^\alpha(1) = 0, \tag{9}$$

where $\varkappa_i$ is equal to the one inside the interval (x_{i-1}, x_i) and is equal to zero ontside of it. Since the Green function of problem (9) is known

$$G_\alpha(x,\xi) = \begin{cases} c_\alpha \mathrm{sh}\frac{x}{\alpha}\mathrm{sh}\frac{1-\xi}{\alpha}, & x \le \xi, \\ c_\alpha \mathrm{sh}\frac{1-x}{\alpha}\mathrm{sh}\,\frac{\xi}{\alpha}, & x \ge \xi, \end{cases}$$

where $c_\alpha = 1/(\alpha \text{ sh } 1/\alpha)$ and the functions k_i^α can be founded in the explicit form.

$$k_i^\alpha(x) = \begin{cases} A_i \mathrm{sh}\ x/\alpha, & x \le x_{i-1}, \\ B_i \mathrm{sh}\ x/\alpha + C_i \mathrm{sh}\ (1-x)/\alpha, & x_{i-1} \le x \le x_i, \\ D_i \mathrm{sh}\ (1-x)/\alpha, & x \ge x_i. \end{cases}$$

Here A_i, B_i, C_i, D_i are certain constants. Thus, every a_α-spline σ_α can be described by the following conditions:

a) In every subinterval $\Delta_i = [x_{i-1}, x_i]$ the spline σ_α can be written in the following way

$$\sigma_\alpha|_{\Delta_i} = A_0^i + A_1^i e^{\beta(x - x_{i-1})} + A_2^i e^{\beta(x_i - x)},$$

where $\beta = 1/\alpha$ and A_0^i, A_1^i, A_2^i are constants,

b) $\sigma_\alpha \in C^1(0,1)$,

c) $\sigma_\alpha(0) = \sigma_\alpha(1) = 0$.

It is easy to construct the local basis functions in the space of a_α-splines, which play the role of the finite elements. We consider a four-point pattern of the mesh $x_{i-1} < x_i < x_{i+1} < x_{i+2}$ and construct local basis a_α-spline concentrated on the interval $[x_{i-1}, x_{i+2}]$. Denote by σ_i its quasi-polynomial representation on the interval $[x_{i-1}, x_i]$ and require the conditions

$$\sigma_i(x_{i-1}) = 0, \quad \sigma_i'(x_{i-1}) = 0, \quad \sigma_i(x_i) = 1.$$

This brings about the following algebraic system

$$\begin{aligned} A_0^i + A_1^i + A_2^i e^{\beta h} &= 0, \\ A_1^i - A_2^i e^{\beta h} &= 0, \\ A_0^i + A_1^i e^{\beta h} + A_2^i &= 1, \end{aligned}$$

with the solution

$$A_0^i = -\frac{2e^{\beta h}}{(e^{\beta h}-1)^2}, \quad A_1^i = \frac{e^{\beta h}}{(e^{\beta h}-1)^2}, \quad A_2^i = \frac{1}{(e^{\beta h}-1)^2}.$$

For $\sigma_{i+1}(x)$ we require conditions

$$\sigma_{i+1}(x_i) = 1, \quad \sigma'_{i+1}(x_i) = \sigma'_i(x_i), \quad \sigma_{i+1}(x_{i+1}) = 1,$$

resulting in the system

$$\begin{aligned} A_0^{i+1} + A_1^{i+1} + A_2^{i+1} e^{\beta h} &= 1, \\ A_1^{i+1} - A_2^{i+1} e^{\beta h} &= \frac{e^{\beta h}+1}{e^{\beta h}-1}, \\ A_0^{i+1} + A_1^{i+1} e^{\beta h} + A_2^{i+1} &= 1, \end{aligned}$$

with the solution

$$A_1^{i+1} = A_2^{i+1} = -\frac{e^{\beta h}+1}{(e^{\beta h}-1)^2}, \quad A_0^{i+1} = 1 + \left(\frac{e^{\beta h}+1}{e^{\beta h}-1}\right)^2.$$

Similarly, on the interval $[x_{i+1}, x_{i+2}]$ we have

$$\sigma_{i+2}(x_{i+1}) = 1, \quad \sigma'_{i+2}(x_{i+2}) = 0, \quad \sigma_{i+2}(x_{i+2}) = 0,$$

and

$$A_0^{i+2} = \frac{-2e^{\beta h}}{(e^{\beta h}-1)^2}, \quad A_1^{i+2} = \frac{1}{(e^{\beta h}-1)^2}, \quad A_2^{i+2} = \frac{e^{\beta h}}{(e^{\beta h}-1)^2}, \tag{10}$$

The basis function is constructed.

The basis functions near the boundary points $x = 0$, $x = 1$ can be constructed using the boundary condition. Let us consider, for example, the end interval $[0, 2h]$. On the interval $[h, 2h]$ basic function is already constructed by (10). On the interval $[0, h]$ the function is constructed according to the rule

$$\sigma_1(0) = 0, \quad \sigma_1(h) = 1, \quad \sigma'_1(h) = \sigma'_2(h),$$

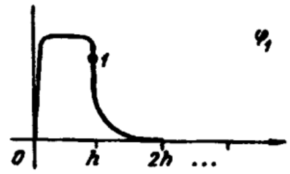

Figure 1

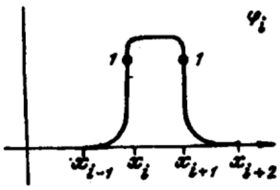

Figure 2

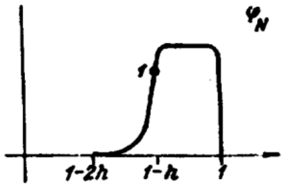

Figure 3

i.e.,

$$\begin{aligned} A_0^1 + A_1^1 + A_2^1 e^{\beta h} &= 0, \\ A_1^1 e^{\beta h} - A_2^1 &= -\frac{e^{\beta h}+1}{e^{\beta h}-1}, \\ A_0^1 + A_1^1 e^{\beta h} + A_2^1 &= 1. \end{aligned}$$

Thus,

$$A_0^1 = \frac{e^{2\beta h} + 4e^{\beta h} + 2}{(e^{\beta h}-1)^2}, \quad A_1^1 = -\frac{e^{\beta h}+2}{(e^{\beta h}-1)^2}, \quad A_2^1 = -\frac{e^{\beta h}+3}{(e^{\beta h}-1)^2}.$$

In Figures 1 - 3 one can see the basic functions near boundary points of the interval $(0,1)$ (see Figures 1, 3) and inside of it (see Figure 2).

Let us find now the corresponding finite element matrix. Under the natural numeration of the basis functions $\varphi_1, \varphi_2, \dots, \varphi_N$ this matrix has 5 diagonals. It is easy to calculate its elements $k_{ij} = a_\alpha(\varphi_i, \varphi_j)$ because

$$a_\alpha(\varphi_i, \varphi_j) = (L_\alpha \varphi_i, \varphi_j)_{L_2},$$

where $L_\alpha u = -\alpha^2 u'' + u$; the function $L_\alpha \varphi_i$ is local piecewise constant, which is concentrated on the interval $[x_{i-1}, x_{i+2}]$, and its values A_0^k on the subintervals $[x_{k-1}, x_k]$, $k = i, \dots, i+2$ are already known. Then

$$a_\alpha(\varphi_i, \varphi_j) == A_0^i \int_{x_{i-1}}^{x_i} \varphi_j + A_0^{i+1} \int_{x_i}^{x_{i+1}} \varphi_j + A_0^{i+2} \int_{x_{i+1}}^{x_{i+2}} \varphi_j.$$

Near the end points $x = 0$, $x = 1$, some natural transformations take place and, finally, we have

$$\begin{aligned} k_{i,i} &= \frac{4}{(c-1)^4}\left[h(c^4 + 4c^2 + 1) - \alpha(c^4 + c^3 - c - 1)\right], \\ & \quad i = 2, 3, \dots, N-1, \end{aligned}$$

$$
\begin{aligned}
k_{i,i\pm1} &= \frac{2}{(c-1)^4}\,[-4hc(c^2+1)+\alpha(c^2-1)(c^2+4c+1)],\\
&\quad i=3,4,\dots,N-2,\\
k_{i,i\pm2} &= \frac{2}{(c-1)^4}\,[2c^22h-\alpha c(c^2-1)],\\
&\quad i=4,5,\dots,N-3,\\
k_{1,1} &= \frac{1}{(c-1)^4}\,[h((c^2+4c+2)^2+4c^2)\\
&\quad -\alpha(2c^4+9c^3+11c^2-12c-10)],\\
k_{1,2} &= \frac{1}{(c-1)^4}\,[-2hc(3c^22+4c+4)\\
&\quad +\alpha(c^22-1)(c^2+8c+2)],\\
k_{1,3} &= \frac{2}{(c-1)^4}\,[2c^2h-\alpha c(c^2-1)],
\end{aligned}
$$

where $c=e^{\beta h}$, $\beta=1/\alpha$.

If $\alpha \ll 1$, $h \gg \alpha$, then $e^{\beta h} \gg 1$ and the following asymptotic equalities take place which are readily used in practice

$$
\begin{aligned}
&k_{i,i} \cong 4h-4\alpha, \quad k_{i,i\pm1} \cong 2\alpha, \quad k_{i,i\pm2} \cong -2\alpha e^{-h/\alpha},\\
&k_{1,1} \cong h-2\alpha, \quad\ \ k_{1,2} \cong \alpha, \quad\ \ k_{1,3} \cong -2\alpha e^{-h/\alpha}.
\end{aligned}
$$

In the case $\alpha \ll h$ the finite element matrix is strongly diagonally dominating and can be efficiently reverted. The right-hand side of this system consists of the elements

$$
g_i = A_0^i \int_{x_{i-1}}^{x_i} f dx + A_0^{i+1} \int_{x_1}^{x_{i+1}} f dx + A_0^{i+2} \int_{x_{i+1}}^{x_{i+2}} f dx.
$$

4. Splines and spectrum equivalent operators

Let us consider in H an other family of symmetric positive defined belinear forms $\overline{a}_\alpha : H \times H \to R^1$, which also depends on the real parameter $\alpha \in (0,1]$, and assume that this family is connected with the initial family (1 - 2) by the relation of equivalence, namely, two constants $C_1 > 0$, $C_2 > 0$ exist and independent of α such that

$$
\forall\, u \in H \quad C_1\overline{a}_\alpha(u,u) \le a_\alpha(u,u) \le C_2\overline{a}_\alpha(u,u). \tag{11}
$$

If we introduce, like in (5), the resolvent operator $\overline{R}_\alpha : H \to H$ by the identity

$$
\forall\, v \in H \quad \overline{a}_\alpha(\overline{R}_\alpha u, v) = (u,v)_H,
$$

then inequality (11) can be written in the following form

$$C_1(\overline{R}_\alpha^{-1}u,u)_H \le (R_\alpha^{-1}u,u)_H \le C_2(\overline{R}_\alpha^{-1}u,u)_H.$$

Taking into account the symmetry of the operators R_α and $\overline{R}_\alpha$, we obtain

$$C_1(\overline{R}_\alpha^{-1/2}u,\overline{R}_\alpha^{-1/2}u)_H \le (R_\alpha^{-1}u,u)_H \le C_2(\overline{R}_\alpha^{-1/2}u,\overline{R}^{-1/2}u)_H.$$

Denote by $v = \overline{R}_\alpha^{-1/2}u$. Then we have

$$\forall\, v \in H \quad C_1(v,v)_H \le (\overline{R}_\alpha^{1/2}R_\alpha^{-1}\overline{R}_\alpha^{1/2}v,v)_H \le C_2(v,v)_H.$$

Thus the operator $\overline{R}_\alpha^{1/2}R_\alpha^{-1}\overline{R}_\alpha^{1/2}$ is spectral equivalent to the unit operator.

Let us consider N-dimensional space $\overline{H}_\alpha^N$ of $\overline{a}_\alpha$-splines and find an approximate solution $\tilde{u}_\alpha^N \in \overline{H}_\alpha^N$ from the condition

$$\forall\, \overline{v}_\alpha^N \in \overline{H}_\alpha^N \quad a_\alpha(\tilde{u}_\alpha^N,\overline{v}_\alpha^N) = f(\overline{v}_\alpha^N). \tag{12}$$

It is clear that the resolvent operator $\overline{R}_\alpha^N : H \to \overline{H}_\alpha^N$ of problem (12) can be written in the form

$$\tilde{u}_\alpha^N = \tilde{R}_\alpha^N f = \overline{R}_\alpha K_N^*(K_N\overline{R}_\alpha R_\alpha^{-1}\overline{R}_\alpha K_N^*)^{-1}K_N\overline{R}_\alpha f.$$

Denote $M_\alpha = \overline{R}_\alpha R_\alpha^{-1}$. Then

$$\tilde{R}_\alpha^N = R_\alpha M_\alpha^* K_N^*(K_N M_\alpha R_\alpha M_\alpha^* K_N^*)^{-1}K_N M_\alpha R_\alpha$$

similar to the operator

$$R_\alpha^N = R_\alpha K_N^*(K_N R_\alpha K_N^*)^{-1}K_N R_\alpha,$$

which realizes the orthogonal projection of the element $R_\alpha f = u_\alpha$ onto the a_α-splines $H_\alpha^N = R_\alpha K_N$ (K_N is the linear span of the elements $k_1,\dots,k_N$) in the scalar product $a_\alpha(u,v)$; the operator R_α^N realizes the orthogonal projection of the same element to the subspace $\overline{H}_\alpha^N = R_\alpha M_\alpha^* K_N = \overline{R}_\alpha K_N$. Let $k \in K_N$, then

$$a_\alpha(R_\alpha k,\overline{R}_\alpha k) = (k,\overline{R}_\alpha k)_H = \overline{a}_\alpha(\overline{R}_\alpha k,\overline{R}_\alpha k) \ge \frac{1}{c_2}\, a_\alpha(\overline{R}_\alpha k,\overline{R}_\alpha k).$$

If we denote $\|u\|_\alpha = a_\alpha(u,u)^{1/2}$, then we have

$$\cos\Theta \ge \frac{1}{c_2}\int\limits_{u\in H}\frac{\|\overline{R}_\alpha u\|_\alpha}{\|R_\alpha u\|_\alpha},$$

where $\cos\Theta$ is the cosine of the angle between the vectors $R_\alpha k$ and $\overline{R}_\alpha k$. It is easy to see that

$$\begin{aligned}\inf_{u\in H}\frac{\|\overline{R}_\alpha u\|_\alpha}{\|R_\alpha u\|_\alpha} &= \inf_{\|v\|_\alpha=1}\|\overline{R}_\alpha R_\alpha^{-1}v\|_\alpha = \inf_{u\in H}\frac{a_\alpha(\overline{R}_\alpha R_\alpha^{-1}v,v)}{a_\alpha(v,v)}\\ &= \inf_{v\in H}\frac{(R_\alpha^{-1}\overline{R}_\alpha R_\alpha^{-1}v,v)_H}{R_\alpha^{-1}v,v)_H} = \inf_{w\in H}\frac{(R_\alpha^{-1/2}\overline{R}_\alpha R_\alpha^{-1/2}w,w)_H}{(w,w)_H}\\ &\geq C_1.\end{aligned}$$

Thus, $\cos\Theta \geq C_1/C_2$ independent of $\alpha > 0$. It means that the angle between the subspaces H_α^N and H_α^N is bounded by the constant $\Theta_0 < 1$ uniformly with respect to α. Finally, for the Rietz approximation, using $\overline{a}_\alpha$-splines instead of a_α-splines as the trial functions, the same uniform (with respect to α) error estimate (8) takes place, but with the other constant $C > 0$.

References

[1] V.A. Vasilenko, A.V. Kovalkov, M.V. Zuzin, Spline functions and digital filters, - Publishing House of Computing Center USSR Ac. Sci., Novosibirsk, 1984, 156 pp. (in Russian).

[2] V.A. Vasilenko, Local basises in the space of L-splines, Computing Algorithms in the Mathematical Physics, - Publishing House of Computing Center USSR Ac. Sci., Novosibirsk, 1983, pp. 28-40 (in Russian).

[3] V.A. Vasilenko, Quasipolynomial finite elements in elliptic problems with small parameter, Variational Difference Methods in the Problems of Numerical Analysis - Publishing House of Computing Center USSR Ac. Sci., Novosibirsk, 1988, pp.12-27 (in Russian).